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RISK-BEARING AND THE THEORY OF INCOME DISTRIBUTION

by

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ABSTRACT

This paper develops a stochastic theory of distribution by introducing a class of dynamic models focusing on the role of incomplete markets in generating inequality. Without complete markets, there is no possibility of perfect insurance against risk; variability in income is therefore attributable to the nature of available contracts rather than to differences in preferences or endowments. Unlike that of previous models, this approach takes explicit account of the reason for market incompleteness in modeling agents' behavior; in particular, the amount of risk borne by agents is endogenous.

The framework we adopt modifies the standard model of growth with altruism: bequests are composed of a safe asset and a risky investment project requiring unobservable effort. Agents are risk-averse and partially insure by issuing equity contracts in their "firms"; incentive compatibility requires that they retain a portion of the equity in their own firms. Lineage wealth follows a Markov chain displaying global convergence to an ergodic distribution which also represents the long-run distribution of wealth for the population.

The paper helps to illuminate the role of particular assumptions (such as availability of production loans and unboundedness of utility) in generating the qualitative properties of the distribution of wealth, the choice of "occupation," and the prevention of poverty traps. The analysis is complicated by the nonconvexities and nonoptimalities introduced by incentive constraints.

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RISK-BEARING AND THE THEORY OF INCOME DISTRIBUTION

1. Introduction

Does a market economy exacerbate the level of inequality in wealth and income, or does it merely reproduce variation in individual attributes? Indeed, under certain circumstances might it even attenuate those differences? One way to approach these questions is to consider a dynamic economy and to ask what relation the long-run distribution of wealth has to the initial distribution of endowments and the idiosyncratic risks that individuals encounter over time.

It turns out, of course, that the answers one obtains will depend on just what one means by a market economy. Under the complete markets of Arrow and Debreu, there are essentially two sources of variation in income, namely differences in preferences and differences in endowments:[1] variations in the outcomes of risky prospects play no role, since agents will insure themselves perfectly.[2] Identical agents would remain identical through time, and under standard assumptions about preferences, the ranking of individuals would be preserved.

This is unsatisfactory for several reasons. It is empirically absurd to rule out social mobility and to claim that people do not bear idiosyncratic risks against which they would like to insure. It is also empirically very

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[1]State-dependent preferences could be another source of variation but they are somewhat beyond the framework of the standard model.

[2]Note that this does not say that risk is eliminated entirely, only that everyone's fortunes will be perfectly correlated. The economy as a whole may follow a stochastic process; this is one interpretation of the model in Brock and Mirman (1972). In particular, the time average for one agent of a function of income need bear no resemblance to its average across all agents at a given time.

well established that to the extent such things can be measured, differences in abilities explain only a small part of the differences in earnings. What is more, as Becker and Tomes (1979, 1986), for example, argue, in a complete-markets world, it should be possible for parents to insure against uncertainty about the abilities of their children, and therefore variations in endowments cannot be taken as purely exogenous. In any case, it is logically incomplete to try to explain the wealth distribution entirely in terms of exogenous differences in tastes and endowments without first asking what the long run distribution would look like in a world of identical agents who start their lives with the same endowment.

One answer to this last question was given a long time ago by Champernowne (1953). He started with a population of identical agents and showed that if each agent bore an idiosyncratic risk proportional to its wealth, then the long-run income distribution would approximate a Paretian distribution.

The problem with this style of analysis is that there is no explanation of why the agents bear this (or any other) kind of uninsured risk. The possibilities of choosing how much risk to bear and obtaining insurance to cover the rest are not considered at all, and as result the structure of the economy (i.e. which markets, what kind of technology and what kind of information are available) plays no role in determining the nature of the wealth distribution.

Thus, this approach suffers from a shortcoming similar to that of the complete-markets approach: it takes the entire question of distribution outside the scope of economic analysis. In fact, it is precisely the relation between what we just described as the structure of the economy and the long run wealth distribution which is the central concern of this paper.

To analyze this question, we develop a model which essentially combines two existing strands of analysis in the broad Arrow-Debreu framework; the theory of incomplete markets and insurance and the neoclassical theory of growth with altruism.[3] This formulation is not entirely novel. Loury (1981) and Eckstein et al. (1985) acknowledge the role of incomplete markets in

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[3]This latter element enables us to focus on the role of savings and bequests in the wealth dynamics.

generating the long run income distribution but do not attempt to take explicit account of the source of market incompleteness in modeling agents' behavior. Galor and Zeira (1988) provides an example of a model which emphasizes the role of imperfect loan markets in analyzing the shape of distributions in general and poverty traps in particular.

While we also look at the role of imperfect loan markets, the key element in our paper, by contrast, is a model of incomplete (rather than absent) insurance due to moral hazard in production. In this sense, our approach resembles that of the simulation study by Phelan and Townsend (1988), which considers moral hazard with an infinite horizon. Green (1987) is another recent attempt to study the stochastic process of consumption, although there the source of imperfect insurance is adverse selection. We have chosen to examine the moral hazard introduced by imperfect monitoring not only for its importance in a production economy, but also for its modeling simplicity.

The general framework we adopt is a modification of the standard theory of growth under altruism as expounded, for instance, in Bernheim and Ray (1987) and Kohlberg (1974). Agents are risk-averse and have identical preferences and labor endowments,[4] deriving utility from consumption and from a bequest which becomes the entire nonlabor endowment ("wealth") of their offspring.

We introduce risk into the model by assuming that there are two primary assets available, one safe and one risky but paying a higher expected rate of return.[5] The risky asset takes the form of an investment project which pays the high stochastic return only if one puts a certain amount of unobservable effort into it. Once the project has been set up, one can sell shares in it, thereby insuring oneself. We assume that there are a very large number of

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[4]The assumption of identical preferences and labor endowments is not just a modeling simplification: in order to study the contribution of particular market conditions to generating differences across agents, it is best not to confound issues by starting out with variations among agents.

[5]Our attitude toward the source of the risk is agnostic. The obvious interpretation is that it is "pure luck" in some production process, but if one prefers one can think of it as a difference in ability which one is not aware of ex ante. This reinterpretation allows us to use our framework to ask questions like the old one of what is the relation between the distribution of abilities and the distribution of wealth in the long run.

agents so that potentially one could insure away all risk. However, the market will never provide such insurance because agents would lose the incentive to put in the requisite amount of effort. Each entrepreneur[6] will have to hold at least a fraction β of the shares of its own project for its equity contracts to be incentive compatible.[7]

In addition to this imperfection, we will also assume that there is an elastic supply of production loans available but no market for consumption loans. In the penultimate section we informally consider the implications of making the supply of production loans endogenous; an earlier version of the present paper (Banerjee and Newman, 1989) considers the implications of allowing consumption loans.

Under the assumptions mentioned above and some strong but relatively standard assumptions on the utility function we can show that

1) There is a unique, positive level of wealth below which everybody would undertake a project. Above this level people will become rentiers, investing in the safe asset only; their children will be poorer than them.

2) β is nondecreasing as a function of wealth and so in this sense the poor will bear less absolute risk (though possibly more relative risk).

3) There is a level of wealth below which people's wealth will increase from the present to the future generation irrespective of the realization of the uncertainty. In fact these low levels of wealth will not persist in the long run. Extreme poverty therefore is a transient phenomenon in this model.

4) A unique ergodic distribution exists for this model of the evolution of lineage income;[8] it is supported on a compact interval. Moreover, any initial distribution of income will converge to the ergodic

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[6] This term is used loosely — it is possible that what we call an entrepreneur is someone who simply invests in her children's education.

[7] For analysis of the firm along these lines see for example Jensen and Meckling (1976).

[8] The idea of modeling the distribution across agents by seeking an ergodic distribution for a single agent goes back at least to Champernowne (1953) and is used more explicitly in Loury (1981).

distribution.

5) Simulations for a range of parameter values suggest that the long-run distribution of income generated by this model is single-peaked and skewed toward low wealth levels, like most empirical distributions.[9] See Appendix C for an example.

None of these results, of course, guarantee that the distribution the model generates is empirically reasonable. Nevertheless, the fact that the distribution is on a connected interval and that there is only one switch point between being an entrepreneur and being a rentier is reassuring, particularly since the presence of the incentive constraint introduces nonconvexities which are a common source of perverse results. The heavy concentration of agents towards the bottom end of the distribution in our simulations is also reassuring since our model is undeniably biased in the direction of making upward mobility easy.[10]

Consequently we feel justified in thinking of the basic model we present in this paper as a "benchmark" for comparison with other models with more complex structures. In particular, we cannot relax the key assumptions of this model without losing some of the "nice" properties we list above.

For example, eliminating the market for production loans may create a poverty trap since the poor will only be able to earn a low return on their assets and as a result may accumulate very little. The assumptions about the utility function, particularly that it is unbounded below also play an important role: with a lower bound on utility, it may be impossible to satisfy incentive compatibility, especially at low levels of wealth, and this would tend to restrict mobility, possibly leading to a poverty trap. Another assumption (assumption (2.2) below) which puts a bound on the strength of the

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[9] There are two reasons for this: first, as wealth increases, the relative share of the safe asset in the agent's portfolio increases; since holding the safe asset tends to cause wealth to decrease (see Proposition (5.5) below), the expected increment in wealth tends to decline as wealth rises. In addition, the amount of risk borne also increases with wealth, so that even without the first effect, there would be some "stretching" of the tail at high wealth levels.

[10] In particular, the assumption that the poor can easily get production loans is clearly empirically dubious.

bequest motive is also crucial in ensuring that an ergodic distribution exists.

However, as we show in the earlier version of this paper, it is also true (and this is reassuring) that the assumption about the absence of a market for consumption loans turns out to be the least crucial. The properties of the long-run wealth distribution listed above remain true even if we drop this assumption, except in certain perverse cases.

The earlier version of this paper also obtains reasonably strong steady-state convergence properties for a complete-markets (perfect insurance) rendering of the model.[11] This underscores our contention that completeness of markets cannot yield an adequate explanation of distribution, and also that our results concerning the existence of a nondegenerate limiting distribution do not depend on peculiar assumptions about technology or preferences, but rather on the "institutional" assumptions about market structure.

The inclusion of endogenous levels of risk-bearing in an otherwise standard growth model is of some independent interest. There is definitely a view in the theory of development, for which Schumpeter is one source but which almost certainly predates him, which draws a strong connection between growth and risk-bearing. By making the extent of risk-bearing endogenous, we open the way for examining questions concerning the distribution of risk-bearing and the effects of policies which influence incentives on the pattern of growth.

To this end we say a few things about the comparative statics of the limiting distribution. In particular, if we measure inequality by the range statistic, some types of productivity growth can be shown to increase inequality while other types will reduce it. A linear profits tax with complete loss offset is shown to be fully neutral. Apart from being of some independent interest, this result shows that making risk-bearing endogenous changes things considerably: in a model in which the degree of risk was taken as given, this policy would have significant effects. Finally, for sufficiently high levels of labor productivity, the rentier class will be

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[11] An exception is the presence of a cycle rather than a stable steady state for one version of the model; from the point of view of distribution theory, however, this is no more adequate than steady-state convergence.

eliminated.

The technique we use to demonstrate convergence also has some independent interest, albeit from a purely technical point of view. Because the Markov chain describing the evolution of lineage wealth turns out to be neither continuous nor monotonic, we cannot employ standard convergence theorems. Instead we first prove existence of an invariant measure using a theorem due to Duffie et al. (1988) and then invoke a theorem due to Doob which provides conditions for convergence once an invariant measure is in hand (specifically, indecomposability of the state space under all iterates; and absolute continuity of the process with respect to the invariant measure). So far as we know, this represents the first application of the Doob theorem in the economics literature.

We consider this essay as a preliminary exercise in the development of a theory of growth and distribution. Its contribution, as we see it, is not so much that it predicts results that we did not expect but that it tells us what is involved in modeling these issues. Specifically, this is a model which cannot be transformed in to a single-agent dynamic optimization problem. Further, the dynamics are stochastic. The analytical tools we need are therefore somewhat different from those customarily used in dynamic macroeconomics. However, despite the problems created by nonconvexities introduced by the incentive constraints, it turns out to be relatively easy to characterize the transition equations, and we can therefore say something about the long-run behavior of the model.

2. Demographics, Preferences and Technology

The economy consists of a large number of agents with identical preferences who are active for one period. Each agent reproduces asexually one agent identical to itself, so the economy is always the same size. An agent's endowment consists of a bequest inherited from its parent. The agent consumes some of the endowment and invests the remainder. The return on the investment becomes the bequest to the agent's offspring.

An agent's preferences are described by the following von Neumann-Morgenstern utility function:

$$E\{u(c_t) + v(b_t) - e_t\},$$

where E is the expectation operator, c_t is the consumption of an agent who lives in period t, b_t is the bequest it leaves to its offspring and e_t is the level of entrepreneurial effort the agent expends on its investment project, should it undertake one. Both $u(\cdot)$ and $v(\cdot)$, defined on the positive reals, are increasing, strictly concave,[12] smooth and satisfy

$$u'(x) \geq Av'(x), \text{ where } A > 1. \qquad (2.1)$$

We further assume they are bounded above and unbounded below. Finally, we allow effort to take on only two values, namely 0 and $\bar{e}$.

The form of the bequest preference is peculiar in two respects. First, we assume that the preference is for the bequest itself, not for the offspring's consumption or utility; it may be supposed that agents desire to adhere to some tradition for bequest-giving. It is not clear to us which formulation is most "realistic," and ours does have the virtue of greater simplicity to recommend it. Second, we have assumed that the bequest utility is unbounded below. While this is hardly tenable for a serious understanding of bequest motives, it turns out to be the easiest model to deal with: it helps to guarantee that the incentive compatibility constraint can be satisfied. We have already suggested some of the difficulties that will be encountered when this assumption is relaxed, but a full treatment awaits further research.

After choosing its consumption level, an agent will wish to invest the remainder of its endowment so as to maximize the expected utility of the bequest. It has three options available. First, it can invest in a safe, perfectly divisible physical asset which yields the gross return $\hat{r}$.[13] This return may exceed unity, but is strictly less than the constant A referred to above. Thus, the following relation holds between the marginal utility of consumption and that of bequest-giving:[14]

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[12] One may be tempted to interpret $v(\cdot)$ as the dynamic-programming value function for an infinite-horizon version of the model. The problem with this formulation, however, is that $v(\cdot)$ would not satisfy concavity — even for a continuous-effort version of the model — because of the incentive compatibility constraint which we introduce below.

[13] Alternatively, one might think of this safe return as a prevailing world gross interest rate which our economy takes as given.

