Discussion Paper No. 870

AGGREGATION OF SEMIORDERS:

INTRANSITIVE INDIFFERENCE MAKES A DIFFERENCE*

by

Itzhak Gilboa** and Robert Lapson***

February 1990

*We wish to thank Tatsuro Ichiishi, Jorge Nieto, Ariel Rubinstein, Efraim Sadka and especially David Schmeidler for stimulating discussions and comments.

**KGSM, Northwestern University, Evanston, IL 60208. Partial financial support from NSF grant IRI-8814672 is gratefully acknowledged.

***KGSM, Northwestern University, Evanston, IL 60208.

ABSTRACT

A semi-order can be thought of as a binary relation P for which there is a utility u representing it in the following sense: xPy iff $u(x) - u(y) > 1$.

Weak orders (for which indifference is transitive) can not be considered a successful approximation of semi-orders; for instance, a utility function representing a semi-order in the manner mentioned above is almost unique, i.e. cardinal and not only ordinal.

In this paper we deal with semi-orders on a product space and their relation to given semi-orders on the original spaces. Following the intuition of Rubinstein we find surprising results: with the appropriate framework, it turns out that a Savage-type expected utility requires significantly weaker axioms than it does in the context of weak orders.

Moreover, our axioms provide a conceptual basis for the weighted average paradigm in general, and, in particular, may be used to justify utilitarianism in a social choice context.

1. INTRODUCTION

The bulk of economic theory literature assumes that the decision maker's preference relation may be represented by a utility function u(.) as follows: xPy iff u(x) - u(y) > 0.

All of the axioms P should satisfy in order to allow for such a representation, as well as the very existence of a binary preference relation, have been criticized over the years on theoretical and empirical grounds alike. Yet, it seems that most of the axioms, including the transitivity of P, are still widely acceptable.

One exception is the transitivity of indifference: a representation as above, trying to retain this property, leads us to contend that the decision maker strictly prefers x to y whenever u(x) is -- even slightly -- larger than u(y). Most theorists would probably not like to take this axiom literally; it does not seem very plausible that an economic agent strictly prefers an annual income of $30,000.01 to $30,000.00 or 2,534 grains of sugar in a cup of coffee to 2,533 grains etc. On the other hand, indifference between very close alternatives would lead, in the presence of transitive indifference, to indefference between any two alternatives, which is absurd.

Moreover, psychology suggests the Weber-Fechner Law (Weber (1834), Fechner (1860)) which says, roughly, that human beings cannot discern between very close objects, and only when the difference in mass, temperature, length, pressure, etc. exceeds a certain "just noticeable difference" does a distinction emerge in people's minds. Weber's Law also says, that for each scale -- mass, for instance,-- there is a certain constant $\lambda > 1$ (depending on the individual and possibly also on the circumstances) such that the individual would discern the difference between two magnitudes only if their ratio exceeds λ (or drops below $1/\lambda$). On the appropriate logarithmic scale, we therefore get the following representation:

$$xPy \text{ iff } u(x) - u(y) > 1. \qquad (1)$$

Semi-orders, defined by Luce (1956), are transitive binary relations the indifference of which may not be transitive. For the sake of the discussion one may take the representation (1) as a definition of a semi-order although it is equivalent to Luce's definition only if the set of alternatives is finite. For discussion of representations of semi-orders -- and the more general class of interval orders -- see Fishburn (1970, 1985), Manders (1981), Gensemer (1987), Bridges (1983), Chateauneuf (1987) and Beja and Gilboa (1989)[1].

From the view point of economic theory it is quite reasonable to argue, then, that in "reality" people have semi-orders (or even less structured preferences,) where indifference is intransitive, but weak orders can still be taken as an approximation, or a mathematical idealization, of reality; an idealization that simplifies matters just like the continuum is a convenient representation of very large, though discrete, sets.

The main point of this paper is that this idealization is far from being innocuous. Surprisingly enough, we have good news: some aspects of economic theory seem to be conceptually simpler with semi-orders than with weak orders. Although the weak order idealization simplifies mathematics, it requires considerably stronger axioms (from a conceptual viewpoint) to derive such results as the expected utility paradigm.

Let us take as an example the following point, which was noted in Beja and Gilboa (1989). Suppose P is a semi-order with a representation (1) by u. Then the function u is almost unique. Should v represent P as well, there will be an f such that $v \equiv f(u)$, but f cannot be any monotone transformation. It can be defined arbitrarily (as long as monotonicity is preserved) on [0, 1], and then one has to extend it in such a way that $f(x + n) = f(x) + n$, for all $x \in [0, 1]$ and $n \in \mathbf{Z}$.

If we think of the just noticeable difference -- normalized to one -- as relatively small, the utility u is almost unique. It thus makes sense to discuss properties such as concavity or convexity of the utility function, properties that the mathematical idealization, namely weak orders, rendered devoid of meaning.

In fact, there is more information in a semi-order than in a weak order. Indeed, every semi-order P induces a weak order of indirectly revealed preferences -- which we will denote by Q -- represented by the same utility function u in the regular sense, i.e., xQy iff $u(x) - u(y) > 0$. However, the semi-order P cannot be reconstructed from its associated Q.

On second thought, this result is not surprising at all: a semi-order implicitly provides not only rankings of the alternatives but also rankings of differences between pairs of alternatives, which is the essence of a cardinal utility functions. The point is that the ranking of differences implied by the classification to "larger than the just noticeable difference (jnd)" and "not larger than the jnd" is naturally given in the original preferences, and it suffices for fixing u almost uniquely, without the additional artificial assumption that the decision maker can answer questions like "do you prefer x to y more than you prefer w to z ?"

In a similar way, this paper shows that when semi-orders are the primitive preferences, one needs relatively weak axioms to derive expected utility (or weighted average) representation for preference relation over a product space of given spaces -- for instance, the space of acts which are functions from the set of states of nature to consequences. But before we explain these results we have to digress and describe Rubinstein's approach.

In Rubinstein (1988) preference orders over simple lotteries are discussed. A lottery is "simple" if it promises a certain monetary prize x (say $x \in [0,1]$) with probability p and zero with probability (1 - p). (For some $p \in [0,1]$.)

Rubinstein assumes that there are two "similarity" relations, $\sim_x$ and $\sim_p$, defined on the set of monetary prizes and the set of probabilities, respectively, with the interpretation that two "similar" magnitudes (prizes or probabilities) are indistinguishable in the decision maker's mind. He then considers weak orders on lotteries (which are simply the product space of prizes and probabilities) and defines the "star" (*) property as follows: A weak order satisfies the (*) property with respect to (w.r.t.) $\sim_x$ and $\sim_p$ if whenever two lotteries are similar in one component but not similar in another -- the other component determines the preference between the two lotteries. Rubinstein proves that for given similarity relations there is an almost unique weak order on lotteries satisfying the (*) property. Contending that this property is a basic feature of any reasonable decision process people actually go through while making decisions, he concludes that there is a basic flaw in axiomatic theories justifying expected utility maximization (and its generalizations), because this decision rule may well be inconsistent with the (*) property.

Rubinstein's similarity relations are, in fact, the indifference relations induced by some semi-orders. Our original intuition was that his results hinge on the fact that in the lottery space a weak order is assumed to be given, while on the more primitive spaces -- only semi-orders. In this case his conclusions seem to be somewhat dubious since it seems unreasonable that a decision maker who cannot discern small differences between monetary prizes as such will have perfect distinction power regarding the more complicated space of lotteries.

