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AGGREGATION OF SEMIORDERS:
INTRANSITIVE INDIFFERENCE MAKES A DIFFERENCE
by
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ABSTRACT

A semi-order can be thought of as a binary relation $P$ for which there is a utility $u$ representing it in the following sense: $xPy$ iff $u(x) - u(y) > 1$.

Weak orders (for which indifference is transitive) can not be considered a successful approximation of semi-orders; for instance, a utility function representing a semi-order in the manner mentioned above is almost unique, i.e. cardinal and not only ordinal.

In this paper we deal with semi-orders on a product space and their relation to given semi-orders on the original spaces. Following the intuition of Rubinstein we find surprising results: with the appropriate framework, it turns out that a Savage-type expected utility requires significantly weaker axioms than it does in the context of weak orders.

Moreover, our axioms provide a conceptual basis for the weighted average paradigm in general, and, in particular, may be used to justify utilitarianism in a social choice context.
1. INTRODUCTION

The bulk of economic theory literature assumes that the decision maker's preference relation may be represented by a utility function \( u(.) \) as follows: \( xPy \iff u(x) \cdot u(y) > 0 \).

All of the axioms \( P \) should satisfy in order to allow for such a representation, as well as the very existence of a binary preference relation, have been criticized over the years on theoretical and empirical grounds alike. Yet, it seems that most of the axioms, including the transitivity of \( P \), are still widely acceptable.

One exception is the transitivity of indifference: a representation as above, trying to retain this property, leads us to contend that the decision maker strictly prefers \( x \) to \( y \) whenever \( u(x) \) is \( \ldots \) even slightly \( \ldots \) larger than \( u(y) \). Most theorists would probably not like to take this axiom literally; it does not seem very plausible that an economic agent strictly prefers an annual income of \( \$30,000 \) to \( \$30,000\) or 2,533 grains of sugar in a cup of coffee to 2,533 grains etc. On the other hand, indifference between very close alternatives would lead, in the presence of transitive indifference, to indifference between any two alternatives, which is absurd.

Moreover, psychology suggests the Weber-Fechner Law (Weber (1834), Fechner (1860)) which says, roughly, that human beings cannot discern between very close objects, and only when the difference in mass, temperature, length, pressure, etc. exceeds a certain "just noticeable difference" does a distinction emerge in people's minds. Weber's Law also says, that for each scale -- mass, for instance, -- there is a certain
constant $\lambda > 1$ (depending on the individual and possibly also on the circumstances) such that the individual would discern the difference between two magnitudes only if their ratio exceeds $\lambda$ (or drops below $1/\lambda$). On the appropriate logarithmic scale, we therefore get the following representation:

$$xPy \iff u(x) \cdot u(y) > 1.$$  \hspace{1cm} (1)

Semi-orders, defined by Luce (1956), are transitive binary relations the indifference of which may not be transitive. For the sake of the discussion one may take the representation (1) as a definition of a semi-order although it is equivalent to Luce's definition only if the set of alternatives is finite. For discussion of representations of semi-orders -- and the more general class of interval orders -- see Fishburn (1970, 1985), Maders (1981), Gensemer (1987), Bridges (1983), Chateauneuf (1987) and Beja and Gilboa (1989).\(^1\)

From the viewpoint of economic theory it is quite reasonable to argue, then, that in 'reality' people have semi-orders (or even less structured preferences.) Were indifference is intransitive, but weak orders can still be taken as an approximation, or a mathematical idealization, of reality; an idealization that simplifies matters just like the continuum is a convenient representation of very large, though discrete, sets.

The main point of this paper is that this idealization is far from being innocuous. Surprisingly enough, we have good news: some aspects of economic theory seem to be conceptually simpler with semi-orders than with weak orders. Although the weak order idealization simplifies mathematics, it requires considerably stronger axioms (from a conceptual viewpoint) to derive such results as the expected utility paradigm.
Let us take as an example the following point, which was noted in Beja and Gilboa (1989). Suppose $P$ is a semi-order with a representation (1) by $u$. Then the function $u$ is almost unique. Should $v$ represent $P$ as well, there will be an $f$ such that $v = f(u')$, but $f$ cannot be any monotone transformation. It can be defined arbitrarily (as long as monotonicity is preserved) on $[0, 1]$, and then one has to extend it in such a way that $f(x + n) = f(x) + n$, for all $x \in [0, 1]$ and $n \in \mathbb{Z}$.

If we think of the just noticeable difference -- normalized to one -- as relatively small, the utility $u$ is almost unique. It thus makes sense to discuss properties such as concavity or convexity of the utility function, properties that the mathematical idealization, namely weak orders, rendered devoid of meaning.

In fact, there is more information in a semi-order than in a weak order. Indeed, every semi-order $P$ induces a weak order of indirectly revealed preferences -- which we will denote by $Q$ -- represented by the same utility function $u$ in the regular sense, i.e., $xQy \iff u(x) - u(y) > 0$. However, the semi-order $P$ cannot be reconstructed from its associated $Q$.

On second thought, this result is not surprising at all: a semi-order implicitly provides not only rankings of the alternatives but also rankings of differences between pairs of alternatives, which is the essence of a cardinal utility functions. The point is that the ranking of differences implied by the classification to "larger than the just noticeable difference (jnd)" and "not larger than the jnd" is naturally given in the original preferences, and it suffices for fixing $u$ almost uniquely, without the additional artificial assumption that the decision maker can answer questions like "do you prefer $x$ to $y$ more than you prefer $w$ to $z$?"
In a similar way, this paper shows that when semi-orders are the
primitive preferences, one needs relatively weak axioms to derive expected
utility (or weighted average) representation for preference relation over a
product space of given spaces -- for instance, the space of acts which are
functions from the set of states of nature to consequences. But before we
explain these results we have to digress and describe Rubinstein's approach.

In Rubinstein (1988) preference orders over simple lotteries are
discussed. A lottery is "simple" if it promises a certain monetary prize $x$
(say $x \in [0,1]$) with probability $p$ and zero with probability $(1 - p)$. (For
some $p \in [0,1]$.)

Rubinstein assumes that there are two "similarity" relations, $\sim_x$ and
$\sim_p$, defined on the set of monetary prizes and the set of probabilities,
respectively, with the interpretation that two "similar" magnitudes (prizes
or probabilities) are indistinguishable in the decision maker's mind. He
then considers weak orders on lotteries (which are simply the product space
of prizes and probabilities) and defines the "star" ($\ast$) property as follow:
A weak order satisfies the ($\ast$) property with respect to ($\sim_x$, $\sim_p$) if
whenever two lotteries are similar in one component but not similar in
another -- the other component determines the preference between the two
lotteries. Rubinstein proves that for given similarity relations there is an
almost unique weak order on lotteries satisfying the ($\ast$) property. Conten-
ding that this property is a basic feature of any reasonable decision
process people actually go through while making decisions, he concludes that
there is a basic flaw in axiomatic theories justifying expected utility
maximization (and its generalizations), because this decision rule may well
be inconsistent with the ($\ast$) property.
Rubinstein's similarity relations are, in fact, the indifference relations induced by some semi-orders. Our original intuition was that his results hinge on the fact that in the lottery space a weak order is assumed to be given, while on the more primitive spaces -- only semi-orders. In this case his conclusions seem to be somewhat dubious since it seems unreasonable that a decision maker who cannot discern small differences between monetary prizes as such will have perfect distinction power regarding the more complicated space of lotteries.