[14] If $v(\cdot) = \delta u(\cdot)$, then this relation assumes the standard form $\delta\hat{r} < 1$.

$$u'(x) > v'(x)\hat{r}. \tag{2.2}$$

The second choice available to the agent is to invest in financial assets of two types: loans of "start-up capital" to other agents and equity shares in other agents' projects (recall there are no consumption loans). In the next section we argue that both assets are safe and therefore yield return $\hat{r}$. Finally, the agent may invest in its own risky investment project.

The project requires a fixed amount I of capital and, in general, is financed jointly by many agents. It yields a risky gross return $\tilde{r}$ with high expected value if the agent puts forth a unit of entrepreneurial effort; should the agent shirk, however, a low return is yielded with certainty. The random variable $\tilde{r}$ has distribution F(r) which is supported on the interval $[r_0, r_1]$ and is mutually absolutely continuous with Lebesgue measure there; we assume that the worst return r_0 is the same as the return from shirking. In our notation:

$$\tilde{r} \begin{cases} \leq r \in [r_0, r_1] & \text{with probability } F(r), \text{ if } e = \bar{e} \\ = r_0 & \text{with probability } 1, \quad \text{if } e = 0 \end{cases}$$

In order to avoid trivialities, the following relations hold:

$$r_0 < \hat{r} < \bar{r},$$

where

$$\bar{r} \equiv \int r dF(r)$$

(integration is understood to be over $[r_0, r_1]$ unless otherwise indicated). Obviously $\bar{r} < r_1$.

A crucial assumption which will underlie most of our results is that the returns on the project are independent across agents and over time.

3. Finance

Each agent who undertakes a project may be thought of as an owner-manager of a firm, à la Jensen and Meckling (1976). Agents will have two incentives for seeking outside finance for their projects: first, since they are risk-averse, they prefer to share risk with other agents by issuing equity; second, through borrowing, poor agents are able to afford to undertake projects, even if their wealth ω is small compared to I.

We assume that agents issue linear equity shares in their project. After choosing its consumption, an agent selects a fraction β of the project for which it is liable.[15] It commits this portion βI to the project (this commitment is observable to other agents) and then goes to the equity market with its shares $(1-\beta)I$, which are sold at a price P. The agent then uses part of the proceeds of equity sales for financing the remaining $(1-\beta)I$ of the project (again, all commitments of capital are observable). In the following period, the agent's offspring pays dividends of $(1-\beta)Ir$, where r is the realized value of the random return $\tilde{r}$, to the offspring of the equity holders.

The remainder of the agent's saving is invested in either the safe asset or the equity of other agent's firms. Observe that all existing firms are essentially alike: they have identical distributions of returns, provided their owner-managers work. Clearly, an agent will wish to hold an equal share of his invested wealth in each of the firms in the economy, at least in those in which the manager's promise to work is credible. We can imagine that there is a mutual fund which buys all issued equity and then resells shares of itself to the agents. A share of the mutual fund costs P, the same as a share of equity in a typical firm. By the law of large numbers, the mutual fund's dividend is exactly the expected project return $\bar{r}$.[16] Thus, a share of the mutual fund is a safe asset yielding gross return $\bar{r}/P$. If the asset market is to be in equilibrium, then, we must have

$$\frac{\bar{r}}{P} = \hat{r}, \quad \text{or } P = \frac{\bar{r}}{\hat{r}} .$$

Thus, the agent is indifferent between holding the safe asset or investing in the rest of the economy. This arbitrage condition greatly simplifies the analysis, because it says that the return to holding *and* issuing equity is independent of distribution (as long as a positive measure of the agents are undertaking projects).

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[15] We recognize that such contracts are not optimal, but they are realistic. Moreover, even the optimal contract will still require that agents bear some risk (if shirking yields a noisy rather than a certain return), so the general flavor of our results would not be greatly affected by respecifying the equity contract.

[16] Strictly speaking, we must require that at any time enough agents are taking the project (for example, a positive measure of a continuum) in order that the law of large numbers apply. This turns out to be true because of Proposition (7.4) below, which implies that once a positive measure of agents takes the project, there will always be a positive measure of project-takers.

We can now write the equation for the bequest in terms of wealth and the choice variables c, β and e. If an agent does not start a firm, it invests in the safe asset and the stock market, earning the return $\hat{r}$. Thus,

$$b_t = (\omega_t - c_t)\hat{r}, \quad \text{if } e_t = 0. \tag{3.1}$$

If the agent does start a firm and conscientiously sets $e = \bar{e}$, then after consuming c_t and laying I into the project, it has $\omega_t - c_t - I + (1-\beta_t)I\frac{\bar{r}}{\hat{r}}$ remaining to invest at the safe return $\hat{r}$. The agent's offspring collects Ir_t from the project and pays out $(1-\beta_t)Ir_t$ to its shareholders, so that the bequest is

$$b_t = (\omega_t - c_t - I)\hat{r} + (1 - \beta_t)I\bar{r} + \beta_t Ir_t, \quad \text{if } e_t = \bar{e}. \tag{3.2}$$

Note that if the agent borrows for production, it has additional assets Z on which the safe return is earned, but also a liability of $Z\hat{r}$, so that the equation is valid whether the agent borrows or not. And, since $\omega_{t+1} = b_t$, these equations also describe the evolution of a lineage's wealth.

Because of the importance of the bequest in studying the dynamics of wealth, we introduce some notation, writing the realized value of the bequest in state r when the project is taken as a function of wealth, consumption and equity holdings:

$$B_r(\omega,c,\beta) \equiv (\omega-c)\hat{r} + I(\bar{r}-\hat{r}) + I\beta(r-\bar{r}),$$

which is essentially a rearrangement of (3.2). The cases corresponding to r_0 and r_1 are written B_0 and B_1. Similarly, we write

$$B_s(\omega,c) \equiv (\omega-c)\hat{r}$$

for the case in which the safe strategy is followed. Sometimes the arguments of the functions B_r and B_s will be dropped.

4. Incentive Compatibility

Since effort is not observable, or at least not enforceable, an agent has an incentive to issue equity, for which it earns a fairly high return, and shirk, thereby saving itself the disutility of effort, even while it accepts the low return r_0 for its offspring. Thus, not any choice of β will do to convince potential buyers of the firm's equity that the manager will work: indeed, it will do so only if the agent has enough of a stake in its own firm.

Formally, the utility of working must exceed that obtained from pretending to undertake the project and then shirking:

$$\int v(B_r)dF(r) - \bar{e} \geq v(B_0) \tag{IC'}$$

(As consumption is committed in advance, the utility of consumption does not enter here; equivalently, the condition may be regarded as following from dynamic consistency.) Note that for a given β, this inequality will hold for some wealth levels, but not others. In order that the inequality be operative as an incentive compatibility constraint, therefore, it is necessary to assume that an agent's inherited wealth,[17] as well as the contribution βI, is observable.

To summarize, we restate the agent's problem as

$$\max_{(c,\beta,e)} \quad E\{u(c) + v(b) - e\} \tag{4.1}$$
$$\text{s.t. } (3.1),\ (3.2),\ (IC),$$
$$c \geq 0,\ 0 \leq \beta \leq 1,\ e \in \{0,\bar{e}\},$$
$$\omega - c \geq 0.$$

The last constraint reflects the fact that the agent cannot borrow to finance consumption. We do not allow consumption loans because such contracts would be unenforceable under the time and information structure of our model: having already consumed, the agent has nothing to hand over to the creditor (except its head). Knowing this, no other agent will be willing to lend for consumption. By contrast, production loans are feasible because the physical investment is observable, and the assets are physically present to be seized by a creditor, should the agent attempt to default on the loan (we will show below [Proposition 5.4] that even under the poorest realization of the project, the bequest will be nonnegative after repayment of the loan, so that the argument for enforceability of production loan contracts is justified). No significant qualitative results depend on our exclusion of consumption loans; a variation of the model in which they are included is analyzed in Banerjee and Newman (1989).

5. Behavior of the Agent

As we remarked above, the agent's problem can be analyzed without considering the distribution of wealth for the whole economy. Note, however, that this "representative agent" arises from the linearity of project returns, and not from any optimality properties of equilibrium. A further departure

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[17] Actually, wealth after consumption, ω-c. The point is to avoid the issue of whether consumption could be a signal of (unobservable) wealth.

from the usual macroeconomic framework is the presence of nonconvexities, not only from the discrete effort choice, but also from the incentive compatibility constraint.

We approach the study of the agent's behavior by breaking the optimization problem (4.1) into two parts corresponding to taking and forsaking the project. Each case is treated as a separate problem, and the optima of the subproblems are compared to yield the global optimum. The following series of propositions summarizes this analysis. Most of the proofs are provided in Appendix A.

Proposition 5.1 Whatever the values of ω and c, the agent chooses β so that IC′ holds with equality, if it holds at all.

This result is easily understood: increasing β acts as a mean-preserving spread on the distribution of bequest size; a risk-averse agent should like to have β as small as possible. On the other hand, if the agent wishes to sell shares in its firm, it must bear enough risk to induce it to work. Consequently, an agent will choose β so that it is just indifferent between shirking and working. Note that the left-hand side of (IC′) is less than the right-hand side for $\beta = 0$ (the integrand and the right-hand side are equal in this case), so that the constraint could only be satisfied for positive β. We will refer to the binding form of the incentive compatibility constraint as (IC).

Proposition 5.2 β is unique for each value of ω-c for which it is defined.

One can therefore consider β as a (differentiable) function of "saving," $s \equiv \omega - c$, (Proposition 5.4 below will guarantee that β exists as long as the agent desires to undertake the project) and write $\beta = \beta(s)$. The bequest functions B_r and B_s can be treated similarly.

If the project is chosen, (IC) may be substituted into the agent's problem and we may be rewrite it as

$$\max_{s} \; u(\omega-s) + v(B_0(s)) \tag{5.1}$$
$$\text{s.t.} \quad B_0(s) = s\hat{r} + I(\bar{r}-\hat{r}) + I(r_0-\bar{r})\beta(s),$$
$$\omega \geq s$$
$$s \geq 0.$$

This procedure is analogous to substituting the budget constraint into the utility function in a standard consumer choice problem. However, unlike that problem, (5.3) does not necessarily remain concave — at least we have been unable to show that it does in the general case.[18] This (possible) lack of concavity does not create particularly severe problems for studying the agent's behavior, but does require us to take a roundabout approach in the proof of convergence of the stochastic process that it generates.

The first-order necessary condition, which as usual will be an important source of information concerning the agent's behavior, is

$$u'(\omega-s) = v'(B_0)[\hat{r} + I(r_0-\bar{r})\beta'(s)] + \lambda, \tag{5.2}$$

where λ is the Lagrange multiplier associated with the constraint $s \geq 0$: $\lambda = 0$ if $s > 0$. (The constraint $\omega \geq s$ never binds, since $u'(\cdot)$ becomes arbitrarily large as its argument approaches zero). Notice we have differentiated β, as the implicit function theorem permits us to do.[19] Explicit calculation which exploits the concavity of $v(\cdot)$, as in the proof of Proposition 5.2, shows that

(1) $\beta'(s) > 0$; and

(2) the term in brackets is positive and less than $\hat{r}$.

Therefore interior solutions are well-defined. Note that $B_0(s)$ is increasing since the term in brackets is just $B_0'(s)$. We shall denote the solution to (5.1) by s^*, often treating it as a function of ω and writing $s^*(\omega)$.

Figure 1 illustrates the solution to the incentive compatibility constraint for $\beta(s)$. Notice that as s increases, the maximum and minimum realizations of the bequest must be spread apart (β must increase) so that the vertical distance between $\int v(B_r(s))dF(r)$ and $v(B_0(s))$ remains constant. This result depends only on the concavity of $v(\cdot)$ and not on any third or

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[18] Concavity *is* preserved if, for instance, the distribution of project returns is supported on the two-point set $\{r_0, r_1\}$ and $v(\cdot)$ belongs to the constant-relative-risk-aversion family of utility functions.

[19] The partial derivative of the (IC) equation with respect to β can be shown to be positive for all β and s.

higher-order derivative condition.

When the agent follows the safe strategy, it solves:

$$\max_{s} \; u(\omega-s) + v(s\hat{r}) \tag{5.3}$$
$$\text{s.t. } \omega \geq s$$
$$s \geq 0.$$

This problem is perfectly standard. Under our assumptions, the solution, denoted s^{**} (with the corresponding function $s^{**}(\omega)$), is determined uniquely by the first-order condition

$$u'(\omega-s^{**}) = v'(s^{**}\hat{r})\hat{r} \tag{5.4}$$

and is always interior.

As an agent becomes wealthier, it will need to bear more risk (β must be larger) in order to convince others that it will conscientiously attend to its project. This result is essentially provided by

<u>Proposition</u> <u>5.3</u> (a) $s^{*}(\omega)$ is nondecreasing; (b) $s^{**}(\omega)$ is increasing.

An immediate corollary of part (a) is that β is nondecreasing in ω, since $\beta(s)$ is increasing.

<u>Proposition</u> <u>5.4</u> If the agent finds it optimal to choose the project, then $\beta \leq \frac{\bar{r}-\hat{r}}{\bar{r}-r_0}$ (equivalently, $I(\bar{r}-\hat{r})+I\beta(r_0-\bar{r}) \geq 0$).

The proof depends on the observation that choosing the project implies

$$u(c^{*}) + v(B_0^{*}) \geq u(c^{**}) + v(s^{**}\hat{r}) \geq u(c^{*}) + v(s^{*}\hat{r}), \tag{5.5}$$

where the first inequality relates the maximized value of utility under the two plans, and the second follows from the fact that c^{**} maximizes $u(c)+v(s\hat{r})$. Then $B_0^{*} \geq s^{*}\hat{r}$, yielding the result.