This intuition seems to have been shared by Aizpurua, Nieto and Urierarte (1988) and Aizpurua, Ichiichi, Nieto and Uriarte (1989). They allowed the preference order on lotteries to have intransitive indifference but found that Rubinstein's results are robust with respect to this generalization. To cope with the "over-determination problem" they introduced "correlated similarities" -- allowing for the similarity relation on the probability space to depend upon the associated monetary prize. Thus, they found that expected utility maximization was not inconsistent with the (*) property, though this result (as well as Rubinstein's) could not be extended to lotteries involving more than two possible prizes.

Our approach is slightly different. We require the binary relation on the product space to be a semi-order (which is a stronger requirement than theirs), and we found that the (*) property -- which we call strong monotonicity -- is too strong a requirement, and there typically will not be any semi-orders on the product space which are strongly monotone with given ones on the original spaces (this is true even for two such spaces.)

We therefore define monotonicity in a weaker form: given two alternatives in the product space $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, if there is a preference in one component ($x_i P_i y_i$) and indirectly revealed preference in all others ($x_j Q_j y_j$) then we require that x will be preferred to y. Lack of preference of y_j over x_j does not suffice to use this monotonicity axiom. Thus, our monotonicity is weaker than the (*) property, and it can be thought of as a "stochastic dominance" or "pareto dominance" axiom.

Weakening the (*) property allows for the existence of monotone semi-orders on the product space, but it also allows for the possibility of the decision maker being able to discern between alternatives which were indistinguishable in the original spaces. Thus we introduce a consistency requirement stating that if all other components are held fixed, and in one component we consider two indistinguishable alternatives x_i and y_i, then the whole vector with x_i is indistinguishable from that with y_i.

We proved that -- under some mild technical assumptions -- there is an almost unique semi-order P on $X_1 x \ldots x X_n$ which is monotone and consistent w.r.t. given semi-orders P_i on X_i, and that each such P has an almost unique representation. Moreover, if u_i represents P_i with a jnd of 1, one of the monotone and consistent semi-orders will be represented by Σu_i, namely

$$xPy \text{ iff } \Sigma u_i(x_i) - \Sigma u_i(y_i) > 1.$$

Thus, if all X_i and P_i are the same -- the set of consequences and a semi-order on it -- we obtain an expected utility representation of a semi-order on actions, namely functions from the n states of nature to X. Moreover, we get equal probabilities to all states of nature, i.e., the Laplace criterion is justified with monotonicity and consistency alone. Comparing these axioms to the main axioms of Savage (1954), the consistency is a very weak version of the "sure thing principle" (in fact, it corresponds to axiom P2 restricted to singleton events,) while monotonicity corresponds to axiom

P3 (which is sometimes also considered as a part of the "sure thing principle".) One need not require the full strength of P2, nor axiom P4 regarding comparability of events, and yet one obtains a stronger result, namely that all states of nature are equiprobable.

Before carrying on to the axiomatizaton of the general expected utility representation, we should comment on alternative interpretations. For instance, if we replace the states of nature by individuals (with possibly different X_i's and P_i's), our monotonicity and consistency axiomatize utilitarianism[2] (see Harsanyi (1955)). Similar justifications of utilitarianism were obtained by Goodman and Markowitz (1952) and Ng (1977). Both models are in quite different frameworks from ours, and we find them considerably less fundamental and intuitive. In particular, they both use axioms involving "counting" which, to a certain extent, presupposes the desired result.

Luce and Raiffa (1957) raise doubts regarding the appropriateness of this criterion: in fact we resolve here the classical problem of inter-personal comparison of utility by setting the jnd of all individuals to be a standard unit of measurement. This means that more sensitive people will get a higher weight in the social welfare function than less sensitive ones. The question of whether this is just (or "just noticeable") is beyond the scope of this paper.

An alternative interpretation will be to consider each space X_i as a product space of probabilities and prizes, namely of simple lotteries as in Rubinstein (1988). Regardless of the preference relation one has on X_i, be it a non-expected utility one as Rubinstein suggests, or an expected utility as in Aizpurua et. al. (1989) our results allow a conceptually consistent extension of the semi-order to general lotteries. If preferences on X_i are represented by $p_i u(x_i)$ we obtain $\Sigma p_i u(x_i)$ as a representing functional (i.e., expected utility as in von Neumann and Morgenstern (1947)). Other preferences such as $g(p_i)u(x_i)$ would give rise to prospect theory (Kahneman and Tversky (1979)) etc. At any rate, Rubinstein-type arguments imply the additive structure which is at the heart of the expected utility representation.

It is somewhat ironic that Kahneman-Tversky prospect theory was strongly criticized for not satisfying (first order) stochastic dominance, while here we find that a different formulation of the same axiom -- our monotonicity -- prefers prospect theory over theories such as Quiggin's (1982), Yaari's (1987) or non-additive probabilities (Schmeidler (1984), Gilboa(1987)).

However, the Laplace criterion is quite restrictive and may point to a flaw in our assumptions: indeed, assuming all X_i's are the same is intrinsic to the problem of decision making under uncertainty, but the assumption that all P_i's are also identical may be too strong. Instead, we could assume that there is another type of "correlated similarities": the decision maker's ability to distinguish between alternatives depends on the likelihood of the associated state of nature. (In a way, this is a complementary approach to the correlated similarities of Aizpurua et al. (1989).)

We are therefore interested in the following question: given several semi-orders P_i on the same space X, when is there a single utility function $u: X \to \Re$ and a constant $\delta_i > 0$ corresponding to each P_i, such that

$$xP_iy \text{ iff } u(x) - u(y) > \delta_i \text{ ?}$$

In this paper we restrict our attention to the case of all δ_i being rational, for which we provide a complete axiomatization. Using this result and the previous ones we obtain an axiomatization of a semi-order P on X^n represented by

$$xPy \text{ iff } \Sigma p_i u(x_i) - \Sigma p_i u(y_i) > 1$$

for rational probabilities p_i.

To sum up, this paper studies semi-orders on product spaces in general, and on a product of identical spaces in particular. Using axioms motivated by Rubinstein (1988), we provide a conceptual basis for additive separability in the general context and for expected utility in the more specific one. Most importantly, this study shows that with aggregation of preferences, as in the case of the numerical representations of preferences on a single space, a lot of information is lost when we choose to work with weak orders rather than with the more realistic semi-orders.

The rest of this paper is organized as follows. Section 2 presents preliminary definitions and quotes some results from the literature. Our main results are stated in section 3. Finally, the proofs and related analysis are to be found in section 4.

2. PRELIMINARIES AND BASIC DEFINITIONS

The central issue of this paper is semi-orders. The formal definition is the following:

A binary relation P on X is a semi-order if for any x,y,z,w in X

1) not xPx (P is irreflexive);

2) if xPy and zPw then xPw or zPy ;

3) if xPy and yPz then xPw or wPz.

For a given semi-order P define binary relations I, Q, E and Q^o as follows: for every x,y in X:

xIy iff not xPy and not yPx;

xQy iff $\exists$ z in X such that either 1) xPz and not yPz

or 2) zPy and not zPx;

xEy iff not xQy and not yQx;

xQ^oy iff xQy or xEy.

Any superscripts, subscripts etc. of P will be carried over to its associated I, Q, E and Q^o.

Q being such defined is a weak order, i.e., satisfies the set of conditions below: for any x,y,z in X

1) not xQx (Q is irreflexive);

2) if xQy and yQz then xQz (Q is transitive);

3) if xQy then xQz or zQy.

Scott and Suppes (1958) proved that if X is finite, then there exists a utility function on X such that

for any x,y in X xPy iff $u(x) > u(y) + 1$

and xQy iff $u(x) > u(y)$.