This intuition seems to have been shared by Aizpurua, Nieto, and Uriarte (1988) and Aizpurua, Ichishi, Nieto and Uriarte (1989). They allowed the preference order on lotteries to have intransitive indifference but found that Rubinstein's results are robust with respect to this generalization. To cope with the "over-determination problem" they introduced "correlated similarities" -- allowing for the similarity relation on the probability space to depend upon the associated monetary prize. Thus, they found that expected utility maximization was not inconsistent with the (*) property, though this result (as well as Rubinstein's) could not be extended to lotteries involving more than two possible prizes.

Our approach is slightly different. We require the binary relation on the product space to be a semi-order (which is a stronger requirement than theirs), and we found that the (*) property -- which we call strong monotonicity -- is too strong a requirement, and there typically will not be any semi-orders on the product space which are strongly monotone with given ones on the original spaces (this is true even for two such spaces.)

We therefore define monotonicity in a weaker form: given two alternatives in the product space \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), if
there is a preference in one component \((x_i, y_i)\) and indirectly revealed preference in all others \((x_j, y_j)\) then we require that \(x\) will be preferred to \(y\). Lack of preference of \(y_j\) over \(x_j\) does not suffice to use this monotonicity axiom. Thus, our monotonicity is weaker than the \((*) \) property, and it can be thought of as a "stochastic dominance" or "pareto dominance" axiom.

Weakening the \((*) \) property allows for the existence of monotone semi-orders on the product space, but it also allows for the possibility of the decision maker being able to discern between alternatives which were indistinguishable in the original spaces. Thus we introduce a consistency requirement stating that if all other components are held fixed, and in one component we consider two indistinguishable alternatives \(x_i\) and \(y_i\), then the whole vector with \(x_i\) is indistinguishable from that with \(y_i\).

We proved that -- under some mild technical assumptions -- there is an almost unique semi-order \(P\) on \(X_1 \times \cdots \times X_n\) which is monotone and consistent w.r.t. given semi-orders \(P_i\) on \(X_i\), and that each such \(P\) has an almost unique representation. Moreover, if \(u_i\) represents \(P_i\) with a jnd of 1, one of the monotone and consistent semi-orders will be represented by \(\Delta u_i\), namely

\[
xPy \text{ iff } E u_i(x_i) - E u_i(y_i) > 1.
\]

Thus, if all \(X_i\) and \(P_i\) are the same -- the set of consequences and a semi-order on \(i\) -- we obtain an expected utility representation of a semi-order on actions, namely functions from the states of nature to \(X\). Moreover, we get equal probabilities to all states of nature, i.e., the Laplace criterion is justified with monotonicity and consistency alone. Comparing these axioms to the main axioms of Savage (1954), the consistency is a very weak version of the "sure thing principle" (in fact, it corresponds to axiom P2 restricted to singleton events), while monotonicity corresponds to axiom
P1 (which is sometimes also considered as a part of the "sure thing
principle"). One need not require the full strength of P2, nor axiom P4
regarding comparability of events, and yet one obtains a stronger result,
namely that all states of nature are equiprobable.

Before carrying on to the axiomatization of the general expected
utility representation, we should comment on alternative interpretaciones.
For instance, if we replace the states of nature by individuals (with
possibly different X₁'s and P₁'s), our monotonicity and consistency
axiomatize utilitarianism (see Marsányi (1955)). Similar justifications of
utilitarianism were obtained by Goodman and Markowitz (1952) and Ng (1977).
Both models are in quite different frameworks from ours, and we find them
considerably less fundamental and intuitive. In particular, they both use
axioms involving "counting" which, to a certain extent, presupposes the
desired result.

Luce and Raiffa (1957) raise doubts regarding the appropriateness of
this criterion: in fact we resolve here the classical problem of inter-
personal comparison of utility by setting the end of all individuals to be a
standard unit of measurement. This means that more sensitive people will get
a higher weight in the social welfare function than less sensitive ones. The
question of whether this is just (or "just noticeable") is beyond the scope of
this paper.

An alternative interpretation will be to consider each space X₁ as a
product space of probabilities and prizes, namely of simple lotteries as in
Rubinstein (1988). Regardless of the preference relation one has on X₁, be
it a non-expected utility one as Rubinstein suggests, or an expected utility
as in Alspuru et. al. (1989) our results allow a conceptually consistent
extension of the semi-order to general lotteries. If preferences on $X_1$ are represented by $p_1u(x_1)$ we obtain $\Delta p_1u(x_1)$ as a representing functional (i.e., expected utility as in von Neumann and Morgenstern (1947)). Other preferences such as $g(p_1)u(x_1)$ would give rise to prospect theory (Kahneman and Tversky (1979)) etc. At any rate, Rubinstein-type arguments imply the additive structure which is at the heart of the expected utility representation.

It is somewhat ironic that Kahneman-Tversky prospect theory was strongly criticized for not satisfying (first order) stochastic dominance, while here we find that a different formulation of the same axiom -- our monotonicity -- prefers prospect theory over theories such as Quiggin's (1982), Yaari's (1987) or non-additive probabilities (Schmeidler (1984), Gilboa(1987)).

However, the Laplace criterion is quite restrictive and may point to a flaw in our assumptions: indeed, assuming all $X_1$'s are the same is intrinsic to the problem of decision making under uncertainty, but the assumption that all $P_1$'s are also identical may be too strong. Instead, we could assume that there is another type of "correlated similarities": the decision maker's ability to distinguish between alternatives depends on the likelihood of the associated state of nature. (In a way, this is a complementary approach to the correlated similarities of Altpeter et al. (1989).)

We are therefore interested in the following question: given several semi-orders $P_1$ on the same space $X$, when is there a single utility function $u: X \to \mathbb{R}$ and a constant $\delta > 0$ corresponding to each $P_1$, such that

$$x \succsim_1 y \iff u(x) - u(y) > \delta$$
In this paper we restrict our attention to the case of all $\delta_i$ being rational, for which we provide a complete axiomatization. Using this result and the previous ones we obtain an axiomatization of a semi-order $P$ on $X^m$ represented by

$$xPy \iff \sum_{i=1}^m u(x_i) - \sum_{i=1}^m u(y_i) > 1$$

for rational probabilities $r_i$.