In effect, this proposition states that the agent undertakes the project only if the profit will surely be nonnegative. There are two important consequences. First, an agent's child will have no problem paying back any loans that may have been taken, since *after* repayment the bequest is still nonnegative, even in the worst state. Second, since the upper bound on optimal β is less than unity, the agent will always be able to satisfy (IC) when it wishes to take the project. In other words, no agent will be

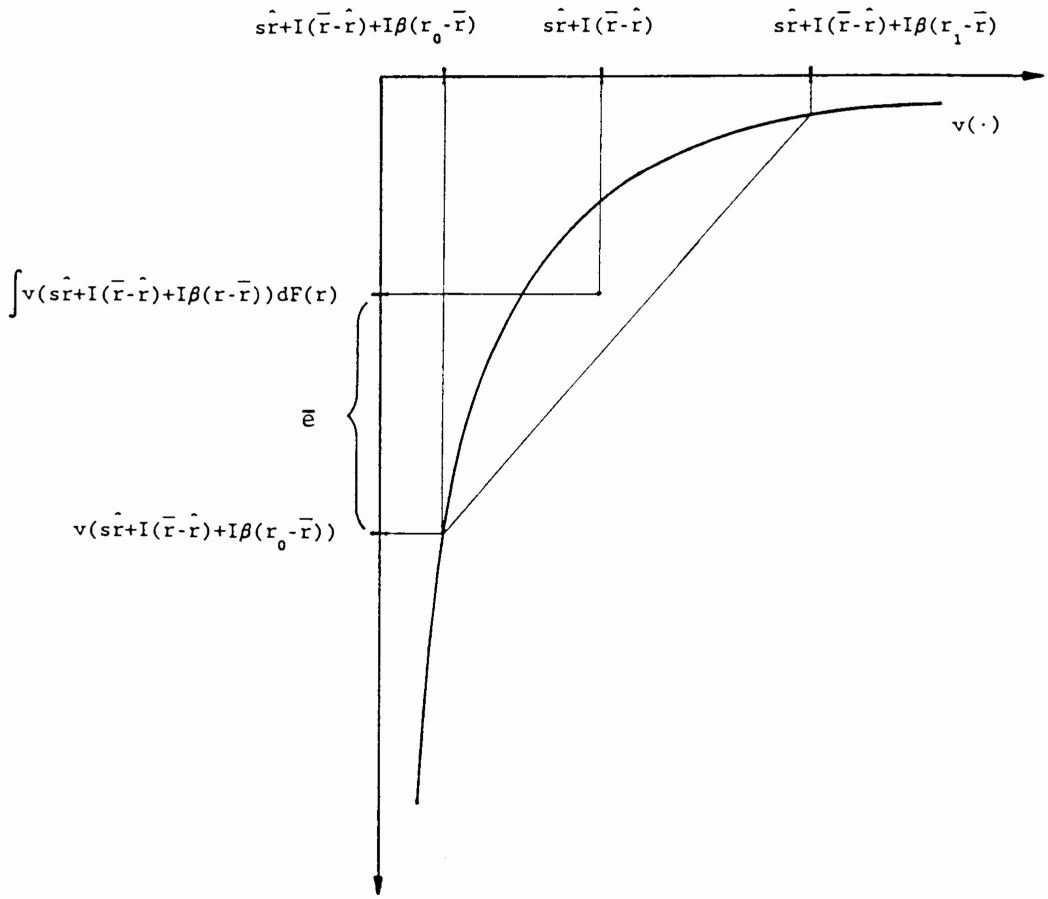

Figure 1. Solution of the incentive compatibility constraint for $\beta(s)$.

prevented from undertaking a project because it is unable to satisfy incentive compatibility; rather, it will simply choose not to take the project because the disutility of effort is not sufficiently compensated by the expected utility of the (possibly) increased size of the bequest.

Assumption (2.1) makes the plausible claim that an agent values consumption (uniformly) more highly on the margin than it does bequests. Along with the assumption that $\hat{r}$ is not too large as reflected in (2.2), it gives us

<u>Proposition</u> <u>5.5</u> For all positive wealth, $s^{**}(\omega)\hat{r} < \omega$. Whenever the project is chosen, $s^{*}(\omega)\hat{r} < \omega$.

The main consequence of this result is that wealth cannot expand indefinitely. In particular, it implies that a lineage of rentiers — those who do not take projects — will find its wealth declining over time.

We turn now to an analysis of the agent's behavior at low wealth levels. Since production loans are available, even a very poor agent can set up a project and need not supply any start-up capital of its own. By undertaking a project, an agent can have a strictly positive bequest, even as it consumes all of its initial wealth. By investing in the safe asset alone, the agent is only able to consume a fraction of its wealth and leave a bequest which is also smaller than its wealth. We should expect that agents who are sufficiently poor will choose the first option, consuming at the corner solution $c = \omega$. The next three propositions verify this intuition.

First define $\hat{\beta}$ to be $\beta(0)$, the equity share retained by an agent which saves zero, that is

$$\int v(I(\bar{r}-\hat{r})+I\hat{\beta}(r-\bar{r}))dF(r) - v(I(\bar{r}-\hat{r})+I\hat{\beta}(r_0-\bar{r})) = \bar{e}. \qquad (5.6)$$

Notice that the unboundedness of the utility function $v(\cdot)$ guarantees that $\hat{\beta}$ is well-defined: as β is increased from zero, the difference of the two terms on the left-hand side of (5.6) increases from zero and becomes arbitrarily large. If utility is bounded below, solutions to (IC) may not exist, particularly at low wealth levels; the result may be a "poverty trap" if agents find undiversifiable projects to be too risky. A full treatment of this case[20] will have to await further research.

Treating $B_r(\cdot)$ as a function of s as we did above, we denote $B_r(0)$ by the constant $\hat{B}_r$. In particular, we have $\hat{B}_0 = I(\bar{r}-\hat{r}) + I\hat{\beta}(r_0-\bar{r}) > 0$.

<u>Proposition</u> <u>5.6</u> At initial wealth $\hat{B}_0$ the agent strictly prefers the risky strategy to the safe strategy and sets $s^* = 0$ there.

From these facts follows

<u>Proposition</u> <u>5.7</u> There exists $\hat{\omega} \geq \hat{B}_0$ such that $c^* = \omega$ ($s^* = 0$) for $\omega < \hat{\omega}$ and $c^* < \omega$ ($s^* > 0$) for $\omega > \hat{\omega}$.

Agents whose wealth lies in $(0,\hat{\omega})$ and who are taking the project are necessarily borrowing to finance it and are passing only their projects (no safe asset earnings) to their children.

Perhaps the most important consequence of Proposition 5.7 is that the bequest passed on in the worst state (i.e. $B_0(\cdot)$ considered as a function of ω) is constant on the interval $(0,\hat{\omega})$ and in fact provides a positive lower bound on the level of wealth that can be sustained in the long run. The existence of this "safety net" is an immediate implication of

<u>Proposition</u> <u>5.8</u> There exists $\underline{\omega} \in (0,\hat{\omega})$ below which the bequest is always larger than initial wealth.

This $\underline{\omega}$ is of course just $\hat{B}_0$. The agent chooses the project not only at this level of wealth but also at all levels below it; the proposition follows from the fact that the function $B_0(\cdot)$ (which is equal to $\hat{B}_0$ on $(0,\hat{\omega})$) determines the lowest possible realization of the bequest given that the project is taken.

Thus, because of the ability to borrow to finance the project and the unboundedness of the utility function, agents who are sufficiently poor have both the wherewithal and the desire to insure that their children will be

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[20] As well as other limitations on downside risk, such as bankruptcy constraints.

better off than they are. Moreover, $\underline{\omega}$ provides an effective lower bound on wealth, since an agent with wealth less than $\underline{\omega}$ will give its children at least $\underline{\omega}$, and a lineage with wealth greater than $\underline{\omega}$ will never have its wealth fall below that level.

Having established that very poor agents will undertake the project, consuming all of their inherited wealth, we now consider the behavior of wealthier agents. The main result is the existence of a wealth level which separates the entrepreneurs from the rentiers.

Proposition 5.9 There exists a unique $\bar{\omega} > \underline{\omega}$ such that the agent takes the project if $\omega < \bar{\omega}$ and does not if $\omega > \bar{\omega}$.

We have already indicated that at low wealth levels, agents choose to take the project; at high levels, however, the utility gain resulting from the increased bequest associated with the project is small compared to the disutility incurred, and agents invest all of their savings at the safe return. Somewhere in between is a "switch point" $\bar{\omega}$, which we show to be unique.

What emerges is the possibility of the existence of two classes of agents: those with $\omega > \bar{\omega}$ are rentiers, bearing no risk, and earning their entire bequests from the safe asset or equity holdings. Everyone else is an entrepreneur, earning at least part of its bequest from the risky project. What we have *not* shown is that there will ever actually be any agent with wealth exceeding $\bar{\omega}$, or, more properly, that such wealth levels will persist; this issue will be discussed below.

One implication of the foregoing concerns the distribution of risk-bearing. Clearly, the wealthier an (entrepreneurial) agent, the greater (more properly, the no lesser) the absolute risk it bears. Relative to initial wealth ω, however, risk need not be increasing, and is likely decreasing. Indeed, since agents with wealth in a neighborhood of $\underline{\omega}$ are all bearing the same absolute risk $I\hat{\beta}$, relative risk must be declining in this neighborhood. Thus, our model suggests that there is a positive minimum level of absolute risk and consequent large relative risk for agents with low wealth. At the other extreme, of course, the very wealthy rentiers bear no risk at all.

6. Dynamics of Lineage Wealth

In this section we begin our analysis of the evolution of a lineage's wealth. Broadly speaking, there are two cases of interest. In the first, ω_t exceeds $\bar{\omega}$ infinitely often; in the second, $\omega_t > \bar{\omega}$ for only a finite number of values of t. In the language of the theory of Markov processes, we distinguish between the case in which a subset of rentier levels of income $\{\omega: \omega > \bar{\omega}\}$ is recurrent and that in which all such subsets are transient. When time averages of the limiting distribution for one lineage are reinterpreted as population averages under ergodicity, this classification corresponds to two rather different pictures of the economy. In the first, there are two classes of economic agents, the entrepreneurs, who bear risk and expend effort; and the rentiers, who do neither. In the second, everyone belongs to the broad middle class of entrepreneurial risk-takers.

The lineage's wealth follows the Markov process

$$\omega_{t+1} = \begin{cases} B_{\tilde{r}}(\omega_t), & \omega_t \leq \bar{\omega} \\ B_s(\omega_t), & \omega_t > \bar{\omega} \end{cases}$$

To define the probability distribution of ω_{t+1} given $\omega_t \leq \bar{\omega}$, it is enough to specify it on the intervals; if J is an interval with endpoints $\omega_1 < \omega_2$, and $\psi(\cdot)$ is the map sending wealth levels into wealth level distributions, then

$$\psi(\omega)[J] \equiv P(\omega_{t+1} \in J \mid \omega_t = \omega)$$

$$= F\left(\frac{\omega_2 - s^*(\omega)\hat{r} - I(\bar{r}-\hat{r})}{I\beta(s^*(\omega))} + \bar{r} \right) - F\left(\frac{\omega_1 - s^*(\omega)\hat{r} - I(\bar{r}-\hat{r})}{I\beta(s^*(\omega))} + \bar{r} \right),$$

since $\omega_{t+1} < \omega_i$ if and only if the realization r_t is less than the argument of $F(\cdot)$. Because the support of F is $[r_0, r_1]$, the support of $\psi(\omega)[\cdot]$ is $[B_0(\omega), B_1(\omega)]$ (note that $B_0(\omega) < B_r(\omega) < B_1(\omega)$ for $r \in (r_0, r_1)$). Of course, if $\omega_t > \bar{\omega}$ then $\psi(\omega)[\cdot]$ is just defined to be the unit mass at $B_s(\omega)$.

Much information concerning the long-run behavior of $\{\omega_t\}$ can be gleaned from the "stochastic policy correspondence," a generalization of the policy function of dynamic programming. This correspondence, denoted $\phi(\cdot)$, simply maps ω_t into the set of possible realizations of ω_{t+1}, that is, into the support of $\psi(\omega)[\cdot]$. Specifically,

$$\phi(\omega) = \begin{cases} [B_0(\omega), B_1(\omega)], & \omega \leq \bar{\omega} \\ B_s(\omega), & \omega > \bar{\omega} \end{cases}$$

By studying the graph of ϕ, it is easy to determine the invariant sets of the

Markov process. The shape of this graph clearly is determined by those of the functions B_r (for which it is enough to examine B_0 and B_1) and B_s. The risky bequests B_r share the common property given in

<u>Proposition</u> <u>6.1</u> For all r, $B_r(\cdot)$ is nondecreasing, and increasing for $\omega > \hat{\omega}$.

Since $s^* = 0$ for $\omega < \hat{\omega}$, the B_r are constant ($= \hat{B}_r$) there. Notice that due to the possible nonconcavity of (5.1), $B_r(\omega)$ may have discontinuities (in fact, at each such discontinuity, $s^*(\cdot)$ is multivalued). It is also clear that for $r' > r$, $B_{r'}(\omega) > B_r(\omega)$ (their difference is $I\beta(r'-r) > 0$).

We observed earlier that $B_0(\cdot)$ has a fixed point, namely $\underline{\omega} = \hat{B}_0$. But in fact a stronger result obtains, namely

<u>Proposition</u> <u>6.2</u> $\underline{\omega}$ is the unique fixed point of $B_0(\omega)$.

The proof depends on showing that $B_0(\omega) < \omega$ for $\omega > \underline{\omega}$. As Proposition (5.8) showed, wealth cannot remain below $\underline{\omega}$ under the risky strategy. The result here helps to imply that arbitrarily small neighborhoods of the form $(\underline{\omega},\omega)$ are visited infinitely often.

The next proposition implies that the project is chosen on a nondegenerate interval and therefore that the dynamics of lineage wealth are governed at least in part by the stochastic transitions $B_r(\cdot)$, even in the long run.

<u>Proposition</u> <u>6.3</u> The switch point exceeds $B_0(\cdot)$'s fixed point: $\bar{\omega} > \underline{\omega}$.

The result follows from the continuity of the value function and the fact that the risky strategy is strictly preferred to the safe strategy at $\underline{\omega}$.

We have stated that $\underline{\omega}$ provides a lower bound on wealth in the long run. This depends on more than just Proposition (5.8), however. In principle, some point below $\underline{\omega}$ might be reached from above. This cannot happen if the project is taken, since Proposition (6.1) guarantees that even the worst realization of the bequest, $B_0(\omega)$ is at least $\underline{\omega}$. The other possibility, that points in $(0,\underline{\omega})$ are reached when the safe strategy is chosen, is precluded by

Proposition 6.4 At the switch point, we have $s^{**}(\bar{\omega})\hat{r} \geq B_0(\bar{\omega})$.

Thus $\underline{\omega}$ is indeed a lower bound on long run wealth, and $(0,\underline{\omega})$ is a transient set. It is in this sense that our model does not exhibit a poverty trap. If ever an agent has wealth below $\underline{\omega}$, its offspring is guaranteed to have at least $\underline{\omega}$, and none of its descendants will ever again have less than $\underline{\omega}$.

Note that the availability of production loans is crucial to this result. If loans were not available at all, then agents with sufficiently little wealth would be unable to undertake a project; with the safe asset as the only available alternative, the lineage's wealth converges to zero. Thus, lack of availability of production loans is one way to generate a poverty trap.[21]

What about the high end? Are sufficiently large wealth levels also transient? Indeed, it is easy to see that this is the case:

Proposition 6.5 There exists $\bar{\bar{\omega}} > \underline{\omega}$ such that if $\omega_t \leq \bar{\bar{\omega}}$ then $\omega_{t+1} \leq \bar{\bar{\omega}}$ and if $\omega_t > \bar{\bar{\omega}}$ then with probability 1, $\omega_{t+n} \leq \bar{\bar{\omega}}$ for all n sufficiently large.