Manders (1981) Beja and Gilboa (1989) showed that for this result to be true for a countable X an additional axiom is needed saying that for every x in X and every infinite sequence $x_1, x_2, \ldots$ in X if $x_i P x_{i+1}$ for i = 1,2,... then for some n xPx_n and if $x_{i+1} P x_i$ for i = 1,2,... then for some n $x_n Px$. Beja and Gilboa (1989) provide characterization of the jnd representation for a general (not necessarily countable) X. We will generally assume in this paper that range(u) = $\mathfrak{R}$.

Let us recall the standard definition of concatenation of binary relations on X: given two binary relations B_1 and B_2 let $B_1 B_2$ be defined by:

For any x,y in X xB_1B_2y iff there exists z in X such that xB_1z and zB_2y.

Note that successive application of this definition render concatenation of more than two relations well-defined.

Let $X_1, \ldots, X_n$ be given sets and let there be semiorders P_i defined on every X_i, i = 1,...,n. Assume that P is a semi-order on $X = X_1 x \ldots x X_n$. For a generic element x in X, x_i will denote its i-th component, x_{-i} will stand for $(x_1, \ldots, x_{i-i}, x_{i+1}, \ldots, x_n)$ and (x_{-i}, y_i) for $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$.

Further suppose that P and Q are such that there exists a utility function u from X onto $\mathfrak{R}$ such that xPy iff u(x) > u(y) + 1 and xQy iff u(x) > u(x) for all x and y in X. We will say that u represents P and call P representable. From here on let us assume that every P_i is representable.

Note. u_i is defined upto a strictly increasing transformation v = f(u), where f(a) = f(a-1) + 1 for all a in $\mathfrak{R}$.

Now we can define some properties P may possess w.r.t. $P_1, \ldots, P_n$:

Definition 1. 1) P on X is <u>P-monotone</u> with respect to the semi-orders

$P_1,\ldots,P_n$ (hereafter P-monotone) if $\forall$ $x,y \in X$ the following holds: if $x_i Q^o{}_i y_i$ for all $i \in N \equiv \{1,\ldots,n\}$ and $\exists$ j such that $x_j P_j y_j$ then xPy. (As above Q_i and Q are the corresponding weak orders.)

2) P is <u>Q-monotone from above</u> with respect to $P_1,\ldots,P_n$ (hereafter Q-monotone from above) if xPy and $x'_i Q^o{}_i x_i$ $\forall$ $i \in N$ imply x'Py.

3) P is <u>Q-monotone from below</u> with respect to $P_1,\ldots,P_n$ (hereafter Q-monotone from below) if xPy and $y_i Q^o{}_i y'_i$ $\forall$ $i \in N$ imply xPy'.

4) P is <u>Q-monotone</u> with respect to $P_1,\ldots,P_n$ (hereafter Q-monotone) if it is both Q-monotone from above and from below.

5) P is <u>monotone</u> with respect to $P_1,\ldots,P_n$ (hereafter monotone) if it is simultaneously P- and Q-monotone.

Definition 2. P is <u>consistent</u> with $P_1,\ldots,P_n$ (hereafter consistent) if for all $i \in N$ and for all $x_i, y_i \in X_i$ if $x_i I_i y_i$ then $(x_{-i}, x_i) I (x_{-i}, y_i)$ for all $x_{-i} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$.

Relations between the two types of monotonicity and the consistency requirements are considered in section 4 (see statements 1 - 5). There one can also find examples of semi-orders which do and do not satisfy various subsets of these properties.

Let us define for each semi-order P' on X' the P'-topology as follows: $x_n \to x$ if for every $y \in X$ for which yQx, there exists M such that $\forall$ $m \geq M$ yQx_m and for $y \in X$ for which xQy there exists M such that $x_m Qy$. The continuity of P we are about to define means that if a sequence $\{x_{ik}\}$ in X_i converges to x_i in X_i in the P_i-topology (for all i), then $\{x_k\}$ converges to x in the P-topology. (In the presence of monotonicity and consistency, this is tantamount to saying that the P-topology on X is the product topology defined by the P_i-topologies on X_i, $i = 1,\ldots,n$.)

Definition 3. 1) P is <u>continuous from above (below)</u> with respect to $\{P_i\}$ $i = 1,\ldots,n$ if for all i, for every sequence $\{x_{ik}\}$ converging to x_i in X_i and for all y in X if $(x_1,\ldots,x_n)P(y_1,\ldots,y_n)$ $((y_1,\ldots,y_n)P(x_1,\ldots,x_n))$ then there exists M such that for every $k > M$ $(x_{1k},\ldots,x_{nk})P(y_1,\ldots,y_n)$ $((y_1,\ldots,y_n)P(x_{1k},\ldots,x_{nk}))$.

2) P is <u>continuous</u> with respect to $P_1,\ldots,P_n$ (hereafter continuous) if it is continuous both from above and from below.

Finally, the symbol ~ will stand for negation.

3. THE MAIN RESULTS

Our main results can be reduced to three theorems.

Theorem A states existence and characterizes monotone, consistent and continuous semi-orders on a product space.

Theorem A. Let $\{P_i\}_{i\in N}$ be semi-orders on $\{X_i\}_{i\in N}$, represented by $\{u_i\}_{i\in N}$. Let P be a semi-order on $X = \Pi_{i\in N}X_i$ represented by u. Then the following are equivalent:

(i) P is consistent, monotone and continuous w.r.t. $P_1,\dots,P_n$;

(ii) There is a strongly monotone and continuous function $f_u: \Re^n \to \Re$ satisfying

$$f_u(a_{-i},a_i) = f_u(a_{-i},a_i+1) - 1,\ \forall\ a \in \Re^n,\ \forall\ i, \qquad (2)$$

such that $u = f_u(u_1,\dots,u_n)$.

In particular, there exists such a P defined by $u = \Sigma u_i$.

Theorem B gives the notion of "almost uniqueness" of such a semi-order on the product space.

Theorem B. Let $\{P_i\}_{i\in N}$ be semi-orders on $\{X_i\}_{i\in N}$ represented by $\{u_i\}_{i\in N}$ respectively. Suppose that P_a and P_b are two representable semi-orders on X which are both consistent, monotone and continuous with respect to $P_1,\dots,P_n$. Then $(P_a)^n \subseteq P_b$ and $(P_b)^n \subseteq P_a$.

Theorem C deals with a joint representation of several semi-orders on the same space. It will need additional axioms. For two semiorders P and P′ on a certain space X′ define the following conditions.

A1. The concatenation of P and P′ is commutative, namely, PP′ = P′P.

A2. For any $k,m \in \mathbf{N}$ either $(P)^k \subseteq (P')^m$ or $(P')^m \subseteq (P)^k$ and for some $k,m \in \mathbf{N}$, we also have $(P)^k = (P')^m$.

Theorem C. Given semi-orders $P_1,\dots,P_n$ on a set X such that P_i is represented by $u_i(.)$, $i = 1,\dots,n$, the following are equivalent:

(i) there exist a function $u: X \to \mathfrak{R}$ with range(u) = $\mathfrak{R}$ and positive rational numbers $\delta_1,\dots,\delta_n$ such that for all $i = 1,\dots,n$, and for all x,y in X

xP_iy iff $u(x) - u(y) > \delta_i$, and

xQ_iy iff $u(x) > u(y)$;

(ii) for all i and j in $\{1,\dots,n\}$ P_i and P_j satisfy A1 and A2.