To sum up, this paper studies semi-orders on product spaces in general, and on a product of identical spaces in particular. Using axioms motivated by Rubinstein (1988), we provide a conceptual basis for additive separability in the general context and for expected utility in the more specific one. Most importantly, this study shows that with aggregation of preferences, as in the case of the numerical representations of preferences on a single space, a lot of information is lost when we choose to work with weak orders rather than with the more realistic semi-orders.

The rest of this paper is organized as follows. Section 2 presents preliminary definitions and quotes some results from the literature. Our main results are stated in section 3. Finally, the proofs and related analysis are to be found in section 4.
2. PRELIMINARIES AND BASIC DEFINITIONS

The central issue of this paper is semi-orders. The formal definition is the following:

A binary relation $P$ on $X$ is a semi-order if for any $x, y, z, w$ in $X$
1) is not $xPz$ ($P$ is irreflexive);
2) if $xPy$ and $zPw$ then $xPw$ or $zPy$;
3) if $xPy$ and $yPz$ then $xPw$ or $wPz$.

For a given semi-order $P$ define binary relations $I$, $Q$, $E$ and $Q^0$ as follows: for every $x, y$ in $X$:

$xPy$ iff not $xPz$ and not $yPz$;
$xQy$ iff $\exists z$ in $X$ such that either 1) $xPz$ and not $yPz$
or 2) $zPy$ and not $zPz$;

$xEy$ iff not $xQy$ and not $yQx$;

$xQ^0y$ iff $wQy$ or $xEy$.

Any superscripts, subscripts etc. of $P$ will be carried over to its associated $I$, $Q$, $E$ and $Q^0$.

$Q$ being such defined is a weak order, i.e., satisfies the set of conditions below: for any $x, y, z$ in $X$

1) not $xQx$ ($Q$ is irreflexive);
2) if $xQy$ and $yQz$ then $xQz$ ($Q$ is transitive);
3) if $xQy$ then $xQz$ or $zQy$.

Scott and Suppes (1958) proved that if $X$ is finite, then there exists a utility function on $X$ such that

for any $x, y$ in $X$ $xPy$ iff $s(x) > u(y) + 1$
and $xQy$ iff $u(x) > u(y)$.
Manders (1981) Beja and Gilboa (1989) showed that for this result to be true for a countable X an additional axiom is needed saying that for every x in X and every infinite sequence \( x_1, x_2, \ldots \) in X if \( x_i \preceq x_{i+1} \) for \( i = 1, 2, \ldots \), then for some \( n \) \( x \preceq x_n \) and if \( x_i \preceq x_{i+1} \) for \( i = 1, 2, \ldots \) then for some \( n \) \( x \preceq x_n \). Beja and Gilboa (1989) provide characterization of the JND representation for a general (not necessarily countable) X. We will generally assume in this paper that range(w) = \( \mathbb{N} \).

Let us recall the standard definition of concatenation of binary relations on X: given two binary relations \( B_1 \) and \( B_2 \) let \( B_1 B_2 \) be defined by:

For any \( x, y \) in X \( x \in B_1 B_2 y \) iff there exists \( z \) in X such that \( x \in B_1 z \) and \( z B_2 y \).

Note that successive application of this definition render concatenation of more than two relations well-defined.

Let \( X_1, \ldots, X_n \) be given sets and let there be semiorders \( P_i \) defined on every \( X_i \), \( i = 1, \ldots, n \). Assume that \( P \) is a semi-order on \( X = X_1 \times \ldots \times X_n \). For a generic element \( x \) in \( X \), \( x_i \) will denote its \( i \)-th component. \( x_i \) will stand for \( (x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \) and \( (x_i, y_i) \) for \( (x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \).

Further suppose that \( P \) and \( Q \) are such that there exists a utility function \( u \) from \( X \) onto \( \mathbb{N} \) such that \( xPy \) iff \( u(x) > u(y) \) + 1 and \( xQy \) iff \( u(x) > u(y) \) for all \( x \) and \( y \) in \( X \). We will say that \( u \) represents \( P \) and call \( P \) representable. From here on let us assume that every \( P_i \) is representable.

Note. \( u_1 \) is defined up to a strictly increasing transformation \( \nu = f(u) \), where \( f(a) = f(a+1) + 1 \) for all \( a \) in \( \mathbb{N} \).

Now we can define some properties \( P \) may possess w.r.t. \( P_1, \ldots, P_n \):

**Definition 1.** 1) \( P \) on \( X \) is \( P \)-monotone with respect to the semi-orders
$P_1, \ldots, P_n$ (hereafter $P$-monotone) if $\forall x, y \in X$ the following holds: if $x_i^{Q_i}y_i$ for all $i \in N = \{1, \ldots, n\}$ and $j$ such that $x_j^{P_j}y_j$, then $xP_jy_j$.

(As above $Q_i$ and $Q'_{i}$ are the corresponding weak orders.)

2) $P$ is $Q$-monotone from above with respect to $P_1, \ldots, P_n$ (hereafter $Q$-monotone from above) if $xP_jy_j$ and $x_i^{Q_i}x_j$ for all $i \in N$ imply $xP_iy_i$.

3) $P$ is $Q$-monotone from below with respect to $P_1, \ldots, P_n$ (hereafter $Q$-monotone from below) if $xP_jy_j$ and $y_i^{Q_i}y_j$ for all $i \in N$ imply $xP_iy_i$.

4) $P$ is $Q$-monotone with respect to $P_1, \ldots, P_n$ (hereafter $Q$-monotone) if it is both $Q$-monotone from above and from below.

5) $P$ is monotone with respect to $P_1, \ldots, P_n$ (hereafter monotone) if it is simultaneously $P$- and $Q$-monotone.

Definition 2. $P$ is consistent with $P_1, \ldots, P_n$ (hereafter consistent) if for all $i \in N$ and for all $x_i, y_i \in X_i$ if $x_i^{P_i}y_i$ then $(x_1, \ldots, x_i, y_1, \ldots, y_i) \in (x_{i-1}, \ldots, x_1, y_{i-1}, \ldots, y_1)$.

Relations between the two types of monotonicity and the consistency requirements are considered in section 4 (see statements 1 - 5). There one can also find examples of semi-orders which do and do not satisfy various subsets of these properties.

Let us define for each semi-order $P$ on $X$ the $P$-topology as follows: $x_n \rightarrow x$ if for every $y \in X$ for which $y \leq x$, there exists $m$ such that $\forall m \geq M$ $y \leq x$ and for $y \in X$ for which $x \leq y$ there exists $M$ such that $x \leq y$. The continuity of $P$ we are about to define means that if a sequence $(x_{ik})$ in $X_i$ converges to $x_i$ in the $P_i$-topology (for all $i$), then $(x_{ik})$ converges to $x$ in the $P$-topology. (In the presence of monotonicity and consistency, this
is tantamount to saying that the $P$-topology on $X$ is the product topology defined by the $P_i$-topologies on $X_i$, $i = 1, \ldots, n$.