In other words, there is also an (almost sure) upper bound on long run wealth.

We can now clarify what distinguishes the two cases mentioned above concerning the persistence of rentier wealth levels. Let ω^* denote the least of the fixed points of $B_1(\cdot)$, assuming it exists. Set $\bar{\bar{\omega}}$ equal to $B_1(\bar{\omega})$ if $\bar{\omega}$ is less than ω^* (if ω^* does not exist then $\bar{\bar{\omega}}$ equals $B_1(\bar{\omega})$ also); the rentier levels $(\bar{\omega},\bar{\bar{\omega}}]$ then recur infinitely often. If instead $\bar{\omega}$ exceeds ω^*, then $\bar{\bar{\omega}}$ equals ω^* and all rentier wealth levels are transient. Figure 2 illustrates

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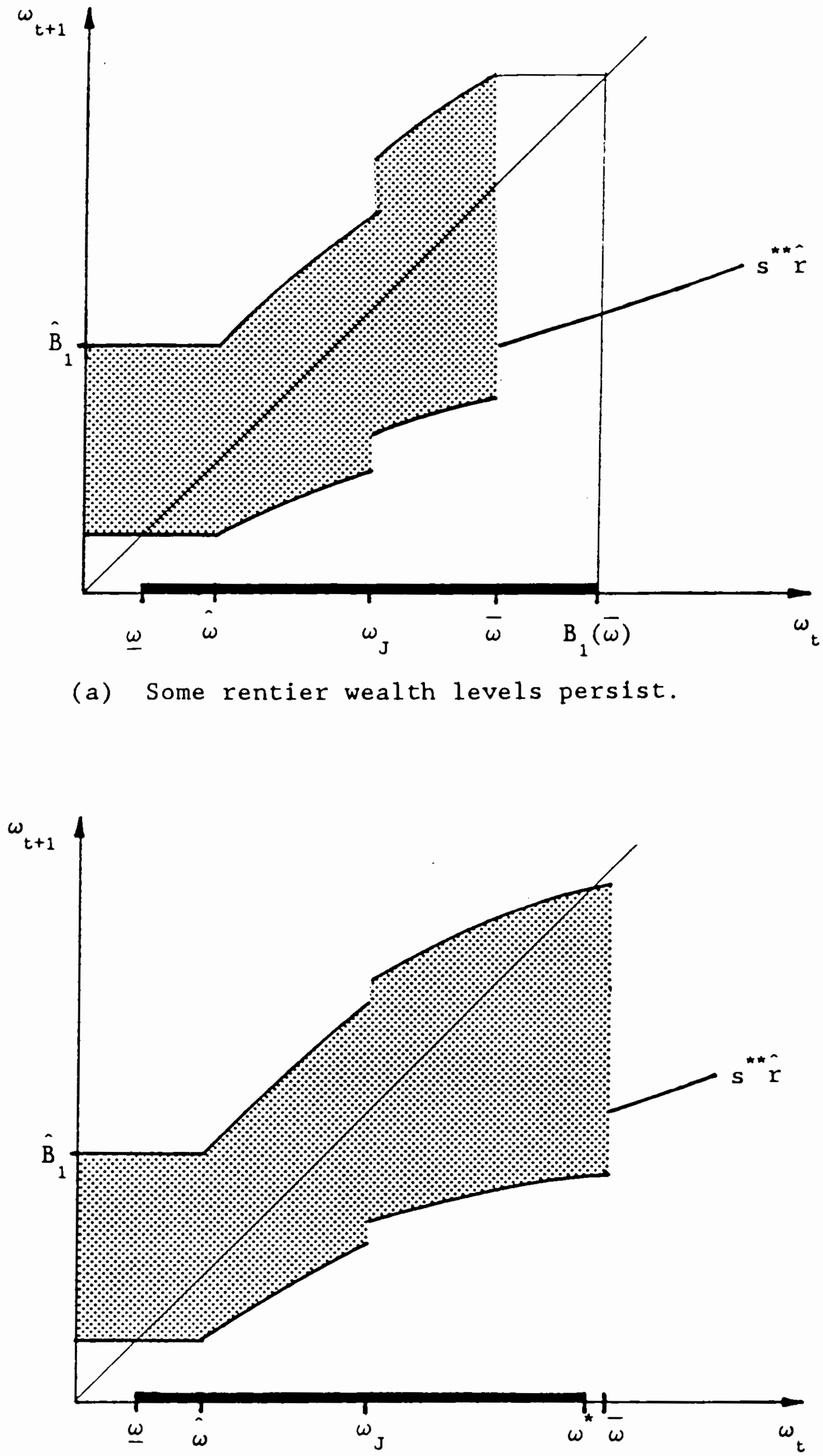
[21] Without production loans, agents might finance their projects with equity and savings alone; this supposition would introduce the additional constraint that $s + I(1-\beta)\bar{r}/r \geq I$, which effectively places an upper bound on β. It is not obvious that both this constraint and (IC) can be simultaneously satisfied, nor even that failure to satisfy both only occurs at low wealth levels. But, supposing that a certain minimum wealth is necessary to finance without loans, and if this level is above B_0, then since with positive probability every lineage will eventually find itself too poor to finance a project, the absence of production loans would lead to complete impoverishment of the entire population!



(a) Some rentier wealth levels persist.

(b) All rentier wealth levels transient.

Figure 2. The stochastic policy correspondence ϕ. Shading indicates possible realizations of $B_{\tilde{r}}(\cdot)$. Heavy line is the support of the limiting distribution.

the graph of ϕ for typical representatives of each case.[22]

The specific shape of ϕ is determined in part by the order of the points $\hat{\omega}$, $\hat{B}_1$, $\bar{\omega}$ and ω^*. However, nothing we have said so far rules out any particular order of these values. For instance, it is perfectly possible that $\hat{B}_1 < \hat{\omega}$, in which case $\bar{\bar{\omega}} = \omega^* = \hat{B}_1$; rentier wealth levels will persist if $\underline{\omega} < \hat{B}_1$, but not otherwise. This case may be interpreted as corresponding to a poor economy in which every project-taker consumes all its inherited wealth; we shall have more to say about this case below. We do not believe that there are weak conditions that will rule out one or another of these cases, although some indication is provided by the comparative statics presented in Section 8. Our preliminary simulation results do suggest that $\hat{\omega} < \hat{B}_1$ is more likely, while it is relatively easy to obtain both the cases $\bar{\bar{\omega}} = \omega^*$ and $\bar{\bar{\omega}} = B_1(\bar{\omega})$ by varying the model's parameters.

Despite our ignorance about the precise shape of the policy correspondence, we have considerable information concerning the long run behavior about the sequence of lineage wealth levels $\{\omega_t\}$. Anything outside of the interval $[\underline{\omega},\bar{\bar{\omega}}]$ is transient. Moreover, this interval is nondegenerate: the minimum value of $\bar{\bar{\omega}}$ is $\hat{B}_1$, which is greater than $\underline{\omega} = \hat{B}_0$. And, it is easy to see, any wealth level in between can be reached from any other. Finally, once in the interval, ω_t remains there for all time. We take up the issue of the existence, uniqueness and convergence to an ergodic distribution in the next section.

7. Toward a Theory of Distribution: Ergodicity

The above discussion suggests that the right place to look for an ergodic distribution of the Markov process governing lineage wealth is on the interval $[\underline{\omega},\bar{\bar{\omega}}]$ (hereinafter denoted Ω): any probability mass distributed outside of Ω will be carried inside with probability one. Ergodicity has two implications.

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[22] We have drawn the correspondence with $B_1(\cdot)$ shown to have single fixed point, although it appears that it may have many. $B_0(\cdot)$ does have a unique fixed point, however, and as a consequence the support of the limiting distribution is correctly described here; in particular, the support is always connected. In Appendix B, where we relax the assumption that there are no consumption loans, the consequences of nonuniqueness (for both $B_0(\cdot)$ and $B_1(\cdot)$) are examined further.

First, over time a lineage will experience all wealth levels[23] in the interval: the descendants of the rich will eventually be poor, and those of the poor will be rich. This tendency for a lineage's wealth to travel all over the interval captures the notion of individual (intergenerational) mobility usually ascribed to market societies (see for instance Becker and Tomes [1979]). Second, since all agents in the economy follow the same process independently of each other, we can reinterpret the long-run time distribution for a single lineage as the population distribution of wealth at a single moment of time.[24]

In this section we show that irrespective of the initial distribution, the process governing lineage wealth converges to a unique ergodic distribution. This result can be reinterpreted in terms of the distribution of wealth for the economy: regardless of how wealth is distributed, over time it will revert to that of the ergodic distribution for our process. This is the right sort of result to look for, rather than the weaker one (usually known as the "ergodic theorem") that time averages converge, because of our interpretation of the ergodic distribution of lineage wealth as identical to the population distribution of wealth: if, for instance, the process were cyclic of period two, time averages would still converge, but the wealth distribution in the economy would be observed to flip back and forth from one generation to the next![25]

We are forced to take a somewhat roundabout approach because most of the available theorems regarding convergence of Markov processes cannot be directly applied to our model. These theorems tend to depend on easily-verified conditions such as continuity or monotonicity, neither of

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[23]By which we mean, of course, that all nonnegligible sets are visited with probability one by the lineage's trajectory.

[24]The large number and independence of agents is also crucial to this reinterpretation.

[25]Moreover, the observed distributions would depend in an essential way on the initial distributions. The weaker result is acceptable if it is assumed that observations occur infrequently relative to the period length, as, for instance, in statistical mechanics or, closer to home, in finance.

which is satisfied here.[26] Continuity fails because there are jumps in the stochastic policy function: the map ψ defined in the previous section is discontinuous, since the corresponding supports change suddenly with small changes in ω (look again at Figure 2(a) and consider a neighborhood of ω_J or $\bar{\omega}$). Monotonicity (that is, the property that $\omega' > \omega$ implies that $\psi(\omega')$ stochastically dominates $\psi(\omega)$) fails whenever the switch point $\bar{\omega}$ is contained in $[\underline{\omega},\bar{\bar{\omega}}]$.[27]

Because of these difficulties, we prove convergence by applying a theorem of Doob (Breiman, 1968, Theorem 7.18):

Theorem Let η be an invariant distribution on a state space Ξ for a Markov process $\pi(\cdot)$. Suppose

(1) Ξ is indecomposable under π^t, t = 1, 2, 3, ...

(2) For all $\xi \in \Xi$, $\pi(\xi) \ll \eta$.

Then for all $\xi \in \Xi$ and Borel sets $A \subset \Xi$, $\lim_{t\to\infty} \pi^t(\xi)[A] = \eta(A)$.

In order to apply this theorem, we establish the existence of an invariant measure on the state space Ω for the process ψ.[28] Next, we show that

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[26] Futia (1982) provides a survey of limit theorems for continuous (there called "stable") Markov processes. Hopenhayn and Prescott (1987) concern themselves with monotone processes.

[27] Observe that for $\bar{\bar{\omega}} = \omega^* < \bar{\omega}$, monotonicity does obtain (this follows from Proposition (6.1)); since their "monotone mixing condition" is also satisfied, Hopenhayn and Prescott's (1987) Theorem 2 may be applied to our model to conclude that there exists a globally stable distribution.

[28] A measure μ on Ω is invariant for the process $\psi(\cdot)$ if $\mu(A) = \int_\Omega \psi(\omega)[A]\mu(d\omega)$ for all measurable $A \subset \Omega$. A measurable set $B \subset \Omega$ is μ-invariant if for μ-almost every $\omega \in B$, $\psi(\omega)[B] = 1$. An invariant measure μ^* on Ω is ergodic if for any invariant set B, we have $\mu^*(B) = 0$ or $\mu^*(B) = 1$.
Roughly speaking, an invariant measure is left unchanged by the process and an invariant set is one which is almost surely mapped into itself. A measure is ergodic if nonnegligible sets (other than almost the whole space) are sent at least partly outside themselves: the process "mixes up" the space.
If the invariant measures form a convex set, then the ergodic measures are the extreme points of this set. If the invariant measure is unique, therefore (as it must be under global stability), then it is ergodic.

Ω is indecomposable under ψ and all of its iterates.[29] Finally, by showing that for T large enough, $\psi^T(\omega)[\cdot]$ is absolutely continuous with respect to the invariant measure for each ω in Ω, the Doob theorem enables us to conclude that the iterates $\psi^{kT}(\omega)$, k = 1, 2, ..., and therefore $\psi^t(\omega)$, t = 1, 2, ..., converge to the (unique and therefore ergodic) measure.

To establish existence, we circumvent the continuity problem by convexifying the process. Note that at any jump point ω' of the correspondence ϕ, the agent is indifferent between two (possibly more) choices (this is true whether ω' occurs because of nonconcavity of (5.1) or because ω' is the switch point $\bar{\omega}$). This observation permits us to make ψ into a convex-valued correspondence Ψ by setting $\Psi(\omega')$ equal to all convex combinations of the distributions in $\psi(\omega')$. Since the agent is indifferent between the extreme points of $\Psi(\omega')$, it is equally happy with any distribution in $\Psi(\omega')$. Note that $\Psi(\cdot)$ so constructed has closed graph.

Now observe that any point in Ω is sent into a distribution supported on (a subset of) Ω; that is, Ω is "self-justified" in the sense of Duffie et al. (1988). Applying their Theorem 1 we obtain

<u>Proposition 7.1</u> An invariant measure μ with support in Ω exists for the convexified process Ψ.

Their Corollary 1 actually gives us existence of an ergodic measure for Ψ, but this does not help us, since we will get it automatically from Doob's theorem.

A potential problem with applying Duffie et al.'s theorem is that the selection from Ψ corresponding to μ need not correspond to the action the agent chooses at a discontinuity. The usual trick of assuming the existence of some mechanism (e.g. an auctioneer) to insure that, for the sake of equilibrium, agents choose the "right" mixture of actions among which they are indifferent, is completely unconvincing in the present context. Fortunately, we have the following fact:

[29] Define the t^{th} iterate of ψ in the usual way, viz.,

$$\psi^t(\omega)[A] = \int_\Omega \psi(\omega')[A]\psi^{t-1}(\omega)[d\omega'].$$

Ω is indecomposable under ψ^t if there do not exist nonempty measurable sets A and $A^c \equiv \Omega \backslash A$ such that $\psi^t(\omega)[A] = 1$ for all $\omega \in A$ and $\psi^t(\omega)[A^c] = 1$ for all $\omega \in A^c$.

Lemma 7.1 If μ is invariant for the process $\psi(\cdot)$, then it is invariant for the process $\psi'(\cdot)$, where $\psi'(\omega) = \psi(\omega)$ for μ-almost every ω.