Corollary D applies the previous results to expected utility representation.

Corollary D. Let X′ be a set and let $P_1,\dots,P_n$ be semi-orders on it represented by $u_1,\dots,u_n$ (respectively), where range(u_i) = $\mathfrak{R}$ for all i. Suppose that for every $i,j \in N$ P_i and P_j satisfy A1 and A2. Define P on X = $(X')^n$ by $u = \Sigma(1/\delta_i)u_i$ for δ_i obtained from Theorem C. Then

(i) P is continuous, consistent and monotone w.r.t $P_1,\dots,P_n$;

(ii) If P′ is another semi-order on X which is continuous, consistent and monotone w.r.t $P_1,\dots,P_n$ then $(P')^n \subseteq P$ and $P^n \subseteq P'$.

Proofs of the main theorems and related results are contained in section 4.

4. PROOFS AND AUXILIARY RESULTS

Let $X_1, \ldots, X_n$ be given sets with semi-orders $P_1, \ldots, P_n$ defined on them respectively. Let $X = X_1 x \ldots x X_n$, and let P be a semi-order on X represented by u. We assume these conditions unless otherwise stated.

Let us first show that strong consistency is too binding a requirement even for n = 2.

We will say that P is strongly consistent with P_1 and P_2 if for all x,y in X, $x_i P_i y_i$ and $x_j I_j y_j$ imply xPy, where $\{i,j\} = \{1,2\}$. In the next lemma P_i are assumed to be representable, whence range$(u_i) = \mathbb{R}$. However note that it suffices that range$(u_i) \supset (a,b)$ for some $b > a + 1$.

Lemma 4.1. If P_1 and P_2 are representable semi-orders. Then there does not exist a representable semi-order P which is strongly consistent with them.

Proof. Let x,y,z in X be such that $u_1(y_1) = u_1(x_1) - \varepsilon + 1$, $u_2(y_2) = u_2(x_2) - \varepsilon - 1$, $u_1(z_1) = u_1(x_1) + \varepsilon + 1$, $u_2(z_2) = u_2(x_2) + \varepsilon - 1$, where ε is a positive number less than 1. It follows from the definition of strong consistency that xPy and zPx. By transitivity of P it implies zPy. Moreover, $u(z) - u(y) > 2$. Denote the interval $[(u_1(y_1),u_2(y_2)),\ (u_1(z_1),u_2(z_2))]$ by d. For any two points v,w in X such that $(u_1(v_1),u_2(v_2)),\ (u_1(w_1),u_2(w_2)) \in d$ and $u_1(v_1) > u_1(w_1)$ we get $u(v) - u(w) > 2$, a contradiction. //

The next four results relate to our concepts of monotonicity and consistency.

Lemma 4.2. A representable semi-order P which is Q-monotone from above (from below) is Q-monotone.

Proof. We will show that Q-monotonicity from above imply Q-monotonicity from below. The second part is proved symmetrically. Let xPy and $y_i Q^o{}_i y'_i$, i = 1,...,n. For every z in X such that y'Pz it follows from Q-monotonicity from above that yPz. As the range of u is $\mathfrak{R}$, $u(y) \geq u(y')$. Hence, xPy'. //

The three statements below show interrelations between monotonicity and consistency. In fact, they illustrate that P-monotonicity, Q-monotonicity and consistency are mutually independent.

Observation 4.3. Consistency and P-monotonicity do not imply Q-monotonicity.

Proof. Consider the following example. n = 2, $X_1 = X_2 = \mathfrak{R}$.

$u_1(x_1) = \{ x_1$, if $x_1<0$;

0, if x_1 is in [0, 1/2];

$x_1 - 1/2$, if $x_1>1/2$ },

$u_2(x_2) = x_2$.

P_1 may be also represented by v defined as follows:

$v(x_1) =$ $x_1/2$, if x_1 is in [0, 3/2[;

$x_1/2 + k/2$, if x_1 is in [1/2+k, 3/2+k[;

$x_1/2 - k/2 - 1/4$, if x_1 is in]-k-1, -k];

$x_1/2 - 1/4$, if x_1 is in]-1, 0[, where k is in **N**.

Define $u(x_1,x_2) = v(x_1) + u_2(x_2)$ and let xPy iff $u(x) > u(y) +1$. Since v represents P_1 and u represents P_2, P is consistent and P-monotone w.r.t. P_1 and P_2. However, P is not Q-monotone: let $x = (1/2,1)$, $y = (0,1)$ and $z = (0,1/5)$. $u(x) = 5/4$, $u(y) = 1$, $u(z) = 1/5$. Hence, xPz and not yPz, namely, xQy. But $y_i Q^o{}_i x_i$, i=1,2. This violates Q-monotonicity. //

Observation 4.4. Q-monotonicity and consistency do not imply P-monotonicity.

Proof. Consider the following example: $n = 2$, $X_i = \Re$, $u_i(x_i) = x_i$, $x_i P_i y_i$ iff $x_i > y_i + 1$, $i = 1,\ldots,n$. Define $u(x_1,x_2) = (x_1 + x_2)/2$ and xPy iff $u(x) > u(y)+1$, i.e. $x_1 + x_2 > y_1+y_2+2$. P is obviously Q-monotone and consistent. Let $x_1 = x_2 = 1.1$, $y_1 = 0$, $y_2 = 1$. This means that $x_1 P_1 y_1$ and $x_2 Q^o{}_2 y_2$, but $x_1+x_2 = 2.2 < 3 = y_1+y_2+ 2$. Hence, not xPy and P is not P-monotone. //

Observation 4.5. Monotonicity does not imply consistency.

Proof. Consider the following example: $n = 2$, $X_i = \Re$, $u_i(x_i) = x_i$, $x_i P_i y_i$ iff $x_i > y_i + 1$, $i = 1,\ldots,n$. Define $u(x_1,x_2) = 2*(x_1 + x_2)$ and xPy iff $u(x) > u(y)+1$, i.e. $x_1 + x_2 > y_1+y_2+1/2$. P is obviously monotone. Let $x_1 = 0.9$, $x_2 = y_1 = y_2 = 0$. This means that $x_1 I_1 y_1$, but $x_1+x_2 = 0.9 > 1/2 = y_1+y_2+ 1/2$. Hence, not xIy and P is not consistent. //

Now we show that continuity also has to be explicitly assumed:

Observation 4.6. Monotonicity and consistency do not imply continuity.

Proof. Consider the following example of a representable semi-order P that is monotone and consistent w.r.t. given $\{P_i\}_{i\in N}$ but is not continuous w.r.t. them.

Let $n = 2$, $X_1 = X_2 = \mathbb{R}$, $u_1(x_1) = x_1$, $u_2(x_2) = x_2$, where $u_i(.)$ represents P_i, $i = 1,2$. P_1 admits also another representation $v_1(x) = x/2 + k/2$, where $x \in [k, k+1)$, $k \in \mathbf{Z}$.

Define $u(x) = v_1(x_1) + u_2(x_2)$ and assume that $u(.)$ represents P.

Let $x = (0, -0.1)$, $y = (1, 0)$ and $y_k = (1-1/k, 0)$, $k \in \mathbf{N}$. $u(x) = -0.1$, $u(y) = 1$. Hence, yPx, but there is no k such that $y_k Px$, since $u(y_k) < 1/2$ for any k. It is easy to see that P is nevertheless monotone and consistent. //

Now let us proceed from the definitions of section 2.2 to our first main result, Theorem A.

The following lemma shows that our concepts of monotonicity are indeed weaker than strong consistency as implied by Rubinstein (1988) and they allow us to achieve some positive results.