Definition 3. 1) $P$ is \textit{continuous from above (below)} with respect to $(P_i)_{i = 1, \ldots, n}$ if for all $i$, for every sequence $(x_{1k}^i)$ converging to $x_i$ in $X_i$ and for all $y$ in $X$ if \((x_1^i, \ldots, x_n^i)P(y_1^i, \ldots, y_n^i)\) \((y_1^i, \ldots, y_n^i)P(x_1^i, \ldots, x_n^i)\) then there exists $M$ such that for every $k > M$ \((x_{1k}^i, \ldots, x_{nk}^i)P(y_1^i, \ldots, y_n^i)\) \((y_1^i, \ldots, y_n^i)P(x_{1k}^i, \ldots, x_{nk}^i)\).

2) $P$ is \textit{continuous} with respect to $(P_1, \ldots, P_n)$ (hereafter continuous) if it is continuous both from above and from below.

Finally, the symbol $\neg$ will stand for negation.
J. THE MAIN RESULTS

Our main results can be reduced to three theorems.

Theorem A states existence and characterizes monotone, consistent and continuous semi-orders on a product space.

Theorem A. Let \( \{P_i\}_{i \in \mathbb{N}} \) be semi-orders on \( \{X_i\}_{i \in \mathbb{N}} \) represented by \( \{u_i\}_{i \in \mathbb{N}} \). Let \( P \) be a semi-order on \( X = \prod_{i \in \mathbb{N}} X_i \) represented by \( u \). Then the following are equivalent:

(i) \( P \) is consistent, monotone and continuous w.r.t. \( P_1, \ldots, P_n \);

(ii) There is a strongly monotone and continuous function \( f_u : \mathbb{R}^n \to \mathbb{R} \) satisfying

\[
 f_u(a_i, a_i) = f_u(a_{i-1}, a_i) - 1, \quad \forall a \in \mathbb{R}^n, \quad \forall i, \tag{2}
\]

such that \( u = f_u(u_1, \ldots, u_n) \).

In particular, there exists such a \( P \) defined by \( u = f_u \).

Theorem B gives the notion of "almost uniqueness" of such a semi-order on the product space.

Theorem B. Let \( \{P_i\}_{i \in \mathbb{N}} \) be semi-orders on \( \{X_i\}_{i \in \mathbb{N}} \) represented by \( \{u_i\}_{i \in \mathbb{N}} \) respectively. Suppose that \( P_a \) and \( P_b \) are two representable semi-orders on \( X \) which are both consistent, monotone and continuous with respect to \( P_1, \ldots, P_n \). Then \( (P_a)^{\cap} \subseteq P_b \) and \( (P_b)^{\cap} \subseteq P_a \).

Theorem C deals with a joint representation of several semi-orders on the same space. It will need additional axioms. For two semiorders \( P \) and \( P' \) on a certain space \( X' \) define the following conditions.
A1. The concatenation of $P$ and $P'$ is commutative, namely, $PP' = P'P$.

A2. For any $k, n \in \mathbb{N}$ either $(P^k)^n \subseteq (P^n)^k$ or $(P^n)^k \subseteq (P^k)^n$ and for some $k, m \in \mathbb{N}$, we also have $(P)^k = (P')^m$.

Theorem C. Given semi-orders $P_1, \ldots, P_n$ on a set $X$ such that $P_i$ is represented by $u_i(\cdot)$, $i = 1, \ldots, n$, the following are equivalent:

(i) there exist a function $u \colon X \to \mathbb{R}$ with range($u$) = $\mathbb{R}$ and positive rational numbers $\delta_1, \ldots, \delta_n$ such that for all $i = 1, \ldots, n$, and for all $x, y \in X$

$\quad xP_i y \iff u(x) - u(y) > \delta_i$,

$\quad xQ_i y \iff u(x) > u(y)$;

(ii) for all $i$ and $j$ in $\{1, \ldots, n\}$, $P_i$ and $P_j$ satisfy A1 and A2.

Corollary D applies the previous results to expected utility representation.

Corollary D. Let $X'$ be a set and let $P_1, \ldots, P_n$ be semi-orders on it represented by $u_1, \ldots, u_n$ (respectively), where range($u_i$) = $\mathbb{R}$ for all $i$. Suppose that for every $i, j \in \mathbb{N}$, $P_i$ and $P_j$ satisfy A1 and A2. Define $u$ on $X = (X')^{\mathbb{N}}$ by $u = \mathbb{E}(i\cdot u_i)$ for $\delta_i$ obtained from Theorem C. Then

(i) $P$ is continuous, consistent and monotone w.r.t. $P_1, \ldots, P_n$;

(ii) If $P'$ is another semi-order on $X$ which is continuous, consistent and monotone w.r.t. $P_1, \ldots, P_n$ then $(P')^\mathbb{N} \subseteq P$ and $P \subseteq (P')^\mathbb{N}$.

Proofs of the main theorems and related results are contained in section 4.
4. PROOFS AND AUXILIARY RESULTS

Let $X_1, \ldots, X_n$ be given sets with semi-orders $P_1, \ldots, P_n$ defined on them respectively. Let $X = X_1 \times \ldots \times X_n$, and let $P$ be a semi-order on $X$ represented by $u$. We assume these conditions unless otherwise stated.

Let us first show that strong consistency is too binding a requirement even for $n = 2$.

We will say that $P$ is strongly consistent with $P_1$ and $P_2$ if for all $x, y$ in $X$, $x_1 P_1 y_1$ and $x_2 P_2 y_2$ imply $x P y$, where $(1, j) = (1, 2)$. In the next lemma $P_i$ are assumed to be representable, whence $\text{range}(u_i) = \mathbb{R}$. However note that it suffices that $\text{range}(u_i) \supset (a, b)$ for some $b > a + 1$.

Lemma 4.1. If $P_1$ and $P_2$ are representable semi-orders. Then there does not exist a representable semi-order $P$ which is strongly consistent with them.

Proof. Let $x, y, z$ in $X$ be such that $u_1(y_1) = u_1(x_1) - \epsilon + 1$, $u_2(y_2) = u_2(x_2) - \epsilon - 1$, $u_1(z_1) = u_1(x_1) + \epsilon + 1$, $u_2(z_2) = u_2(x_2) + \epsilon - 1$, where $\epsilon$ is a positive number less than 1. It follows from the definition of strong consistency that $x P y$ and $y P z$. By transitivity of $P$ it implies $z P y$. Moreover, $u(z) - u(y) > 2$. Denote the interval $[(u_1(y_1), u_2(y_2)), (u_1(z_1), u_2(z_2))]$ by $d$.

For any two points $v, w$ in $X$ such that $(u_1(v_1), u_2(v_2)), (u_1(w_1), u_2(w_2)) \in d$ and $u_1(v_1) > u_1(w_1)$ we get $u(v) = u(w) > 2$, a contradiction. //

The next four results relate to our concepts of monotonicity and consistency.
Lemma 4.2. A representable semi-order \( P \) which is \( Q \)-monotone from above (from below) is \( Q \)-monotone.