The reason is simple. Denote by $\psi(\omega)[A]$ the measure of the Borel set A according to $\psi(\omega)$; fixing A, this is a measurable function of ω. Then invariance of μ means that for all $A \subset \Omega$, $\mu(A) = \int_\Omega \psi(\omega)[A]\mu(d\omega)$. But this expression is unaffected by the value of $\psi(\cdot)[A]$ on a set of measure zero.

In Appendix B we prove:

Proposition 7.2 An invariant measure of Ψ is atomless.

Consequently, the set of discontinuity points has measure zero (since it is finite), and the choice of action by the agent on that set is irrelevant: the same distribution is invariant for any selection from Ψ.[30]

One further property our invariant distribution is provided in

Proposition 7.3 Let λ denote Lebesgue measure on Ω. If μ is an invariant measure on Ω under ψ, then $\lambda \ll \mu$.

This result is of more than technical interest, since it says that the invariant measure has no "holes": every subset of Ω having positive Lebesgue measure (in particular, every subinterval) is reached with positive probability, so each such set of wealth levels will be occupied by a positive fraction of the population. This seems a desirable property for any continuous approximation to an empirical income distribution.

To sum up, we have (at least one) invariant measure μ on $[\underline{\omega},\bar{\bar{\omega}}]$ for the process governing lineage wealth. It assigns positive probability to every (Lebesgue) nonnegligible subset of Ω. If μ happens to be the initial distribution of wealth in the economy, it will continue to represent the distribution of wealth for all time.

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[30] Note that it does not even matter that the agent is supposed to be indifferent among the mixtures at jump points; thus convexification can be viewed as a purely technical contrivance with no behavioral implications.

To establish our stronger result, namely that starting with any distribution ν, we should expect μ to be representative of the eventual distribution of wealth in the economy, we need to show that Ω is indecomposable under all iterates of ψ. The basic idea is the following. Notice first of all that if ω is a point with a stochastic transition, then all iterates of $\psi(\omega)$ are nicely behaved in the following sense:

<u>Proposition 7.4</u> Let $\underline{\omega} \le \omega < \bar{\omega}$. Define $J_t(\omega) = [B_0^t(\omega), B_1^t(\omega)] \cap \Omega$ and let λ_t be Lebesgue measure on $J_t(\omega)$. Then $\psi^t(\omega) \ll \gg \lambda_t$.

<u>Sketch of Proof</u> To see the first relation, note that $\psi(\cdot)$ inherits from the distribution of project returns $F(\cdot)$ the property that if a set A has Lebesgue measure zero, then $\psi(\omega)[A] = 0$. From the definition of an iterate,

$$\psi^2(\omega)[A] = \int_\Omega \psi(\omega')[A]\psi(\omega)[d\omega'] =$$
$$= \int_{[\underline{\omega},\bar{\omega}]} \psi(\omega')[A]\psi(\omega)[d\omega'] + \int_{[\underline{\omega},\bar{\bar{\omega}}]} \psi(\omega')[A]\psi(\omega)[d\omega'].$$

The first integral on the second line vanishes because the integrand is zero, and the second does likewise because the probability under $\psi(\omega)$ of being in $B_s^{-1}(A)$ is zero ($B_s(\cdot)$ is strictly increasing). Now proceed by induction.

The second relation follows from the fact (see Figure 3)[31] that $J_t(\omega)$ is the support of $\psi^t(\omega)$. $\square$

Now suppose that Ω could be partitioned into nonempty sets A and A^c with $\psi^t(\omega)[A] = 1$ for all $\omega \in A$ and $\psi^t(\omega)[A^c] = 1$ for all $\omega \in A^c$. In Appendix B we show (Lemma 7.5) that A must contain a point $\omega_0 < \bar{\omega}$. But Proposition (7.4) implies that λ-almost every point in $J_t(\omega_0)$ is also in A. Therefore we can choose from $A \cap J_t(\omega_0)$ a point ω_1^0 arbitrarily close to $B_0^t(\omega_0)$ and a point ω_1^1 arbitrarily close to the lesser of $\bar{\omega}$ and $B_1^t(\omega_0)$. Define $J_t(\omega_1^0)$ and $J_t(\omega_1^1)$ analogously to $J_t(\omega_0)$; almost every ω in the union of these three intervals must be in A. Continue in this way, obtaining a sequence of intervals, almost every point in which is also in A. It should be easy to see from Figure 3

[31] Figure 3 actually represents the case in which there are consumption loans. This is the harder case, and our proof is valid for both.

that the union of these intervals, $\bigcup_{k}^{\infty} [J_t(\omega_k^0) \cup J_t(\omega_k^1)]$, is just Ω. Consequently almost every point of Ω is in A and $\lambda(A) = \lambda(\Omega)$. But exactly the same construction can be applied to A^C, so $\lambda(A^C) = \lambda(\Omega)$. This can only be true if $\lambda(\Omega) = 0$, a contradiction. We have therefore shown that $\psi(\cdot)$ satisfies condition (1) of Doob's theorem.

The second requirement of Doob's theorem is that every point in Ω be sent into a measure which is absolutely continuous with respect to the invariant measure μ. Now this clearly is not true for $\psi(\cdot)$, since any point in Ω greater than $\bar{\omega}$ is sent into a unit mass, while Proposition (7.2) tells us that μ is atomless. However, it is true that every such point is sent below $\bar{\omega}$ in a finite number of steps, and in fact this number is bounded by the number of periods it takes ω to fall below $\bar{\omega}$ under $B_s(\cdot)$. If we define

$$T = \min \{t \geq 1: B_s^{t-1}(\bar{\bar{\omega}}) \leq \bar{\omega}\},$$

then Proposition (7.4) tells us that for all $\omega \in \Omega$, $\psi^T(\omega) \ll \lambda$, since every iterate of points below $\bar{\omega}$ is absolutely continuous with respect to Lebesgue measure and all points above $\bar{\omega}$ pass below $\bar{\omega}$ at least once under ψ^T. Now use Proposition (7.3) to conclude that $\psi^T(\omega) \ll \mu$.

What we have shown is that while ψ need not satisfy the conditions of Doob's theorem, ψ^T does. Thus, the iterates of ψ^T converge to μ regardless of the initial distribution ν on Ω:

$$\lim_{k\to\infty} \int_{\Omega} \psi^{kT}(\omega)[A]\nu(d\omega) = \mu(A).$$

The final step in the proof is to recall the following

<u>Fact</u> If π^t, $t = 1, 2, \ldots$ is an iterated map and the subsequence π^{kT}, $k = 1, 2, \ldots, T \geq 1$, converges globally to a limit η, then π^t also converges globally to η.

To see this, note that for a given initial point ν, there is an integer k_0 such that $\pi^{kT}\nu$ is "close" to η for $k > k_0$; similarly, there are integers k_1, k_2, ..., k_{T-1} for the initial points $\pi\nu$, $\pi^2\nu$, ..., $\pi^{T-1}\nu$ so that $\pi^{kT}\pi^m\nu$ is close to η for $k > k_m$. Letting $K = \max_m \{k_m\}_{m=0}^{T-1}$, any $t > KT$ can be written as $kT+m$, with $k \geq K$ and $0 \leq m < T$; then $\pi^t\nu = \pi^{kT}\pi^m\nu$ is close to η, as required.

Applying this argument to ψ, we have our desired result, namely

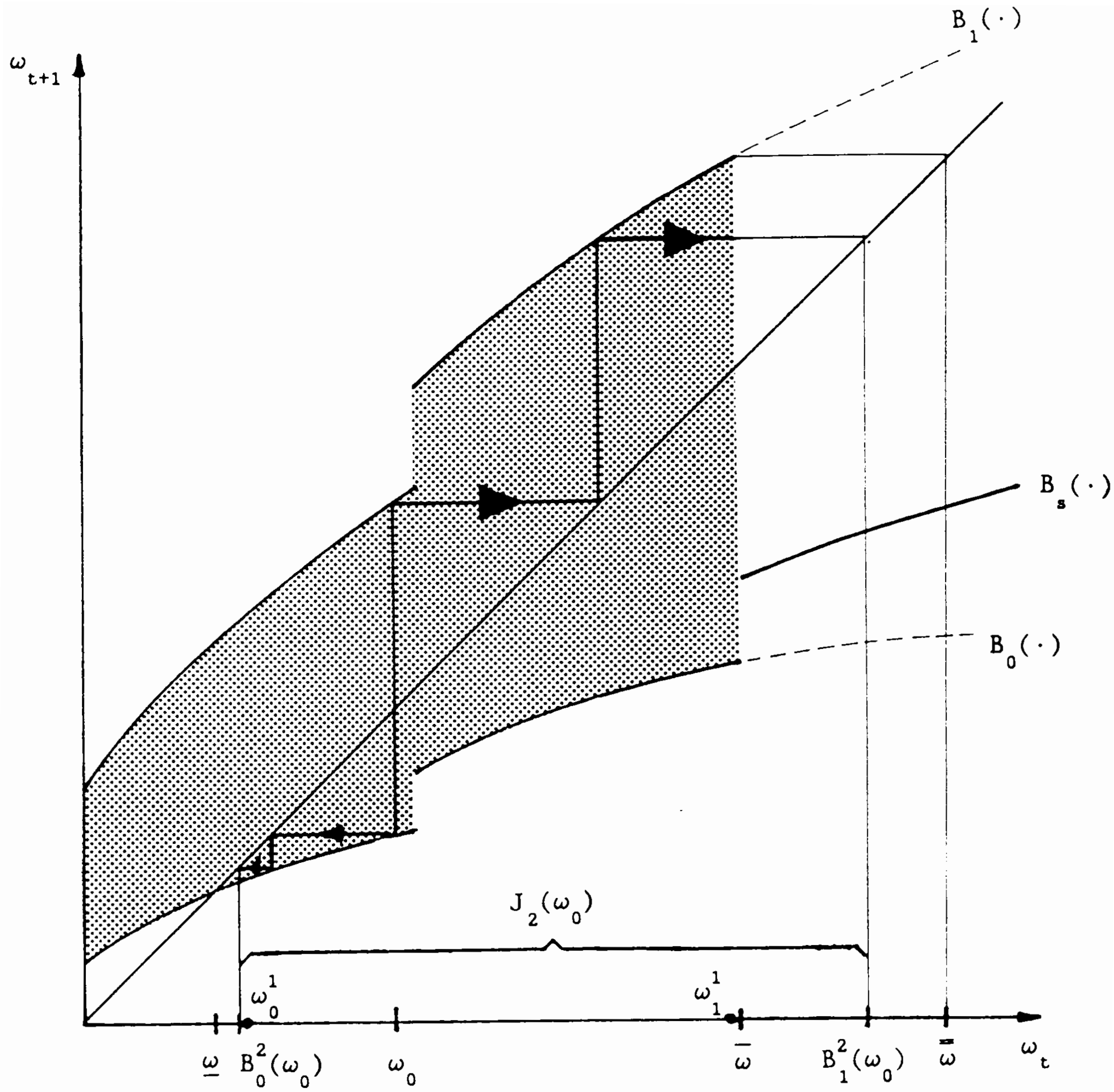

Figure 3. Construction of intervals $J_t(\omega_0)$ (t = 2 shown).

Proposition 7.6 For any distribution ν on Ω and $A \subset \Omega$,

$$\lim_{t\to\infty} \int_{\Omega} \psi^{t}(\omega)[A]\nu(d\omega) = \mu(A).$$

As pointed out in a footnote, the uniqueness implied by this global stability implies that μ is ergodic.

Recall that any wealth level outside of Ω is carried into it in finite time, so the proposition extends to any initial wealth distribution on $\mathbf{R}_{+}$. In particular, we obtain the nondegenerate distribution μ even if all lineages start out with the same wealth. One-time redistributions have only temporary effect. It is for good reason that the jubilee (Leviticus 25: 10-16) was to be repeated every fifty years.

8. Some Rudimentary Comparative Statics

In the previous sections we have established the broad characteristics of the wealth distribution generated by our model. In a sense, the predictions of our model are rather strong: given any initial distribution of wealth, in the long run only one particular distribution, having certain broad characteristics (support on a compact interval, no "spikes", bounded away from extreme poverty, lineage mobility throughout the interval), will obtain. The global stability of this distribution provides theoretical justification for comparative-static analysis: it insures that questions about how the distribution of wealth responds (after sufficient time) to changes in the parameters reflecting productivity, project size and the like, or to various government policies, are well-posed.

Unfortunately, any attempt to generate a closed form characterization of the stationary distribution appears to be quite hopeless, so we are restricted in what we can expect to do analytically. For instance, it is difficult to determine how standard measures of inequality are affected by changes in the parameters. A more complete analysis — one which takes full advantage of the ergodicity properties of the limiting distribution — will have to await future research.

It does turn out to be fairly easy to calculate changes in $\underline{\omega}$ (= $\hat{B}_0$), and as this is the lowest wealth level in the stationary distribution of ω, we can at least say something about the situation of the poorest people in the economy by looking at changes in $\underline{\omega}$.[32]

Looking at changes in the highest point in the stationary distribution is, however, much more complicated since, as we saw above, the equation that defines this point will be different depending on the order of the points $\hat{B}_1$, $\hat{\omega}$, $\bar{\omega}$ and ω^*. We are able to say something about the comparative statics of this point in some of these cases and in particular we will focus on how $\hat{B}_1$ and $\bar{\omega}$ change as we change the parameters of the problem. For these cases we can therefore determine changes in inequality as measured by the range, and although this statistic has well-known defects, it is at least indicative of the kinds of results that a fuller analysis might obtain.

We shall also be concerned with the institutional interpretations of our model as changes in parameters cause the economy to move from one of our cases to another. In particular, we seek conditions guaranteeing the emergence of a class of rentiers in the stationary distribution. This seems to be of considerable interest from the point of view of understanding the historical evolution of market economies.

Most of the proofs are routine, and we omit them. They can be found in the appendix of Banerjee and Newman (1989).

Changes in $\bar{e}$

An increase in $\bar{e}$ represents a reduction in labor productivity (or an increase in the disutility of labor). We expect that this will make it more difficult for incentive compatibility to hold, and in fact it is easy to see (look at Figure 1) that for a given level of s, an increase in $\bar{e}$ necessitates an increase in $\beta(s)$ if (IC) is to remain satisfied. This is true in particular if s = 0, from which we conclude that

$$d\underline{\omega}/d\bar{e} = I(r_0-\bar{r})d\hat{\beta}/d\bar{e} < 0 \quad \text{and} \quad d\hat{B}_1/d\bar{e} = I(r_1-\bar{r})d\hat{\beta}/d\bar{e} > 0.$$

As one would expect a decrease in productivity reduces the level of wealth of the poorest people in the economy. What it does to the richest people depends on which case we are in.