Lemma 4.7. For any representable semi-orders $P_1,\ldots,P_n$ on $X_1,\ldots,X_n$ respectively there exists a consistent, monotone and continuous representable semi-order P.

Proof. Let u_i be such that for all $x_i, y_i \in X_i$, $x_i P_i y_i$ iff $u_i(x_i) > u_i(y_i) + 1$ and $x_i Q_i y_i$ iff $u_i(x_i) > u_i(y_i)$, $i = 1,\ldots,n$.

Define $u(x) = \Sigma_i u_i(x_i)$ and xPy iff $u(x) > u(y) + 1$. Then P is obviously consistent, monotone and continuous.

Note that this P is representable, by construction. //

Lemma 4.8. Let P be a semi-order which is Q-monotone, and representable by u, then there exists $f_u: \mathfrak{R}^n \to \mathfrak{R}$ such that $u = f_u(u_1,\ldots,u_n)$, i.e. $u_i(x_i) = u_i(y_i)$, i= 1,...n, imply $u(x) = u(y)$. Moreover, f_u is unique.

Proof. Suppose that $u(x) > u(y)$, i.e. xQy. Thus, there exists z in X such that

$u(y) \leq u(z) + 1$ and $u(x) > u(z) + 1$;

Note that $y_i Q^o{}_i x_i$, $i = 1,\ldots,n$. Hence, by Q-monotonicity from above, yPz, which contradicts the condition $u(y) \leq u(z) + 1$. //

Now we are in a position to prove Theorem A.

Proof of the Theorem A. Let us first show that (ii) implies (i). Assume, then, that $u = f_u(u_1,\ldots,u_n)$ with f_u as in (ii). Q-monotonicity of P follows from monotonicity of f_u. As for P-monotonicity one should only use (2). P has to be consistent because of (2) and the monotonicity of f_u. Finally, let us show that the continuity of f_u implies that of P. Assume that a sequence $\{x_{ik}\}$ converges to x_i as $k \to \infty$ in the P_i-topology on X_i. Since range$(u_i) = \mathfrak{R}$, this implies that $u_i(x_{ik}) \to u_i(x_i)$ as $k \to \infty$. By continuity of f_u, $u(x_k)$ converges to $u(x)$ which implies that x_k converges to x in the P-topology.

We now wish to show that (i) implies (ii). By Lemma 4.8, we know that

Q-monotonicity of P implies the existence of a unique $f_u\colon \mathbb{R}^n \to \mathbb{R}$ such that $u = f_u(u_1,\ldots,u_n)$. We will now prove it satisfies all requirements.

Strict monotonicity. Let $a_i, b_i \in \mathbb{R}$ satisfy $a_i > b_i$. We will show that for every $c_{-i} \in \mathbb{R}^{n-1}$ $f_u(c_{-i},a_i) > f_u(c_{-i},b_i)$. Since range$(u_i) = \mathbb{R}$, we can find $x,y,z \in X$ such that $u_i(x_i) = a_i$, $u_i(y_i) = b_i$ and $b_i - 1 < u_i(z_i) < a_i - 1$. Similarly, let $w_j \in X_j$, $(j \neq i)$ satisfy $u_i(w_i) = c_i$. Note that xP_iz but $\sim(yP_iz)$. By P-monotonicity, $(w_{-i},x)P(w_{-i},z)$ and, by consistency, $\sim((w_{-i},y)P(w_{-i},z))$ whence $(w_{-i},x)Q(w_{-i},y)$ and $f_u(c_{-i},a_i) > f_u(c_{-i},b_i)$.

Continuity. Assume the contrary, i.e., that f_u is not continuous at some point $a \in \mathbb{R}^n$. Then there is an $\varepsilon > 0$ and a sequence $\{a_k\}$ converging to a as $k \to \infty$ such that $f_u(a_k) > f_u(a) + \varepsilon$ for all k or $f_u(a_k) < f_u(a) + \varepsilon$ for all k. Let us assume the former, i.e., $f_u(a_k) > f_u(a) + \varepsilon$. Find $x_k \in X$ such that $(u_1(x_{1k}),\ldots,u_n(x_{nk})) = a_k$, an $x \in X$ for which $(u_1(x_1),\ldots,u_n(x_n)) = a$ and a $z \in X$ such that $f_u(a) + \varepsilon + 1 > u(z) > f_u(a) + 1$. Since $a_k \to a$ as $k \to \infty$ for each i, $u_i(x_{ik}) \to u_i(x_i)$, which implies that $x_{ik} \to x_k$ as $k \to \infty$ in the P_i-topology on X_i. However, for all k $\sim(zPa_k)$ while zPa, so that x_k does not converge to x in the P-topology. A contradiction to the continuity of P.

Condition (2). Let there be given a vector $a \in \mathbb{R}^n$ and an index $1 \le i \le n$ and consider $f_u(a_{-i},a_i+1)$. By P-monotonicity, $f_u(a_{-i},a_i+1+\varepsilon) > f_u(a) + 1$ for all $\varepsilon > 0$, and consistency implies $f_u(a_{-i},a_i+1-\varepsilon) \le f_u(a) + 1$. Strict monotonicity of f_u also implies that the latter inequality is strict, namely that $f_u(a_{-i},a_i+1-\varepsilon) < f_u(a) + 1$ for all $\varepsilon > 0$. The continuity of f_u implies $f_u(a_{-i},a_i+1) = f_u(a) + 1$. //

Our next objective is to see to what extent semi-orders on product spaces which are consistent, monotone and continuous with respect to given ones on the original spaces are unique. First we want to examine the possibility of reverse preference: could it happen that two distinct semi-orders P_a and P_b satisfying the conditions of Theorem A rank two alternatives in opposite direction, namely xP_ay and yP_bx? We first consider the case $n = 2$:

Lemma 4.9. Let P be a semi-order on X which is consistent, monotone and continuous with respect to P_1 and P_2. And let $x,y \in X$ be such that $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) < u_2(y_2)$, but $\sim(x_1P_1y_1)$ and $\sim(y_2P_2x_2)$. Then $\sim(xPy)$.

Proof. By Theorem A, P may be represented by $u = f_u(u_1,u_2)$. Suppose xPy. Then, by Q-monotonicity, zPy for z in X with $u_1(z_1) = u_1(x_1)$ and $u_2(z_2) = u_2(y_2)$. But, by consistency, $\sim(zPy)$, a contradiction. //

Lemma 4.10 generalizes the previous result for $n > 2$, but falls short of excluding preference reversal. Note that while the statement of Lemma 4.9 is symmetric with respect to x and y, this is not true of Lemma 4.10, where x and y play different roles.

Lemma 4.10. Let $\{P_i\}_{i\in N}$ be representable semi-orders on $\{X_i\}_{i\in N}$ and let P be a representable semi-order on X which is consistent, monotone and continuous with respect to $P_1,\ldots,P_n$, $n > 2$. Suppose that x,y in X are such that for any $i\neq j$ $u_i(x_i) \geq u_i(y_i)$ and $u_j(x_j) < u_j(y_j)$, but for every i, $x_iI_iy_i$. Then $\sim(yPx)$.

The proof is similar to the proof of Lemma 4.9.

Corollary 4.11. Suppose that P_a and P_b are two semi-orders on X which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$, $n < 4$. Then for any $x, y \in X$ $xP_a y$ implies $\sim(yP_b x)$.

The proof follows from monotonicity and Lemma 4.10.