Proof. We will show that \( Q \)-monotonicity from above imply \( Q \)-monotonicity from below. The second part is proved symmetrically. Let \( xPy \) and \( y_iQ'y_i' \), \( i = 1, \ldots, n \). For every \( z \) in \( X \) such that \( y'Pz \) it follows from \( Q \)-monotonicity from above that \( y'Pz \). As the range of \( u \) is \( \mathbb{N} \), \( u(y) \geq u(y') \). Hence, \( xPy' \). //

The three statements below show interrelations between monotonicity and consistency. In fact, they illustrate that \( P \)-monotonicity, \( Q \)-monotonicity and consistency are mutually independent.

Observation 4.3. Consistency and \( P \)-monotonicity do not imply \( Q \)-monotonicity.

Proof. Consider the following example. \( n = 2 \), \( X_1 = X_2 = \mathbb{N} \).

\[
u_1(x_1) = \begin{cases} x_1, & \text{if } x_1 < 0; \\ 0, & \text{if } x_1 \text{ is in } [0, 1/2]; \\ x_1 \cdot 1/2, & \text{if } x_1 > 1/2. \end{cases}
\]

\[
u_2(x_2) = x_2.
\]

\( P_1 \) may be also represented by \( v \) defined as follows:

\[
v(x_1) = \begin{cases} x_1/2, & \text{if } x_1 \text{ is in } [0, 3/2]; \\ x_1/2 + k/2, & \text{if } x_1 \text{ is in } [1/2+k, 3/2+k]; \\ x_1/2 - k/2 - 1/4, & \text{if } x_1 \text{ is in } [-k-1, -k]; \\ x_1/2 - 1/4, & \text{if } x_1 \text{ is in } [-1, 0[ \text{, where } k \text{ is in } \mathbb{N}. \end{cases}
\]
Define \( u(x_1, x_2) = v(x_1) + u_2(x_2) \) and let \( xPy \) iff \( u(x) > u(y) + 1 \). Since \( v \) represents \( P_1 \) and \( u \) represents \( P_2 \), \( P \) is consistent and \( P \)-monotone w.r.t. \( P_1 \) and \( P_2 \). However, \( P \) is not \( Q \)-monotone: let \( x = (1/2, 1) \), \( y = (0, 1) \) and \( z = (0, 1/5) \). \( u(x) = 5/4 \), \( u(y) = 1 \), \( ufz = 1/5 \). Hence, \( xPz \) and not \( yPz \), namely, \( xQy \). But \( y_i Q y_j \), \( i = 1, 2 \). This violates \( Q \)-monotonicity. //

**Observation 4.4.** \( Q \)-monotonicity and consistency do not imply \( P \)-monotonicity.

Proof. Consider the following example: \( n = 2 \), \( x_i = \mathbb{R} \), \( u_i(x_i) = x_i \), \( x_i P_i y_i \) iff \( x_i > y_i + 1 \), \( i = 1, \ldots, n \). Define \( u(x_1, x_2) = (x_1 + x_2)/2 \) and \( xPy \) iff \( u(x) > u(y) + 1 \), i.e., \( x_1 + x_2 > y_1 + y_2 + 2 \). \( P \) is obviously \( Q \)-monotone and consistent. Let \( x_1 = x_2 = 1, y_1 = 0, y_2 = 1 \). This means that \( x_1 P_1 y_1 \) and \( x_2 Q_2 y_2 \), but \( x_1 x_2 > 2 \). \( 2 < 3 = y_1 y_2 + 2 \). Hence, not \( xPy \) and \( P \) is not \( P \)-monotone. //

**Observation 4.5.** \( Q \)-monotonicity does not imply consistency.

Proof. Consider the following example: \( n = 2 \), \( x_i = \mathbb{R} \), \( u_i(x_i) = x_i \), \( x_i P_i y_i \) iff \( x_i > y_i + 1 \), \( i = 1, \ldots, n \). Define \( u(x_1, x_2) = 2*(x_1 + x_2) \) and \( xPy \) iff \( u(x) > u(y) + 1 \), i.e., \( x_1 + x_2 > y_1 + y_2 + 1/2 \). \( P \) is obviously monotone. Let \( x_1 = 0, x_2 = y_1 = y_2 = 0 \). This means that \( x_1 P_1 y_1 \), but \( x_1 y_2 = 0 \). \( 0 > 1/2 = y_1 y_2 + 1/2 \). Hence, not \( xPy \) and \( P \) is not consistent. //

Now we show that continuity also has to be explicitly assumed:

**Observation 4.6.** Monotonicity and consistency do not imply continuity.
Proof. Consider the following example of a representable semi-order \( P \) that is monotone and consistent w.r.t. given \( \{ P_i \}_{i \in \mathbb{N}} \) but is not continuous w.r.t. them.

Let \( n = 2 \), \( X_1 = X_2 = \mathbb{R} \), \( u_1(x_1) = x_1 \), \( u_2(x_2) = x_2 \), where \( u_1(.) \) represents \( P_1 \), \( i = 1, 2 \). \( P_1 \) admits also another representation \( v_1(x) = x/2 + k/2 \), where \( x \in \mathbb{R}, k = 1, 2 \) and \( k \in \mathbb{Z} \).

Define \( u(x) = v_1(x_1) + u_2(x_2) \) and assume that \( u(.) \) represents \( P \).

Let \( x = (0, -0.1) \), \( y = (1, 0) \) and \( y_k = (1-1/k, 0) \), \( k \in \mathbb{N} \). \( u(x) = -0.1, u(y) = 1 \). Hence, \( y_k \) and \( y \) are not comparable, since \( u(y_k) < 1/2 \) for any \( k \). It is easy to see that \( P \) is nevertheless monotone and consistent. //

Now let us proceed from the definitions of section 2.2 to our first main result, Theorem A.

The following lemma shows that our concepts of monotonicity are indeed weaker than strong consistency as implied by Rubinstein (1988) and they allow us to achieve some positive results.

Lemma 4.7. For any representable semi-orders \( P_1, \ldots, P_n \) on \( X_1, \ldots, X_n \) respectively there exists a consistent, monotone and continuous representable semi-order \( \hat{P} \).

Proof. Let \( u_1 \) be such that for all \( x_1, y_1 \in X_1 \), \( x_1 \leq y_1 \) iff \( u_1(x_1) > u_1(y_1) \). \( 1 \) and \( 1 \leq y_1 \) iff \( u_1(x_1) > u_1(y_1) \). \( i = 1, \ldots, n \).
Define $u(x) = \sum u_i(x_i)$ and $x \preceq y$ iff $u(x) > u(y) + 1$. Then $P$ is obviously consistent, monotone and continuous.

Note that this $P$ is representable, by construction. //

Lemma 4.8. Let $P$ be a semi-order which is $Q$-monotone, and representable by $u$, then there exists $f_u : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $u = f_u(u_1, \ldots, u_n)$, i.e. $u_i(x_i) = u_i(y_i)$, $i = 1, \ldots, n$, imply $u(x) = u(y)$. Moreover, $f_u$ is unique.