However, we do know that as $\bar{e}$ increases, $\bar{\omega}$ must decline, since it makes

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[32] If one takes seriously the Rawls criterion (in either its utility or commodity form) for assessing social welfare, then $\underline{\omega}$ provides all the information one needs.

taking the project strictly worse while keeping the returns from the alternative unchanged. In fact we can show that the value of $\bar{\omega}$ can be made arbitrarily close to 0 by making $\bar{e}$ large. To see this, note that for any fixed wealth level, say ω', which can be arbitrarily small, we know that

$$\int v(B_r(s^*(\omega'))dF(r) - v(B_0(s^*(\omega'))$$

is bounded, say by $M(\omega')$. Take $\bar{e} > M(\omega')$. Clearly there is no solution β less than 1 to (IC′) for this value of $\bar{e}$ and this value of ω'. Proposition (5.4) then allows us to conclude that the value of $\bar{\omega}$ corresponding to this value of $\bar{e}$ must be lower than ω'. Now recall that, as $\bar{e}$ increases, $\hat{B}_1$ increases, so that by choosing an $\bar{e}$ large enough we must be able to make $\hat{B}_1 > \bar{\omega}$. Thus we have

<u>Proposition</u> <u>8.1</u> Economies with low enough levels of labor productivity will always have rentiers.

Notice that since $\hat{B}_1$ is a lower bound on the value of $B_1(\cdot)$, the appearance of rentiers as $\bar{e}$ is increased will generally not be limited to cases in which $\hat{B}_1$ exceeds $\bar{\omega}$ (the simulation example in Appendix C is a case in point).

Consider now an economy in which $\hat{B}_1 < \hat{\omega}$; we may interpret this case as representing a "poor" economy since all project-takers consume all of their inherited wealth. The highest point in support of the stationary distribution will be $\hat{B}_1$, and as we have seen this will increase as we increase $\bar{e}$. We also established that $\underline{\omega}$ decreases as we increase $\bar{e}$ so that, as long as we are in this case, we find that

<u>Proposition</u> <u>8.2</u> In the poor economy, inequality (as measured by the range) declines when labor productivity is increased.

If we think of the economy undergoing exogenous productivity increases starting from a very low level of productivity one will therefore expect to see an initial reduction in the level of inequality. This suggests that there are plausible models in which the Kuznets hypothesis that development will initially cause an increase in inequality will not hold.

Changes in I

An increase in I, like a decrease in $\bar{e}$, represents an increase in the productivity of the economy, though it is evident from looking at the way they enter the expressions for the returns that they do not have exactly the same effect. Given β, increasing I increases the payoffs to taking the project and putting in the effort regardless of the realized return since $(\bar{r}-\hat{r}) + (r_0-\hat{r})\beta$ exceeds zero by Proposition (5.4). But it also makes the payoffs to the option of taking the project and not putting in the effort larger, and the net effect on β may be ambiguous. However, the effect of increasing I on $B_0(s)$ *for a fixed s* is given by

$$dB_0(s)/dI = (\bar{r}-\hat{r}) + (r_0-\bar{r})\beta + I(r_0-\bar{r})d\beta/dI.$$

We know that the sum of the first two terms is unambiguously positive, and while the third term may be negative, it can be shown that the net effect is always positive.

One implication of this conclusion is that $d\underline{\omega}/dI$ is positive which tells us that an increase in the average productivity of the project makes the poorest people better off. More generally, since each agent has the option of keeping the level of savings unchanged, $dB_0/dI > 0$ gives us an unambiguous result concerning the welfare effects of an increase in I, namely

Proposition 8.3 An increase in capital productivity I implies that in the long run all agents are better off.

This is of some interest since it is a property of the equilibrium level of risk-sharing. One can easily think of situations with no risk-sharing where people are so risk-averse that an increase in I makes everybody worse off.

It is also straightforward to show that increasing I makes $\hat{B}_1$ go up by more than $\underline{\omega}$. We have therefore

Proposition 8.4 In the poor economy, increases in I result in increases in inequality, as measured by the range.

This result contrasts with the other case of productivity increase. In fact it might seem paradoxical that it is the labor-replacing productivity growth (reduction in $\bar{e}$) which reduces inequality while the capital-augmenting productivity growth (increase in I) actually increases inequality. The paradox is only apparent, however, since in this very simple economy even the poorest people are effectively "capitalists".

A Tax Policy Example

Here we are concerned with the effects of a linear profits tax-subsidy scheme on the distribution of wealth and welfare. It is well known (Domar and Musgrave, 1944; Stiglitz, 1969) that under exogenous risk such a scheme ought to reduce the level of inequality and should also be welfare-improving, since it acts as a kind of insurance; moreover, since there is no effect on the expected returns, it should increase the capital stock by making projects more attractive.[33] We will show that such a result need not obtain when the amount of risk borne is endogenous.

Suppose the government imposes a profits tax of the form

$$\text{tax} = \alpha\cdot(r-\bar{r}),$$

wher α is positive and less than one. It is easy to check that such a scheme is self-financing. The policy gives rise to a new random variable ρ which replaces r in our equations, where $\rho = (1-\alpha)r + \alpha\bar{r}$. (This can be also be interpreted simply as an exogenous reduction in the riskiness of the project due to better technology.)

Notice now that the modified value of $B_r(s)$ corresponding to ρ will be

$$B_\rho(s)= s\hat{r} + I(\bar{r}-\hat{r}) + \gamma(s)(1-\alpha)(r-\bar{r}),$$

where γ is the retained equity share under the tax scheme. If we now pick γ such that $\gamma(s)(1-\alpha) = \beta(s)$, where $\beta(s)$ is the corresponding share for the original random variable r, we would have $B_r(s) = B_\rho(s)$ and therefore the incentive compatibility constraint will be satisfied. From the uniqueness of β we know that $\gamma = \beta/(1-\alpha)$ must be the right share for the modified problem. But if this is the value of γ that people will choose it is evident that $B_r(s)$ will be unchanged by the change in the random variable. We have therefore shown

<u>Proposition</u> <u>8.5</u> A linear profits tax or risk reduction is fully neutral.

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[33] In our model the capital stock is measured by the fraction of the population which takes projects.

That is, the tax policy will leave the entire path of the distribution of wealth and welfare unchanged. The positive welfare effect of the insurance policy is completely offset by the negative incentive effect that it has.

This is not to say that tax policies in general will always be neutral; but it does emphasize that explicit consideration of the endogeneity of risk-bearing can yield very different conclusions concerning the effects of particular policies from those reached with exogenous risks.

9. Endogenous Supply in the Loan Market

Our analysis in the previous sections is based on the assumption that there is a perfectly elastic supply of capital available in the economy at a certain, fixed interest rate. A possible alternative would be to assume that the interest rate moves to equate the supply and demand for loans within the economy. Thus, we will now regard all those who want to save at the safe rate of return as potential suppliers of loans and those who want to invest in the project but cannot afford to finance it as potential demanders. The safe return is determined by the requirement that these agents will be in equilibrium.

It is easy to see that in this case the safe rate of return in each period will depend on the particular distribution of wealth at the beginning of the period and will typically vary over time. This will make the dynamics of the distribution, even for a single lineage, considerably more complicated. An in-depth analysis of this case is beyond the scope of this paper; therefore we will limit ourselves to some simple, informal comments.

Note first that the rate of return on loans $\hat{r}$ is necessarily less than $\bar{r}$, since at the latter rate no one would take the project and so there will be an excess supply of loans. Second, note that $\hat{r}$ is necessarily at least as large as r_0, since if it was any less than that, people would borrow in order to invest in the project and then put in no effort. As this would still yield a positive rate of return at no cost in utility, the demand for loans at this rate must be infinite; $\hat{r} < r_0$ therefore cannot be an equilibrium.

In other words, the equilibrium value of $\hat{r}$ must be less than $\bar{r}$ and not less than r_0. Note however since at r_0 the agents are indifferent between lending and investing in the project (and choosing not to work) there is either net excess demand at r_0 or the market for loans clears. The question of whether a market-clearing $\hat{r}$ exists then boils down to showing that the excess demand for loans which is negative at $\bar{r}$ and positive at r_0 passes through 0 somewhere in between.

This would be obvious if the individual's excess demand was continuous in $\hat{r}$, but it is easy to see that we cannot assume that it is continuous.[34] However, if the aggregate excess demand jumps from positive to negative at some value $\hat{r}^*$ then it is easy to check (applying the theorem of the maximum) that at $\hat{r} = \hat{r}^*$ a positive mass of agents must be indifferent between the two investment strategies. By allocating an appropriate number of these agents to each of the two alternative strategies we can therefore always clear the market for loans.[35]

This informal argument should persuade the reader that each period, starting from an arbitrary initial distribution of wealth, there is always an equilibrium in which the market for loans clears in that period; which will generate a new initial distribution of wealth for the next period. There is thus a well-defined Markov process which maps from a wealth distribution today to a wealth distribution tomorrow.[36]

The question of whether this map yields an ergodic distribution and whether it has any desirable convergence properties are beyond the scope of this paper. We can, however, say something about the ergodic distribution, if it exists.

In our analysis of the model in Section 5 we made the assumption that $u'(x) > v'(x)\hat{r}$. Since in this section we are no longer assuming that $\hat{r}$ is a constant, this assumption does not really make sense. One rather restrictive alternative , which is certainly well-defined, is to assume that $u'(x) > v'(x)\bar{r}$. This will guarantee that our original assumption is valid for all

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[34] Consider what would happen if the wealth distribution had an atom in a neighborhood of the switch point.

[35] This essentially amounts to convexifying the excess demand correspondence.

[36] By appeal to some law of large numbers (what is meant by this in the present context is subtle, however — see e.g. Green (1989)), the map from the space of distributions to itself is actually deterministic.

relevant $\hat{r}$. We can then use all our results in Section 5 to describe the ergodic distribution; there will be only one switch point and the support of the distribution will be a connected, compact, non-trivial interval starting strictly above $\omega = 0$.

One may feel, however, that the condition $u'(x) > v'(x)\bar{r}$ is too strong. A weaker alternative is $u'(x) > v'(x)r_0$. While this does not allow us to apply all the results of Section 5 to the ergodic distribution, it can be checked that the proof of Proposition (5.9) and the lemmas leading up to do not rely on any assumption of this type. The ergodic distribution will still involve only one switch between being an investor and being a rentier.

Further it can be shown that the ergodic distribution will be non-trivial, i.e. it will not be a point mass. To see this, assume instead the contrary. In this case it must be that no one (actually no positive mass of agents) is doing the project, hence there is no demand for loans. But then the rate of return on the safe asset must be r_0. But now the condition assumed above, that $u'(x) > v'(x)r_0$ implies that $\omega_t > \omega_{t+1}$ and so the initial (trivial) distribution is not reproduced in the next period. This contradicts invariance entailed in the assumption of ergodicity. So, if an ergodic distribution exists, it must be nondegenerate.

10. Conclusion: Directions for Further Research

In this paper we have introduced a class of models which point toward a well-developed positive theory of economic distribution. Once the notion of perfect insurance entailed in the assumption of complete markets is dispensed with, an interesting economic theory of distribution can be derived, and the way is opened for analyzing ramifications of different institutional structures and sources of market incompleteness. We hardly claim yet to have such a complete theory; the following are possible lines of future research.

1) We have skirted the issue of the functioning of the loan market. In a world in which moral hazard is problematic in the equity market, it would likely present similar difficulties in the loan market. To eliminate such loans entirely would be to err in the other extreme (and may lead to the unsatisfactory result that everyone's wealth goes to zero); it would be better to consider the intermediate case in which loans are available, but the effective borrowing cost is decreasing in wealth.

2) Similarly, we might dispense with the questionable assumption that bequest utility is unbounded below. We have already suggested the difficulties that relaxing this assumption may raise: incentive compatibility may not be satisfied if the utility loss from moral hazard cannot be made arbitrarily large. This raises the realistic possibility that agents may need a certain minimum wealth before they can finance projects, or that only smaller projects might be available to the poor. Bounding utility may also affect behavior in the loan market in much the same way as a bankruptcy constraint.

3) A realistic labor market is conspicuously absent. Under the assumptions of this paper, nothing prevents one agent from working for another; but whether it is a manager in its own firm or someone else's, the same problem of eliciting effort arise, and the "employment" contract that would arise would be indistinguishable from the share contract we have examined here.

Perhaps the simplest extension would introduce a technology which allows agents to expend their effort monitoring other agents who in turn would provide their labor services in exchange for a wage. Agents could also be allowed to choose larger project sizes. This approach would lead to a more plausible institutional characterization of market economies.

APPENDIX A: PROPOSITION PROOFS FOR SECTION 5

Proposition 5.2 β is unique for each value of ω-c for which it is defined.

Proof: It is enough to show that the function $\int v(B_r)dF(r) - v(B_0) - 1$, zeros of which correspond to optimal β, is strictly increasing in β. Then there is at most one zero for each value of ω-c. By straightforward calculation:

$$\frac{d}{d\beta}\left\{\int v(B_r)dF(r) - v(B_0) - 1\right\}$$
$$= \int v'(B_r)I(r-\bar{r})dF(r) - v'(B_0)I(r_0-\bar{r}).$$

Perform the integration on the intervals $[r_0,\bar{r})$ and $[\bar{r},r_1]$ separately. On the first interval, the integrand is negative, but strict concavity of v implies that $|v'(B_r)(r-\bar{r})| < |v'(B_0)(r_0-\bar{r})|$, so that the expression is positive. On the second interval, both terms are positive, yielding the result. □

Proposition 5.3 (a) $s^*(\omega)$ is nondecreasing; (b) $s^{**}(\omega)$ is increasing.

Proof: (a) There are two cases:

(1) the constraint ω-c ≥ 0 binds; the result is trivial since s is constant (= 0).

(2) the agent chooses the project; suppose the proposition is false, that is suppose $\omega_1 < \omega_2$ but the corresponding optimal values of s have the relation $s_1 > s_2$. By definition of optimality

$$u(\omega_1-s_1) + v(B_0(s_1)) \geq u(\omega_1-s_2) + v(B_0(s_2)), \tag{A.1}$$
$$u(\omega_2-s_2) + v(B_0(s_2)) \geq u(\omega_2-s_1) + v(B_0(s_1)), \tag{A.2}$$

where s_2 is available at ω_1 since it is less than s_1, and s_1 is available at ω_2, since ω_2 is greater than ω_1. Because $u(\cdot)$ is strictly concave, the following relation holds:

$$u(\omega_1-s_2) - u(\omega_1-s_1) > u(\omega_2-s_2) - u(\omega_2-s_1). \tag{A.3}$$

Now rearrange (A.1) and (A.2) and add to obtain

$$u(\omega_1-s_2) - u(\omega_1-s_1) \leq u(\omega_2-s_2) - u(\omega_2-s_1),$$

a contradiction.