However prefernce reversal is possible for large enough n:

Observation 4.12. Suppose that P_a and P_b are two semi-orders on X which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$, $n > 3$. Then there may be x,y in X such that $xP_a y$ and $yP_b x$.

Proof. Consider the following example. Let $n = 4$, $X_i = \Re$, $u_i(x_i) = x_i$, $i = 1, \ldots, 4$. Define $u_a(x) = \Sigma u_i(x_i)$. As usual suppose that all semi-orders are induced by corresponding utility functions.

Choose another representation of $P_1, \ldots, P_4$. Let

$$v_i(x_i) = \{ (k-x_i)/100 + k, \text{ if } x_i \in [k, k+0.1];$$
$$19.96(k-x_i) + k - 1.995, \text{ if } x_i \in]0.1+k, 0.15+k[;$$
$$(k-x_i)/850 + k + 849/850, \text{ if } x_i \in [0.15+k, k+1[\},$$

where k is an integer, $i = 1, \ldots, 4$.

Define $u_b(x) = \Sigma v_i(x_i)$.

By Theorem A, both P_a and P_b are consistent, monotone and continuous with respect to $P_1, \ldots, P_4$. Let $x = (0.1, 0.1, 0.95, 0.95)$ and $y = (0.2, 0.2, 0.3, 0.3)$. Then $u_a(x) = 2.1$, $u_a(y) = 1$. Hence, $xP_a y$. But $u_b(x) < 0.002 + 2 = 2.002$ and $u_b(y) > 4 * 0.999 = 3.996$. Hence, $yP_b x$. //

Hence, we see that "preference reversal" in the sense of xP_ay but yP_bx is possible. However, $x(P_a)^ny$ and $y(P_b)^nx$ is impossible. In fact, our Theorem B shows that for any such P_a and P_b, $(P_a)^n \subseteq P_b$ and $(P_b)^n \subseteq P_a$. Let us turn to its proof.

Proof of Theorem B. Suppose $y(P_a)^nx$. Let x' be an alternative in X such that $u_i(x'_i) = u_i(x_i) + 1$ for $1 \leq i \leq n$, so that $f_u(u_1(x'_1),\ldots,u_n(x'_n)) = f_u(u_1(x_1),\ldots,u_n(x_n))$ for every f_u that corresponds to a representable semi-order P on X which is consistent, continuous and monotone w.r.t $P_1,\ldots,P_n$.

By Theorem A, yE_az and yE_bz for any $z \in X$ with $u_i(z_i) = u_i(y_i) + m_i$, where $m_i \in \mathbf{Z}$ and $\Sigma m_i = 0$. For each $z \in X$ define $d^z = (d_1^z,\ldots,d_n^z) \in \Re^n$ by $d_i^z = u_i(z) - u_i(x')$. One can find a z with the following properties:

(i) yE_az and yE_bz

(ii) (a) $d_i^z \geq 0$ for all $1 \leq i \leq n$

or (b) $d_i^z \leq 0$ for all $1 \leq i \leq n$ (but not (a))

or (c) $|d_i^z| < 1$ for all $1 \leq i \leq n$ (but not (a) nor (b))

Such a z would be, for instance, one minimizing $\Sigma|d_i^z|$ over the set $\{ z \mid u_i(z) - u_i(x) = m_i$, where $m_i \in \mathbf{Z}$ and $\Sigma m_i = 0\}$.

In case (ii)(a) we have $u_i(z) \geq u_i(x')$ whence $z_iQ^o_ix'_i$ and, by monotonicity $zQ^o_bx'(P_b)^nx$, which implies $z(P_b)^nx$ and $y(P_b)^nx$.

In case (ii)(b) $x'_iQ^o_iz_i$ whence $x'Q_az$ and $x'Q_ay$. However, $f_u(u_1(y_1),\ldots,u_n(y_n)) \geq f_u(u_1(x_1),\ldots,u_n(x_n)) + n$ whence we also get yE_ax' and zE_ax'. But this is possible only if $d_i^z = 0$ for all i which boils down to (ii)(a).

Finally, consider case (ii)(c). Since $d_i^z > -1$ we know that

$u_i(z) > u_i(x)$ for all i. However, we also know that for some i $d_i^z > 0$ which means that $u_i(z_i) > u_i(x_i) + 1$. Monotonicity of P_b means zP_bx whence yP_bx also holds. //

Corollary 4.13. Let $\{P_i\}$ on $\{X_i\}$ be representable semi-orders. Suppose that P_a and P_b are two representable semi-orders on X which are both consistent, monotone and continuous with respect to $P_1,\ldots,P_n$. Then $I_a \subseteq (I_b)^n$ and $I_b \subseteq (I_a)^n$.

The proof follows from Theorem B and the fact that for representable semi-orders $I(P^n)$ induced by P^n coincides with I^n, where I is induced by P.

Let us turn to the proof of Theorem C. For two given representable semi-orders P and P′ on the same space X define:

$$A(P,P') = \{ k/m : k,m \in \mathbf{N},\ P^k \subseteq (P')^m \}.$$

Lemma 4.14. Suppose P and P′ are representable semi-orders on a space X satisfying A1 and A2. Then A(P,P′) is homogeneous, i.e., for every $t \in \mathbf{N}$ $P^k \subseteq (P')^m$ iff $P^{tk} \subseteq (P')^{tm}$.

Proof. Throughout the proof let u and u′ represent P and P′ respectively.

<u>"Only if" part.</u>

$P^k \subseteq (P')^m$ means that for any x,y in X $u(x) - u(y) > k$ implies $u'(x) - u'(y) > m$. If $x^1 P^{tk} x^{t+1}$ then there exists a sequence $(x^1,\ldots,x^t)$ in X such that $x^i P^k x^{i+1}$ for all $i = 1,\ldots,t$. Then $u(x^i) - u(x^{i+1}) > k$ for all i =

1,...,t. This, in turn, implies $u'(x^i) - u'(x^{i+1}) > m$ for all $i = 1,\dots,t$, or $x^1(P')^{tm}x^{t+1}$.

"If" part.

Assume, then, that $P^{tk} \subseteq (P')^{tm}$ for some $t > 1$. Since A2 holds there are two possible cases: 1) $P^k \subseteq (P')^m$, in which the proof is complete, or 2) $(P')^m \subset P^k$. In this case, by the if part, $(P')^{tm} \subseteq P^{tk}$. But, by assumption, $P^{tk} \subseteq (P')^{tm}$. Thus, $P^{tk} = (P')^{tm}$.

To show that $P^k \subseteq (P')^m$, let there be given $x,y \in X$ with xP^ky; we will show that $x(P')^my$ has to hold. Suppose not. Then $u(x) - u(y) > k$ but $u'(x) - u'(y) \le m$. Choose a sequence $y_o,\dots,y_t \in X$ with $y_0 = x$ and $u(y_i) - u(y^{i+1}) = k$ for $0 \le i \le t-1$. This is possible since range(u) = $\mathfrak{R}$. Note that $u'(x) - u'(y_1) < m$ since y_1Qy and this is equivalent to $y_1Q'y$.

By our construction, $\sim(y^iP^ky^{i+1})$ for $0 \le i \le t-1$ which implies, since $(P')^m \subseteq P^k$, that $\sim(y^i(P')^my^{i+1})$. However, for every z satisfying y_tQz we get -- again, using the fact that range(u) = $\mathfrak{R}$,-- $xP^{tk}z$. The latter means that $x(P')^{tm}z$.