Proof. Suppose that $u(x) > u(y)$, i.e. $x \preceq y$. Thus, there exists $z$ in $X$ such that

$u(y) \leq u(z) + 1$ and $u(x) > u(z) + 1$;

Note that $y_i \leq x_i$, $i = 1, \ldots, n$. Hence, by $Q$-monotonicity from above, $y \preceq z$ which contradicts the condition $u(y) \leq u(z) + 1$. //

Now we are in a position to prove Theorem A.

Proof of the Theorem A. Let us first show that (ii) implies (i). Assume, then, that $u = f_u(u_1, \ldots, u_n)$ with $f_u$ as in (ii). $Q$-monotonicity of $P$ follows from monotonicity of $f_u$. As for $P$-monotonicity one should only use (2). $P$ has to be consistent because of (2) and the monotonicity of $f_u$. Finally, let us show that the continuity of $f_u$ implies that of $P$. Assume that a sequence $(x_{ik})$ converges to $x_i$ as $k \rightarrow \infty$ in the $P_i$-topology on $X_i$. Since range$(u_i) = \mathbb{R}$, this implies that $u_i(x_{ik}) = u_i(x_i)$ as $k \rightarrow \infty$. By continuity of $f_u$, $u(x_k)$ converges to $u(x)$ which implies that $x_k$ converges to $x$ in the $P$-topology.

We now wish to show that (i) implies (ii). By Lemma 4.8, we know that
Q-monotonicity of $P$ implies the existence of a unique $f_u : \mathbb{R}^n \to \mathbb{R}$ such that $u = f_u(u_1, \ldots, u_n)$. We will now prove it satisfies all requirements.

**Strict monotonicity.** Let $a_{1,1} \in \mathbb{R}$ satisfy $a_{1,1} > b_{1,1}$. We will show that for every $c_{1,1} \in \mathbb{R}^{n-1}$, $f_u(c_{1,1}, a_{1,1}) > f_u(c_{1,1}, b_{1,1})$. Since range$(u_1) = \mathbb{R}$, we can find $x, y, z \in X$ such that $u_1(x_1) = a_{1,1}$, $u_1(y_1) = b_{1,1}$, and $b_{1,1} < u_1(z_1) < a_{1,1}$.

Similarly, let $w_j \in X_j$, $(j \neq 1)$ satisfy $u_k(w_k) = c_{1,1}$. Note that $x, y, z$ but $(y, z)$ by P-monotonicity, $(w_1, x) P (w_1, z)$ and, by consistency, $-(w_1, y) \neg P (w_1, z)$ whence $(w_1, x) P (w_1, y)$ and $f_u(c_{1,1}, a_{1,1}) > f_u(c_{1,1}, b_{1,1})$.

**Continuity.** Assume the contrary, i.e., that $f_u$ is not continuous at some point $a \in \mathbb{R}^n$. Then there is an $\epsilon > 0$ and a sequence $(a_k)$ converging to $a$ as $k \to \infty$ such that $f_u(a_k) > f_u(a) + \epsilon$ for all $k$ or $f_u(a_k) < f_u(a) - \epsilon$ for all $k$. Let us assume the former, i.e., $f_u(a_k) > f_u(a) + \epsilon$. Find $x_k \in X$ such that $(u_1(x_k), \ldots, u_n(x_k)) = a_k$, an $x \in X$ for which $(u_1(x), \ldots, u_n(x)) = a$ and $a \in X$ such that $f_u(a) + \epsilon + 1 > u(z) > f_u(a) + 1$. Since $a_k \to a$ as $k \to \infty$ for each $i$, $u_i(x_k) \to u_i(x)$, which implies that $x_k \to x$ as $k \to \infty$ in the $P_1$-topology on $X_1$. However, for all $k$ and $z \in \mathbb{R}$, $z \neq a$, so that $x_k \to \mathbb{R}$ does not converge to $x$ in the $P_1$-topology. A contradiction to the continuity of $P$.

**Condition (2).** Let there be given a vector $a \in \mathbb{R}^n$ and an index $1 \leq i \leq n$ and consider $f_u(a_{1}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. By P-monotonicity, $f_u(a_{1}, a_{i}+1, a_{i+1}, \ldots) > f_u(a_{1}, a_{i}, a_{i+1}, \ldots) + 1$ for all $\epsilon > 0$, and consistency implies $f_u(a_{1}, a_{i}+1, a_{i+1}, \ldots) \geq f_u(a_{1}, a_{i}, a_{i+1}, \ldots) + 1$. Strict monotonicity of $f_u$ also implies that the latter inequality is strict, namely $f_u(a_{1}, a_{i}+1, a_{i+1}, \ldots) > f_u(a_{1}, a_{i}, a_{i+1}, \ldots) + 1$ for all $\epsilon > 0$. The continuity of $f_u$ implies $f_u(a_{1}, a_{i}+1, a_{i+1}, \ldots) = f_u(a_{1}, a_{i}, a_{i+1}, \ldots) + 1$. //

Our next objective is to see to what extent semi-orders on product spaces which are consistent, monotone and continuous with respect to given
ones on the original spaces are unique. First we want to examine the possibility of reverse preference: could it happen that two distinct semi-orders $P_a$ and $P_b$ satisfying the conditions of Theorem A rank two alternatives in opposite direction, namely $x_{P_a}^y$ and $y_{P_b}^x$? We first consider the case $n = 2$.

Lemma 4.9. Let $P$ be a semi-order on $X$ which is consistent, monotone and continuous with respect to $P_1$ and $P_2$. And let $x, y \in X$ be such that $u_1(x_1) > u_1(y_1)$ and $u_2(x_2) < u_2(y_2)$, but $\neg(x_{P_1}^1 y_1)$ and $\neg(y_{P_2}^2 x_2)$. Then $\neg(x_{P} y)$.

Proof. By Theorem A, $P$ may be represented by $u = f(u_1, u_2)$. Suppose $x_{P} y$. Then, by Q-monotonicity, $z_{P} y$ for $z$ in $X$ with $u_1(z_1) = u_1(x_1)$ and $u_2(z_2) = u_2(x_2)$. But, by consistency, $\neg(z_{P} y)$, a contradiction. //

Lemma 4.10 generalizes the previous result for $n > 2$, but falls short of excluding preference reversal. Note that while the statement of Lemma 4.9 is symmetric with respect to $x$ and $y$, this is not true of Lemma 4.10, where $x$ and $y$ play different roles.

Lemma 4.10. Let $\{L_i\}_{i \in \mathbb{N}}$ be representable semi-orders on $\{X_i\}_{i \in \mathbb{N}}$ and let $P$ be a representable semi-order on $X$ which is consistent, monotone and continuous with respect to $P_1, \ldots, P_n$, $n > 2$. Suppose that $x, y \in X$ are such that for any $i \neq j$, $u_i(x_i) \geq u_j(x_j)$ and $u_i(y_i) < u_j(y_j)$, but for every $i$, $x_i \not\leq y_i$. Then $\neg(x_{P} y)$.