(b) Implicitly differentiate (5.4) to obtain $\frac{ds^{**}}{d\omega} = \frac{u''}{u''+v''\hat{r}^2} > 0$. □

Lemma A.1 Whenever it is optimal for the agent to choose the project, $c^* \geq c^{**}$.

<u>Proof:</u> First consider interior solutions, and suppose, contrary to the proposition, that $c^{**} > c^{*}$. Then

$$u(c^{**}) > u(c^{*}) \Rightarrow v(B_0) > v(s^{**}\hat{r}) \Leftrightarrow v'(B_0) < v'(s^{**}\hat{r}) \Rightarrow$$
$$v'(B_0)(\hat{r}+I(r_0-\bar{r})\beta'(s^{*})) < v'(s^{**}\hat{r})\hat{r} \Leftrightarrow u'(c^{*}) < u'(c^{**}) \Leftrightarrow c^{*} > c^{**},$$

a contradiction.

For corner solutions, entrepreneurial consumption is ω, while rentier consumption, which is always interior, is less than ω. □

<u>Proposition</u> <u>5.5</u> For all positive wealth, $s^{**}(\omega)\hat{r} < \omega$. Whenever the project is chosen, $s^{*}(\omega)\hat{r} < \omega$.

<u>Proof:</u> (1) Trivial if $s^{*} = 0$.

(2) Suppose $s^{**}\hat{r} \geq \omega$. Then $u'(\omega-s^{**}) = v'(s^{**}\hat{r})\hat{r} \leq v'(\omega)\hat{r}$. However, $u'(\omega) > v'(\omega)\hat{r}$ so that $u'(\omega-s^{**}) < u'(\omega) \Leftrightarrow \omega-s^{**} > \omega$, a contradiction.

(3) By the lemma $c^{*} \geq c^{**} \Leftrightarrow s^{*} \leq s^{**}$ when the project is taken; thus, $s^{*}\hat{r} \leq s^{**}\hat{r} < \omega$. □

<u>Proposition</u> <u>5.6</u> The agent strictly prefers the risky strategy to the safe strategy at $\hat{B}_0$ and sets $s^{*} = 0$ there.

<u>Proof:</u> First observe that regardless of the optimal choice of s^{*} the agent can consume all of its initial wealth and undertake the project, achieving a utility of $u(\hat{B}_0) + v(\hat{B}_0)$. On the other hand, forgoing the project yields at most $u(c^{**}) + v(s^{**}\hat{r})$. Now $c^{**} < \hat{B}_0$, since it is interior, and $s^{**}\hat{r} < \hat{B}_0$ by Proposition (5.5). Thus taking the project is strictly preferred.

To see that $s^{*} = 0$, suppose instead that it were positive. Then necessarily the first-order condition (5.2) holds with $\lambda = 0$, and we have

$$u'(\hat{B}_0) < u'(\hat{B}_0-s^{*}) = v'(B_0(s^{*}))[\hat{r}+I(r_0-\bar{r})\beta']$$
$$< v'(B_0(s^{*}))\hat{r} < v'(\hat{B}_0)\hat{r},$$

where we have used the facts that the term in brackets is less than $\hat{r}$ and that $B_0(s)$ is increasing; the first and third inequalities follow from strict concavity. But we know that $u'(\hat{B}_0) > v'(\hat{B}_0)\hat{r}$, a contradiction. □

<u>Proposition</u> <u>5.7</u> There exists $\hat{\omega} \geq \hat{B}_0$ such that $c^{*} = \omega$ $(s^{*} = 0)$ for $\omega < \hat{\omega}$ and $c^{*} < \omega$ $(s^{*} > \omega)$ for $\omega > \hat{\omega}$.

<u>Proof:</u> We proceed in four steps.

(1) By Proposition (5.3a), if $s^*(\omega') = 0$ then $s^*(\omega) = 0$ for all $\omega < \omega'$. Since $s^*(\hat{B}_0) = 0$ by the previous proposition, the set $Z = \{\omega : s^*(\omega)=0\}$ is nonempty.

(2) Define $\tilde{\omega}$ to be the unique (from strict concavity) solution to

$$u'(\omega) = v'(\hat{B}_0)[\hat{r}+I(r_0-\bar{r})\beta'(0)],$$

which exists, since $\lim_{\omega\to 0} u'(\omega) = \infty$ and $\lim_{\omega\to\infty} u'(\omega) = 0$.

(3) We claim that $\tilde{\omega}$ is an upper bound for Z. To see this, suppose the contrary, that is, there is $\omega' > \tilde{\omega}$ with $s^*(\omega') = 0$. Then $u'(\omega') < u'(\tilde{\omega})$. On the other hand,

$$u'(\omega') = v'(\hat{B}_0))[\hat{r}+I(r_0-\bar{r})\beta'(0)] + \lambda > u'(\tilde{\omega}),$$

where $\lambda > 0$ by uniqueness of $\tilde{\omega}$. Contradiction.

(4) Z is nonempty and has an upper bound, so define $\hat{\omega} = \sup Z$. Since $\hat{B}_0 \in Z$, $\hat{\omega} \geq \hat{B}_0$. □

<u>Proposition 5.8</u> There exists $\underline{\omega} \in (0,\hat{\omega})$ below which the bequest is larger than initial wealth.

<u>Proof:</u> Simply set $\underline{\omega} = \hat{B}_0$. By step (1) of the previous proposition, $s^* = 0$ for $\omega < \hat{B}_0$, so the bequest is $B_r(0)$. But by the same reasoning as in the proof of Proposition (5.6) the agent chooses the project in all of $(0,\underline{\omega})$. Thus, for $\omega < \underline{\omega}$, the lowest possible realization of the bequest is $\hat{B}_0 = \underline{\omega}$ and the proposition is proved. □

<u>Lemma A.2</u> As $\omega \longrightarrow \infty$, $s^{**}(\omega) \longrightarrow \infty$ and $c^{**}(\omega) \longrightarrow \infty$.

<u>Proof:</u> Suppose, to the contrary, that $s^{**}(\omega)$ is bounded, say by M. Then $v'(s^{**}\hat{r})\hat{r}$ is bounded below by $v'(M\hat{r})\hat{r} > 0$. But by the first-order condition, $u'(\omega-s^{**}) = v'(s^{**}\hat{r})\hat{r}$. Thus, the LHS approaches zero (since u is bounded above), while the RHS is bounded away from zero, a contradiction. The proof for $c^{**}(\omega)$ is similar. □

<u>Proposition 5.9</u> There exists a unique $\bar{\omega} > \underline{\omega}$ such that the agent takes the project if $\omega < \bar{\omega}$ and does not if $\omega > \bar{\omega}$.

<u>Proof:</u> By Lemma (A.2), the utility of the safe strategy approaches its upper bound, say $\bar{u} + \bar{v}$, as $\omega \longrightarrow \infty$, since $c^{**} \longrightarrow \infty$ and $s^{**} \longrightarrow \infty$. The utility of the risky strategy is bounded by $\bar{u} + \int \bar{v}dF - 1 = \bar{u} + \bar{v} - 1$. Thus

the safe strategy is strictly preferred for high enough wealth. On the other hand, Proposition (5.6) says that the risky strategy is strictly preferred for wealth low enough, viz., at $\underline{\omega}$. The value functions for the two programs are continuous and are therefore equal at some $\bar{\omega} > \underline{\omega}$.

We show that $\bar{\omega}$ is unique, and therefore that the agent does indeed take the project below $\bar{\omega}$ and declines it above $\bar{\omega}$. Define the value functions

$$W_R(\omega) = u(c^*(\omega)) + v(B_0^*(\omega)),$$
$$W_S(\omega) = u(c^{**}(\omega)) + v(s^{**}(\omega)\hat{r}).$$

Assume first that right- and left-hand derivatives of $W_R(\cdot)$ are equal at $\bar{\omega}$, i.e. that $\bar{\omega}$ is not also a jump point for the risky strategy consumption function $c^*(\omega)$. By the envelope theorem, $W_R'(\omega) = u'(c^*(\omega))$ and $W_S'(\omega) = u'(c^{**}(\omega))$ (this remains true even if $\omega < \hat{\omega}$, i.e. $c^*(\omega) = \omega$). We show that

(1) if $W_R(\bar{\omega}) = W_S(\bar{\omega})$, then $W_R'(\bar{\omega}) \leq W_S'(\bar{\omega})$;

(2) if $W_R(\bar{\omega}) = W_S(\bar{\omega})$ and $W_R'(\bar{\omega}) = W_S'(\bar{\omega})$ then for any $\omega' > \bar{\omega}$ with

$$W_R(\omega') = W_S(\omega') \text{ we must have } W_R'(\omega') < W_S'(\omega').$$

Thus, W_S cannot cut W_R from above, and they intersect with common slope at no more than one point.Then there are only two possibilities: there is a unique intersection, and it is transverse; there are two intersections, one of which is transverse, the other of which is a tangency (which keeps one value function below the other). In the second case, we may suppose that at the tangency, the agent resolves its indifference by maintaining the strategy corresponding to the higher value function. In either case, the switch point $\bar{\omega}$ corresponds to the unique transverse intersection.

To show (1), note that $W_R'(\bar{\omega}) > W_S'(\bar{\omega}) \Leftrightarrow u'(c^*(\bar{\omega})) > u'(c^{**}(\bar{\omega})) \Leftrightarrow c^*(\bar{\omega}) < c^{**}(\bar{\omega})$, which contradicts Lemma (A.1).

For (2), $W_R'(\bar{\omega}) = W_S'(\bar{\omega}) \Rightarrow c^*(\bar{\omega}) = c^{**}(\bar{\omega}) \Leftrightarrow s^*(\bar{\omega}) = s^{**}(\bar{\omega})$ by strict concavity of $u(\cdot)$; equality of the value functions implies $B_0^*(\bar{\omega}) = s^{**}(\bar{\omega})\hat{r}$ and therefore $I(\bar{r}-\hat{r})+I\beta(r_0-\bar{r}) = 0$. Now $\beta(\cdot)$ is nondecreasing in ω; $W_R(\omega') = W_S(\omega')$ and $W_R'(\omega') = W_S'(\omega')$ for $\omega' > \bar{\omega}$ would imply that $\beta(\cdot)$ remains constant over the interval $[\bar{\omega},\omega']$. Since $\beta(\cdot)$ is increasing in s, it must be the case that $s^*(\omega)$ is constant there and therefore $s^{**}(\bar{\omega}) = s^{**}(\omega')$. But this is a contradiction, because $s^{**}(\cdot)$ is strictly increasing in ω, by

Proposition (5.3b). (In fact, it is easy to show using Proposition (5.4) that the tangency, if it exists, must occur above the transverse intersection at $\bar{\omega}$.)

In case that $\bar{\omega}$ is a jump point for c^*, essentially the same argument goes through: the right-hand derivative of W_R, which is at least as large as the left-hand derivative (since consumption can only jump down), cannot exceed the derivative of W_S. If it does, then $W_R > W_S$ in a right neighborhood of $\bar{\omega}$, while $u'(c^*) > u'(c^{**}) \Longleftrightarrow c^* < c^{**}$ there, contradicting Lemma (A.1). □

APPENDIX B: PROPOSITION PROOFS FOR SECTIONS 6 AND 7

Proposition 6.1 For all r, $B_r(\omega)$ is nondecreasing, and increasing for $\omega > \hat{\omega}$.

Proof: Since $B_r(\omega) = B_0(\omega) + \beta(s^*(\omega))(r-r_0)$, and $\beta(\cdot)$ is nondecreasing, it suffices to prove the proposition for B_0. But, as we have said, B_0 is increasing in s (its derivative is positive), we need only show that $s^*(\omega)$ is increasing for $\omega > \hat{\omega}$. On this set, s^* satisfies the first order condition $u'(\omega-s^*) = v'(B_0(s^*))B_0'(s^*)$. If $s^*(\cdot)$ were not increasing, then there would exist two different values of ω for which s^* was the same. But this clearly cannot happen, since $u(\cdot)$ is strictly concave. □

Proposition 6.2 $\underline{\omega}$ is the unique fixed point of $B_0(\omega)$.

Proof: Since we have already established that $\underline{\omega}$ is a fixed point of $B_0(\omega)$ and that $B_0(\omega) = \hat{B}_0 > \omega$ for $\omega < \underline{\omega}$, it suffices to show that $B_0(\omega) < \omega$ for all $\omega > \underline{\omega}$. This is obvious if $B_0(\omega) = \hat{B}_0$, so suppose that $B_0(\omega) > \hat{B}_0$, i.e. $s^*(\omega) > 0$. Then $u'(\omega-s^*(\omega)) = v'(B_0(s^*(\omega)))B_0'(s^*(\omega)) < v'(B_0(s^*(\omega)))\hat{r} < u'(B_0(s^*(\omega)))$; the first inequality follows from the relation $B_0'(s^*) < \hat{r}$ asserted in Section 5, and the second follows from (2.2). Now if $\omega \leq B_0(s^*(\omega))$, then we would have by the strict concavity of $u(\cdot)$, $u'(\omega) > u'(\omega-s^*)$, or $\omega < \omega - s^*$, a contradiction. □

Proposition 6.4 At the switch point, we have $s^{**}(\bar{\omega})\hat{r} \geq B_0(\bar{\omega})$.

Proof: Since $u(c^*(\bar{\omega})) + v(B_0(\bar{\omega})) = u(c^{**}(\bar{\omega})) + v(s^{**}(\bar{\omega})\hat{r})$, and by Lemma (A.1) $c^*(\bar{\omega}) \geq c^{**}(\bar{\omega})$, we have $v(B_0(\bar{\omega})) \leq v(s^{**}(\bar{\omega})\hat{r})$ and the result is immediate. □

Proposition 7.2 An invariant measure of Ψ is atomless.

Proof: Suppose the proposition is false, and let x^* be an atom for an invariant measure μ with $\mu(x^*) = \epsilon$. By invariance, $\mu(x^*) = \int_\Omega \psi(\omega)[x^*]\mu(d\omega)$. Now consider $I_{x^*} = \{\omega \in \Omega\colon \exists \text{ r s.t. } B_r(\omega) = x^*\}$. Then

$$\mu(x^*) = \begin{cases} \int_{I_*} \psi(\omega)[x^*]\mu(d\omega) + \mu(B_s^{-1}(x^*)), \text{ if } B_s^{-1}(x^*) > \bar{\omega} \\ \int_{I_*} \psi(\omega)[x^*]\mu(d\omega), \text{ otherwise} \end{cases}$$

But since $F(\cdot)$ is atomless, $\psi(\omega)[x^*] = P(r=(x^*-s^*(\omega)-I(\bar{r}-\hat{r})+I\beta\bar{r})/I\beta) = 0$, and the integrals vanish. Thus, $x^* > B_s(\bar{\omega})$ and $\mu(B_s^{-1}(x^*)) = \epsilon$. Since $B_s(\cdot)$ is increasing, we can repeat the argument and conclude that μ has an infinity of atoms of measure ϵ, a contradiction. □

<u>Proposition</u> <u>7.3</u> Let λ denote Lebesgue measure on Ω. If μ is an invariant measure on Ω under ψ, then $\lambda \ll \mu$.