Considering the u' scale, we obtain $u'(x) - u'(z) > tm$ for every z satisfying y_tQz (equivalently, $y_tQ'z$). Hence, $u'(x) - u'(y_t) \ge tm$. On the other hand, $\sim(y^i(P')^my^{i+1})$ for $0 \le i \le t-1$ implies $u'(y_i) - u'(y^{i+1}) \le m$ whence $u'(x) - u'(y_t) \le tm$. Combining the inequalities one obtains $u'(y_i) - u'(y_t) = m$ in contradiction to the choice of y_i. //

Lemma 4.15. If two semi-orders P and P' satisfy A2, then Q = Q'.

Proof. Let k and m satisfy $(P)^k = (P')^m$. $(P)^k$ is a semi-order on X and so is

$(P')^m$. Since they are identical, the weak orders $(Q)^k$ and $(Q')^m$ are also identical. However, $(Q)^k = Q$ and $(Q')^m = Q'$. //

We now proceed to our third main result, Theorem C.

Proof of Theorem C. (i) => (ii)

Let us begin with A2. Consider P_i, P_j and $k,m \in \mathbf{N}$. By (i), $x(P_i)^k y$ iff $u(x) - u(y) > k\delta_i$, whence $(P_i)^k \subseteq (P_j)^m$ iff $k\delta_i \le m\delta_j$. Hence, $(P_i)^k \subseteq (P_j)^m$ or $(P_j)^m \subseteq (P_i)^k$. Since $\{\delta_i\}$ are rational equality would hold for some $k,m \in \mathbf{N}$.

As for A1, note that $x(P_iP_j)y$ iff $u(x) - u(y) > \delta_i + \delta_j$, which means that $P_iP_j = P_jP_i$.

(ii) => (i)

Let us first introduce some additional definitions. For a semi-order P on X, let P^* be the binary relation defined as follows: xP^*y iff xIy and for every z satisfying zQx we have zPy. Intuitively, xP^*y means that x is the "supremum" of { w | wIy }. By the usual concatenation of binary relations $(P^*)^k$ is well-defined for $k \ge 1$. Let us also define $(P^*)^0$ to be E (which corresponds to equal u-values) and $(P^*)^{-k}$ for $k \ge 1$ as the inverse of $(P^*)^k$. Similarly, we will refer to the expressions of the type $(P^*_{i1})^{k_1} \ldots (P^*_{is})^{k_s}$, where $i_r \in \{ 1,\ldots,n \}$ and $k_r \in \mathbf{Z}$ for $1 \le r \le s$.

The proof will be simpler to carry out by induction on n. Let us begin with n = 2.

Choose any point x_0 in X and set $u(x_0) = 0$. By A2, there are $m,t \in \mathbf{N}$ such that $(P_1)^m = (P_2)^t$. Assume without loss of generality that g.c.d.(m,t) = 1, where g.c.d. stands for the greatest common divider. This assumption can be made thanks to Lemma 4.14. Define $M = t * m$. We will construct a function u such that

$$u(x) - u(y) > \delta_1 = t \text{ iff } xP_1y \text{ and} \qquad (3)$$

$$u(x) - u(y) > \delta_2 = m \text{ iff } xP_2y$$

For every integer k let us define $V(k) = \{ y \in X \mid$ there exist sequences $k_1,\ldots,k_s$ and $i_1,\ldots,i_s$ such that $y(P^*_{i1})^{k_1}\ldots(P^*_{is})^{k_s}x$ and $\Sigma k_r\delta_{ir} = k \}$. Intuitively, V(k) is the set of all y's for which we have to assign the value $u(y) = k$. Note that $V(k) \neq \emptyset$ for every $k \in \mathbf{Z}$.

Claim 1. For every k and every $y,z \in V(k)$ it is true that yEz.

Proof. For $k \in \mathbf{Z}$ there are unique a_k, b_k, c_k such that $k = a_kM + b_kt + c_km$ with $a_k \in \mathbf{Z}$, $0 \leq b_k \leq m$ and $0 \leq c_k \leq t$. Note that A1 and A2 imply that $(P^*_1)^m = (P^*_2)^t$ and that $P^*_1P^*_2 = P^*_1P^*_2$. Hence, every $y \in V(x)$ satisfies $y(P^*_1)^{ma_k+b_k}\ldots(P^*_2)^{c_k}x$, which implies the desired conclusion. //

Claim 2. Suppose $y \in V(k)$ and $z \in V(g)$ with $k > g$. Then yQz.

Proof. Since u_i represents P_i $(i = 1, 2)$, for every $w_1,w_2,t_1,t_2 \in X$, if $w_1P^*_iw_2$ and $t_1P^*_it_2$ $(i = 1,2)$ then w_1Qt_1 iff w_2Qt_2. Using this argument inductively, for every $w_1,w_2,t_1,t_2 \in X$, and every $k,h \in \mathbf{Z}$, if $w_1(P^*_1)^k(P^*_2)^hw_2$ and $t_1(P^*_1)^k(P^*_2)^ht_2$ then w_1Qt_1 iff w_2Qt_2.

Consider $g = k - 1$. There are k and h such that $kt + hm = -1$. For $y \in V(k)$ and $z \in V(g)$ choose $w \in V(g-1)$. Then $y(P^*_1)^k(P^*_2)^hz$ and $z(P^*_1)^k(P^*_2)^hw$. Hence, yQz iff zQw. It turns out that one of the following is true:

(i) for every k,g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies yQz.

(ii) for every k,g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies zQy.

(iii) for every k,g, $y \in V(k)$ and $z \in V(g)$ with $k > g$ implies yEz.

One only needs to know that for $k = t$ and $g = 0$ $y \in V(k)$ and $z \in V(0)$ satisfy yQz to conclude that (i) is the case. //

At this point one can define u on $\cup_{k \in \mathbf{Z}} V(k)$ by $u(y) = k$ for $y \in V(k)$. It is obvious that u satisfies (3) for $x,y \in \cup_{k \in \mathbf{Z}} V(k)$.

Next, choose $x_1 \in V(1)$. Denote $I = \{ x \mid x_1 Q x Q x_0 \}$. For every x in I define $u(x) = u_1(x) / u_1(x_1)$. For every $k \in \mathbf{Z}$ define a set $V(x,k)$ as $V(k)$ was defined for $x = x_0$. Note that for $y \in V(x,k)$ and $z \in V(k)$, $w \in V(k+1)$ we have wQyQz. Furthermore, for every $y \in X$ there are $x \in I$ and $k \in \mathbf{Z}$ such that $y \in V(x,k)$. Hence, we define $u(y) = k + u(x)$.

It is easy to see that for every $x,y \in X$ and $i = 1,2$ $xP^*_i y$ iff $u(x) - u(y) = \delta_i$.

We now turn to the induction step. Suppose $n > 2$. We already know that for $P_1,\dots,P_n$ there is a function u and positive rational numbers $\delta_1,\dots,\delta_n$ such that $xP_i y$ iff $u(x) - u(y) = \delta_i$ and xQy iff $u(x) > u(y)$. Without loss of generality assume that $\delta_i \in \mathbf{N}$. Define P' by xP'y iff $u(x) - u(y) > 1$, so that $P_i = (P')^{\delta_i}$ for $1 \le i \le n-1$. Let u', δ' and δ'_u represent P and P_u, namely,

$$u'(x) - u'(y) > \delta' \text{ iff } xP'y;$$

$$u'(x) - u'(y) > \delta_u \text{ iff } xP_u y;$$

$$\text{and } u'(x) - u'(y) > 0 \text{ iff } xQy,$$

for every $x,y \in X$ (The existence of those is guaranteed by the proof for the case $n = 2$.). Furthermore, δ' and δ'_u may be assumed to be integer w.l.o.g. Hence, u' also satisfies $u'(x) - u'(y) > \delta'_i = \delta_i * \delta'$ iff $xP_i y$ for $1 \le i \le n$. this completes the proof of the theorem. //

Next we note that axioms A1 and A2 are independent.