The proof is similar to the proof of Lemma 4.9.
Corollary 4.11. Suppose that $P_a$ and $P_b$ are two semi-orders on $X$ which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$, $n < 4$. Then for any $x, y \in X$ $xP_a y$ implies $-(yP_b x)$.

The proof follows from monotonicity and Lemma 4.10.

However preference reversal is possible for large enough $n$:

Observation 4.12. Suppose that $P_a$ and $P_b$ are two semi-orders on $X$ which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$, $n > 3$. Then there may be $x, y \in X$ such that $xP_a y$ and $yP_b x$.

Proof. Consider the following example. Let $n = 4$, $X_i = \mathbb{R}$, $u_i(x_i) = x_i$, $i = 1, \ldots, 4$. Define $u_a(x) = \sum_{i=1}^{4} u_i(x_i)$. As usual suppose that all semi-orders are induced by corresponding utility functions.

Choose another representation of $P_a$. Let

$v_i(x_i) = \begin{cases} (k-x_i)/100 + k, & \text{if } x_i \in [k, k+0.1]; \\ 19.96(k-x_i) + k - 1.995, & \text{if } x_i \in [0.1+k, 0.15+k]; \\ (k-x_i)/850 + k + 849/850, & \text{if } x_i \in [0.15+k, k+1]. \end{cases}$

where $k$ is an integer, $i = 1, \ldots, 4$.

Define $u_b(x) = \sum_{i=1}^{4} v_i(x_i)$.

By Theorem A, both $P_a$ and $P_b$ are consistent, monotone and continuous with respect to $P_1, \ldots, P_n$. Let $x = (0.1, 0.1, 0.95, 0.95)$ and $y = (0.2, 0.2, 0.3, 0.3)$. Then $u_a(x) = 2.1$, $u_a(y) = 1$. Hence, $xP_a y$. But $u_b(x) < 0.002 + 2 = 2.002$ and $u_b(y) > 4 \ast 0.999 = 3.996$. Hence, $yP_b x$. //
Hence, we see that "preference reversal" in the sense of \( x_P y \) but \( y_{P_b} x \) is possible. However, \( x_{(P_a)^k} y \) and \( y_{(P_b)^k} x \) is impossible. In fact, our Theorem B shows that for any such \( P_1 \) and \( P_2 \), \( (P_a)^k \subseteq P_2 \) and \( (P_b)^k \subseteq P_1 \). Let us turn to its proof.

**Proof of Theorem B.** Suppose \( y_{(P_a)^k} x \). Let \( x' \) be an alternative in \( X \) such that \( u_i(x') = u_i(x_i) + 1 \) for \( 1 \leq i \leq n \), so that \( f_u(u_1(x_1), \ldots, u_n(x_n)) = f_u(u_i(x_1), \ldots, u_i(x_n)) \) for every \( f_u \) that corresponds to a representable semi-order \( P \) on \( X \) which is consistent, continuous and monotone w.r.t. \( P_1, \ldots, P_n \).

By Theorem A, \( y_{E_a} z \) and \( y_{E_b} z \) for any \( z \in X \) with \( u_l(z) = u_l(y_l) + m_i \), where \( m_i \in Z \) and \( \sum m_i = 0 \). For each \( z \in X \) define \( d_i := d_1, \ldots, d_n \in \mathbb{R}^n \) by \( d_i = u_i(z) - u_i(x') \). One can find a \( z \) with the following properties:

(i) \( y_{E_a} z \) and \( y_{E_b} z \)

(ii) (a) \( d_i \leq 0 \) for all \( 1 \leq i \leq n \)

or (b) \( d_i \geq 0 \) for all \( 1 \leq i \leq n \) (but not (a))

or (c) \( d_i \leq 1 \) for all \( 1 \leq i \leq n \) (but not (a) or (b))

such a \( z \) would be, for instance, one minimizing \( \sum |d_i| \) over the set \( \{ z \mid u_i(z) = u_i(x) = m_i \} \) where \( m_i \in Z \) and \( \sum m_i = 0 \).

In case (ii)(a) we have \( u_i(z) = u_i(x') \) whence \( z = (P_a)^k x' \) and, by monotonicity, \( z \in (P_a)^k x' \) which implies \( y_{(P_a)^k} x \) and \( y_{(P_b)^k} x \).

In case (ii)(b) \( x' \in (P_a)^k x \), whence \( x' \in (P_a)^k x \). However, \( f_u(u_1(x_1), \ldots, u_n(x_n)) \geq f_u(u_1(x_1), \ldots, u_n(x_n)) + n \) whence we also get \( y_{E_a} x' \) and \( z \in (P_a)^k x' \). But this is possible only if \( d_i \geq 0 \) for all \( i \) which boils down to (ii)(a).

Finally, consider case (ii)(c). Since \( d_i > 0 \) we know that
$u_i(z) > u_j(x)$ for all $i$. However, we also know that for some $1 \leq i < j \leq 0$ which means that $u_i(z) > u_j(x) + 1$. Monotonicity of $P_b$ means $y^P_b x$ whence $y^P_b x$ also holds. //

Corollary 4.13. Let $(P_i)$ on $(X_i)$ be representable semi-orders. Suppose that $P_a$ and $P_b$ are two representable semi-orders on $X$ which are both consistent, monotone and continuous with respect to $P_1, \ldots, P_n$. Then $I_a \subseteq (I_b)^n$ and $I_b \subseteq (I_a)^n$.

The proof follows from Theorem 3 and the fact that for representable semi-orders $I(P^n)$ induced by $P^n$ coincides with $I^n$, where $I$ is induced by $P$.

Let us turn to the proof of Theorem C. For two given representable semi-orders $P$ and $P'$ on the same space $X$ define:

$A(P, P') = \{ k/m : k, m \in \mathbb{N}, x^k \subseteq (P')^{cm} \}$

Lemma 4.14. Suppose $P$ and $P'$ are representable semi-orders on a space $X$ satisfying A1 and A2. Then $A(P, P')$ is homogeneous, i.e., for every $t \in \mathbb{N}$ $P^t \subseteq (P')^{tm}$ iff $P^{ct} \subseteq (P')^{ctm}$.

Proof. Throughout the proof let $u$ and $u'$ represent $P$ and $P'$ respectively.

"Only if" part.

$P^t \subseteq (P')^{tm}$ means that for any $x, y$ in $X$, $u(x) - u(y) > k$ implies $u'(x) - u'(y) > m$. If $x^t \subseteq (P')^{ct}$ then there exists a sequence $(x^1, \ldots, x^k)$ in $X$ such that $x^i \subseteq x^{i+1}$ for all $1 \leq i \leq k$. Then $u(x^i) - u(x^{i+1}) > k$ for all $1 \leq $
1, \ldots, t. This, in turn, implies \( u'(x^i) - u'(x^{i+1}) > m \) for all \( 1 \leq i \leq t - 1 \), or \( k(p_{(P')}^{i+1})_{x^i} < t^m \).