<u>Sketch</u> <u>of</u> <u>Proof:</u> We show that if $E_0 \subset \Omega$ with $\lambda(E_0) > 0$, then $\mu(E_0) > 0$. Suppose instead that $\mu(E_0) = \int_\Omega \psi(\omega)[E_0]\mu(d\omega) = 0$. Since $\psi(\omega)[E_0]$ is nonnegative, it must equal zero for μ-almost every ω: letting $E_1 = \{\omega \in \Omega: \psi(\omega)[E_0] > 0\}$, $\mu(E_1) = 0$. Now look at Figure 4. It is clear that $E_1 = \phi^{-1}(E_0) = \bigcup_{e \in E_0} \phi^{-1}(e)$, where ϕ is the stochastic policy correspondence defined in Section 6, and $\phi^{-1}(\cdot)$ denotes inverse image. Notice that for any point $\omega \in (\underline{\omega},\omega)$, $\phi^{-1}(\omega)$ contains an open interval. Thus, E_1 contains the union of open intervals, and $\lambda(E_1) > 0$, while, as we said, $\mu(E_1) = 0$. Continue in a like manner, constructing the sequence $E_k = \{\omega \in \Omega: \psi(\omega)[E_{k-1}] > 0\}$ and concluding $\mu(E_k) = 0$.

It is easy to see (refer to Figure 4) that the sequence so constructed converges to $(\underline{\omega},\omega)$ in the sense that $E_{k-1} \subset E_k$ and for all $\omega \in (\underline{\omega},\omega)$, there is a k with $\omega \in E_k$. (Taking the inverse images of points, let alone of a λ-nonnegligible set of them, gives a sequence of intervals, the inverse images of which rapidly converge to the whole space.) But since $\mu[(\underline{\omega},\omega)] = 1$, and $\mu(E_k) = 0$, we conclude that μ must contain an atom, which contradicts the previous proposition. □

<u>Lemma</u> <u>7.5</u> If $\psi^t(\omega)[A] = 1$ for all $\omega \in A$, then there exists $\omega_0 \in A$ such that $\omega_0 < \bar{\omega}$.

<u>Proof:</u> For t = 1, suppose instead $\omega > \bar{\omega}$, all $\omega \in A$, and choose $\omega' \in A$. Then $\psi(\omega')[A] = 1$ implies $B_s(\omega') \in A$. If $B_s(\omega') < \bar{\omega}$, we are done. Otherwise, we must have $B_s^2(\omega') \in A$. One can see that $K = \min \{k: B_s^k(\omega') < \bar{\omega}\}$ is well defined. Set $\omega_0 = B_s^K(\omega')$. Then $\omega_0 < \bar{\omega}$ and $\omega_0 \in A$, as desired.

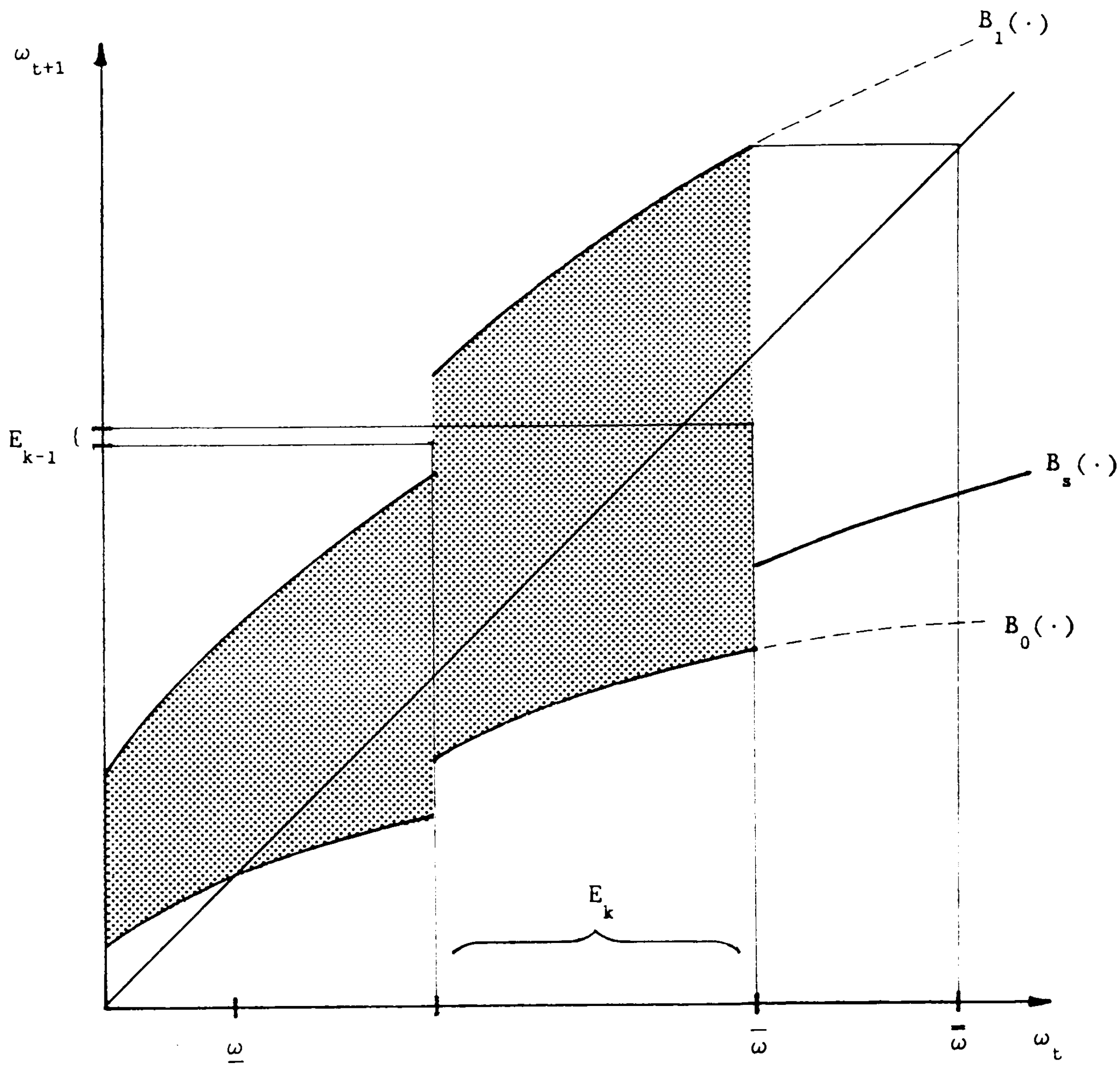


Figure 4. Construction of sets E_k. $E_k = \{\omega \in \Omega: \psi(\omega)[E_{k-1}] > 0\}$.

For $t > 1$, if K is a multiple of t, define ω_0 as before. Otherwise, note that we must have $\psi^{jt}(\omega')[A] = 1$, for all j. Now choose m to be the smallest integer making mt-K positive. Then $\omega'_0 = \psi^K(\omega') < \bar{\omega}$. Now note that the support of every iterate of ω'_0, in particular $\psi^{mt-K}(\omega'_0)$, will intersect $[\underline{\omega},\bar{\omega}]$ in a nondegenerate interval J. But $\psi^{mt-K}(\omega'_0) = \psi^{mt}(\omega')$, so λ-almost every point in J must be in A. So choose ω_0 from $J \cap A$. $\square$

APPENDIX C: SIMULATIONS

The purpose of this appendix is to provide an illustration of the kind of wealth distribution that our model predicts, rather than to provide a substitute for comparative statics.

We ran a version of the model in which utility took the form

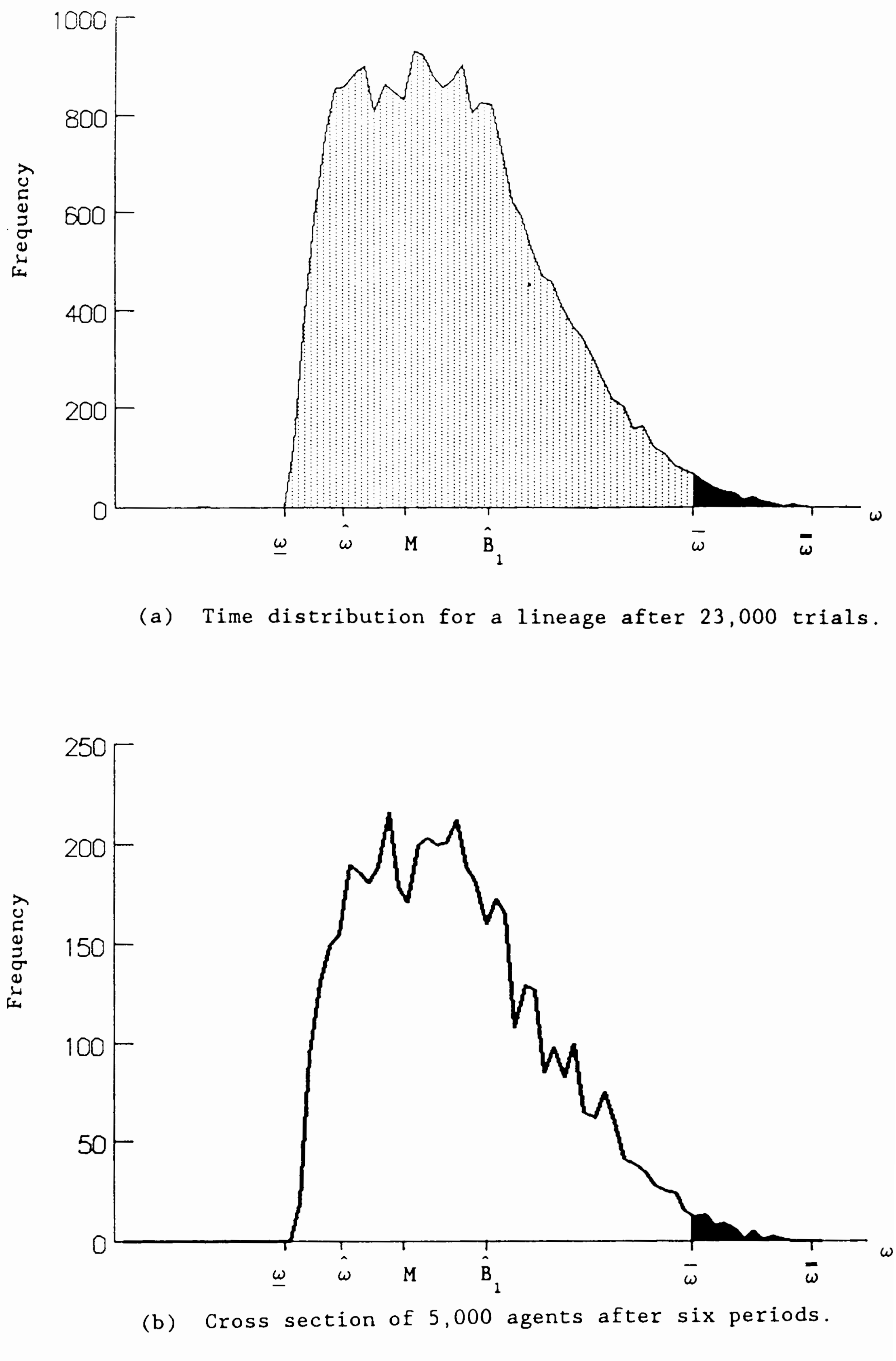
$$u(x) = Ax^{1-\gamma}/(1-\gamma), \; v(x) = x^{1-\gamma}/(1-\gamma),$$

and the distribution of project returns was uniform on $[r_0, r_1]$ (uniform distributions are reasonably far-removed from empirical income distributions, so this was thought not to bias things in our favor). For the runs in Figure 5, the parameter values are given in Table 1, and the computed values for the distribution are in Table 2.

The parameter values were varied somewhat, and all gave the same general shape; the most striking variation was in the narrowing of the range (and the elimination of rentiers, as suggested in Section 8) when effort disutility was decreased from 1.5 to 1.0.

Two kinds of experiments were run. In the first, the evolution of one lineage's wealth was followed for several thousand trials. In the second, 5,000 independent lineages were followed for a few trials. The frequency distribution of the first experiment is shown in Figure 5(a), the second in 5(b). The reader may judge whether these two distributions look the same, as our convergence theorem implies they should (two statistical tests, the chi-square and the Kolmogorov-Smirnov could not reject the hypothesis that they are). The jaggedness in the distributions should probably be discounted, for the reported number of trials is relatively "small" (there is a noticeable increase in smoothness as one increases the number of trials for the temporal wealth distribution).

One obvious shortcoming of our simulations is that the right tail is too short, i.e., the distribution is not sufficiently skewed. For the sake of comparison, values of the coefficient of variation and skewness for the U. S. income distribution (estimated from Internal Revenue Service summary data on individual tax returns) are presented in Table 3. These measures are somewhat fragile, though: two sets of values are given, the first corresponding to all returns, the second with the very high income bracket ($500,000 and up, constituting about 0.09 percent of the returns) excluded. Either way, though, they suggest a good deal more empirical variation than do the simulations.



(a) Time distribution for a lineage after 23,000 trials.

(b) Cross section of 5,000 agents after six periods.

Figure 5. Simulated wealth distributions. Dark regions represent rentier wealth levels. M is the mean wealth.

Given the restrictions of our model, however, particularly the inability of an agent to take projects of more than unit size, this should not surprise us terribly.

γ	A	r_0	r_1	$\hat{r}$	I	$\bar{e}$	initial wealth
2.1	1.2	0.5	2.5	1.1	1.0	1.5	0.1

Table 1. Parameter values for simulations in Figure 5.

$\underline{\omega}$	$\hat{\omega}$	$\hat{B}_1$	$\bar{\omega}$	$\bar{\bar{\omega}}$ ($=B_1(\omega)$)	mean	c.v.	skewness	% rentier
0.25	0.34	0.55	0.82	0.97	0.47	0.28	+0.57	0.9

Table 2. Statistics for simulated distribution.

	c.v.	skewness
all returns	2.06	+6.47
top bracket excluded	1.07	+1.74

Table 3. Statistics for U.S. income distribution

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