Observation 4.16. A2 does not imply A1.

Proof. Consider the following example: $n = 2$, $X = \mathbb{R}$, $u_1(x) = x$,

$$u_2(x) = \{ \begin{array}{ll} 2k + 5/8(x - 3k), & 3k \leq x < 3k+2, \quad k \in \mathbf{Z}; \\ 2k + 5/4 + 3/4(x - 3k - 2), & 3k+2 \leq x < 3k+3, \ k \in \mathbf{Z}. \} \end{array}$$

Define $xP_1y \iff u_1(x) - u_1(y) > 1$ and $xP_2y \iff u_2(x) - u_2(y) > 1$. P_1 and P_2 satisfy A2: $(P_1)^3 = (P_2)^2$ and for every $k,l \in \mathbf{N}$ $k \geq (3/2)l$ implies $(P_1)^k \subseteq (P_2)^l$. However, to see that A1 fails to hold take $x = 1$, $z = 3.45$. zP_2P_1x but $\sim(zP_1P_2x)$. //

Observation 4.17. A1 does not imply A2.

Proof. Again consider an example with $n = 2$, $X = \mathbb{R}$, $u_1(x) = x$. For $0 \leq x \leq 2$ define

$$u_2(x) = \{ \begin{array}{ll} (1/2)x, & 0 \leq x < 0.1; \\ 0.05 + 2(x - 0.1), & 0.1 \leq x < 0.2; \\ 0.25 + (x - 0.2)/2, & 0.2 \leq x < 0.3; \\ x, & 0.3 \leq x \leq 2. \quad \} \end{array}$$

Extend u_2 to $\mathbb{R}$ in the unique way that will satisfy

$$u_2(x + 1) + 1 = u_2(u^{-1}{}_2[u^{-1}{}_2(x) + 1] + 1)$$

(it is easy to see that there exists a unique continuous and strongly monotone u_2 which satisfies this condition).

Finally, define P_1 and P_2 by u_1 and u_2 respectively with a just-noticeable difference of 1.

By definition, $P_1P_2 = P_2P_1$. However, A2 does not hold: for $x = 0.1$ and $y = 1.12$ we have yP_1x but $\tilde{}(yP_2x)$ while for $z = 0.2$ and $w = 0.76$ zP_2w holds while zP_1w <u>does not</u>. //

FOOTNOTES

1. Interval order is an irreflexive binary relation R such that xRy and zRw imply either xRw or zRy. It is easy to see that any semi-order is an interval order but not vice versa. Viewing semi-orders as particular interval orders we may suggest another interpretation of semi-orders.

Suppose that X is a set of signals about real quality of alternatives (like test score signals about students' knowledge). When decision maker observes x he/she does not know for sure that the true quality is x, but he/she may have in mind for every x a range of qualities that can generate the signal x. Suppose further that a decision maker prefers x to y if and only if he prefers any true quality that can generate x to any true quality that can produce y. Then if ranges above are the same for all x in X such a model induces a semi-order on X.

For instance, in many cases one may assume a fixed error rate of measured quantity (with respect to the true one.) On a logarithmic scale we thus get a fixed range length.

Furthermore, in Lapson, Lugachev (1983) there are several sectors to each of which there corresponds an "error rate" induced by applied technology. Thus, it may serve as an example of "correlated semi-orders".

In these examples semi-ordered structures arise not as a result of psychological peculiarities but rather as a result of imprecise measurement or lack of information.

2. This observation is due to David Schmeidler.

REFERENCES

Aizpurua, J.M., Nieto, J and Uriarte, J.R. (1988), "Choice Procedure under Risk Consistent with Similarity Relations," Southern European Economics Discussion Series, 61.

Aizpurua, J.M., Ichiishi, T., Nieto, J and Uriarte, J.R. (1989), "Decision-making under Risk: Non-Transitive Preferences and Correlated Similarities," Manuscript.

Beja, A. and Gilboa, I. (1989), "Numerical Representation of Imperfectly Ordered Preferences: A Unified Geometric Exposition," Manuscript.

Bridges, D.S. (1983), "A Numerical Representation of Preferences with Intransitive Indifference," Journal of Mathematical Economics, 11, 25 - 42.

Chateauneuf, A. (1987), "Continuous Representation of a Preference Relation on a Connected Topological Space," Journal of Mathematical Economics, 16, 139 - 146.

Fechner, G.T. (1860), Elements of Psychophysics, (H.E. Adler, trans.), Holt, Rinehart & Winston, New York, Reprinted 1966.

Fishburn, P.C. (1970), "Intransitive Indifference in Preference Theory: A Survey," Operations Research, 18, 207 - 228.

Fishburn, P.C. (1985), Interval Orders and Interval Graphs, Wiley, New York.

Gensemer, S. (1987), "Continuous Semiorder Representation," Journal of Mathematical Economics, 16, 275 - 290.

Gilboa, I. (1987), "Expected Utility with Purely Subjective Non-Additive Probabilities," Journal of Mathematical Economics, 16, 65 - 88.

Goodman, L.A. and Markowitz, H. (1952), "Social Welfare Functions Based on Individual Rankings," American Journal Sociology, 58, 257 - 262.

Harsanyi, J.C. (1955), "Cardinal Welfare, Individualistic Ethics and Interpersonal Comparison of Utility," Journal of Political Economy, 63, 309 - 321.

Kahneman, D. and Tversky, A. (1979), "Prospect Theory: An analysis of Decision under Risk,", Econometrica, 47, 263 - 292.

Lapson, R. and Lugachev, M.I. (1983) "One Approach towards Grains' Productivity Forecast," II All-Union Conference on System Analysis of Socio-Economic Processes, 201 - 202, (In Russian).

Luce, R.D. (1956), "Semiorders and a Theory of Utility Discrimination," Econometrica, 24, 178 - 191.

Luce, R.D. and Raiffa, H. (1957), Games and Decisions: Introduction and Critical Survey, Wiley, New York.

Manders, K.L. (1981), "On JND Representation of Semiorders," Journal of Mathematical Psychology, 24, 224 - 248.

von Neumann, J and Morgenstern, O. (1947), Theory of Games and Economic Behavior, Princeton.

Ng, Y.K. (1975), "Bentham or Bergson? Finite Sensibility, Utility Functions and Social Welfare Functions," Review of Economic Studies, 42, 545 - 569.

Quiggin, J. (1982), "A Theory of Anticipated Utility," Journal of Economic Behavior and Organization, 3, 223 - 243.

Rubinstein, A. (1988), "Similarity and Decision-making under Risk: Is There a Utility Theory Resolution to the Allais Paradox?," Journal of Economic Theory, 49, 145 - 153.

Savage, L.J. (1954), The Foundations of Statistics, Wiley, New York.

Scott, D. and Suppes, P. (1958), "Foundational Aspects of Theories of Measurement," Journal of Symbolic Logic, 23, 113 - 128.

Schmeidler, D. (1986), "Integral Representation without Additivity," Proceedings of the American Mathematical Society, 97.

Weber, E.H. (1834), Concerning Touch, (Reprinted 1978, H.E. Ross, trans.), Academic Press, New York.

Yaary, M.E. (1987), "Risk Aversion without Diminishing Marginal Utility and the Dual Theory of Choice under Risk," Econometrica, 55, 95 - 116.