"If" part.

Assume, then, that \( P^k \subseteq (P')^{m i} \) for some \( t > 1 \). Since \( A2 \) holds there are two possible cases: 1) \( P^k \subseteq (P')^{m i} \) in which the proof is complete, or 2) \( (P')^{m i} \subseteq P^k \). In this case, by the if part, \( (P')^{m i} \subseteq P^k \). But, by assumption, \( P^k \subseteq (P')^{m i} \). Thus, \( P^k = (P')^{m i} \).

To show that \( P^k \subseteq (P')^{m i} \), let there be given \( x, y \in X \) with \( x \neq y \); we will show that \( (x^i)_{(P')}^y \) has to hold. Suppose not. Then \( u(x) - u(y) > k \) but \( u'(x) - u'(y) \leq m \). Choose a sequence \( y_0, \ldots, y_j \in X \) with \( y_0 = x \) and \( u(y_{j-1}) - u(y_j) = k \) for \( 0 \leq i \leq t - 1 \). This is possible since \( \text{range}(u) = \mathbb{X} \). Note that \( u'(x) - u'(y_j) < m \) since \( y_j Q x \) and this is equivalent to \( y_j Q y \).

By our construction, \( (y^i_k)_{(P')}^{y_{i+1}} \) for \( 0 \leq i \leq t - 1 \) which implies, since \( (P')^{m i} \subseteq P^k \), that \( (y^i_k)_{(P')}^{y_i} \). However, for every \( z \) satisfying \( y_{i-1} Q z \) we get again, using the fact that \( \text{range}(u) = \mathbb{X} \), \( x \neq y \). The latter means that \( (x^i)_{(P')}^{y_i} \).

Considering the \( u' \) scale, we obtain \( u'(x) - u'(z) > m \) for every \( z \) satisfying \( y_{i-1} Q z \). Hence, \( u'(x) = u'(y_i) = m \). On the other hand, \( (y^i_k)_{(P')}^{y_{i+1}} \) for \( 0 \leq i \leq t - 1 \) implies \( u'(y_i) - u'(y_{i+1}) = m \) whence \( u'(x) \neq u'(y_i) \). Combining the inequalities one obtains \( u'(y_i) - u'(y_j) = m \) in contradiction to the choice of \( y_i \).

Lemma 4.15. If two semi-orders \( P \) and \( P' \) satisfy \( A2 \), then \( Q = Q' \).

Proof. Let \( k \) and \( m \) satisfy \( (P)^k = (P')^m \). \( (P)^k \) is a semi-order on \( X \) and so is
(P')^m. Since they are identical, the weak orders \((Q)^k\) and \((Q')^m\) are also identical. However, \((Q)^k = (Q')^m = Q'\). //

We now proceed to our third main result, Theorem C.

Proof of Theorem C. (i) \(\Rightarrow\) (ii)

Let us begin with A1. Consider \(P_1, P_2\) and \(k, m \in \mathbb{N}\). By (i), \(x(P_1)^k y \iff u(x) - u(y) > k\delta_1\), whence \((P_1)^k \subseteq (P_1)^m\) iff \(k\delta_1 \leq m\delta_1\). Hence, \((P_1)^k \subseteq (P_2)^m\) or \((P_2)^m \subseteq (P_1)^k\). Since \(\{\delta_1\}\) are rational equality would hold for some \(k, m \in \mathbb{N}\).

As for A1, note that \(x(P_1)^k y \iff u(x) - u(y) > \delta_1 + \delta_1\), which means that \(F_1 \subseteq F_2\).

(ii) \(\Rightarrow\) (i)

Let us first introduce some additional definitions. For a semi-order \(P\) on \(X\), let \(P^k\) be the binary relation defined as follows: \(xP^k y \iff xy^k\) and for every \(z\) satisfying \(zx\) we have \(zPy\). Intuitively, \(xP^k y\) means that \(x\) is the "supremum" of \(\{w \mid w^ky\}\). By the usual concatenation of binary relations \((P^k)^k\) is well-defined for \(k \geq 1\). Let us also define \((P^k)^0\) to be \(E\) (which corresponds to equal \(u\)-values) and \((P^k)^{-k}\) for \(k \geq 1\) as the inverse of \((P^k)^k\).

Similarly, we will refer to the expressions of the types \((P^k_{11} \ldots (P^k_{1s})^k_z\), where \(s \in \{1, \ldots, n\}\) and \(k \in \mathbb{Z}\) for \(1 \leq r \leq s\).

The proof will be simpler to carry out by induction on \(n\). Let us begin with \(n = 2\).

Choose any point \(x_0\) in \(X\) and set \(u(x_0) = 0\). By A2, there are \(m, t \in \mathbb{N}\) such that \((P_1)^m = (P_2)^t\). Assume without loss of generality that \(g.c.d. (m, t) = 1\), where \(g.c.d\) stands for the greatest common divider. This assumption can
be made thanks to Lemma 4.14. Define $M = t \ast m$. We will construct a function $u$ such that

$$u(x) - u(y) > \delta_1 \ast t \quad \text{iff} \quad x \in P_1 \quad \text{and}$$
$$u(x) - u(y) > \delta_2 \ast m \quad \text{iff} \quad x \in P_2 \quad \text{and}$$

For every integer $k$ let us define $V(k) = \{ y \in X \mid \text{there exist sequences} \quad k_1, \ldots, k_s \quad \text{and} \quad i_1, \ldots, i_s \quad \text{such that} \quad y(P_{i_1}^{k_1}, \ldots, P_{i_s}^{k_s}) \subseteq x \quad \text{and} \quad \exists k' \mid i_1 = k' \}$. Intuitively, $V(k)$ is the set of all $y$'s for which we have to assign the value $u(y) = k$. Note that $V(k) \neq \emptyset$ for every $k \in Z$.

Claim 1. For every $k$ and every $y, z \in V(k)$ it is true that $y \equiv z$.

Proof. For $k \in Z$ there are unique $a_k, b_k, c_k$ such that $k = a_k \ast m + b_k \ast t + c_k \ast t$ with $a_k \in Z$, $0 \leq b_k \leq m$ and $0 \leq c_k \leq t$. Note that $A_1$ and $A_2$ imply that $(P_1 \ast P_2)^{\ast} = (P_2 \ast P_1)^{\ast}$ and that $P_1 \ast P_2 = P_2 \ast P_1$. Hence, every $y \in V(k)$ satisfies $y(P_2 \ast P_1, \ldots, P_1 \ast P_2)^{\ast} \subseteq x$ which implies the desired conclusion. //

Claim 2. Suppose $y \in V(k)$ and $z \in V(g)$ with $k > g$. Then $y \equiv z$.

Proof. Since $u_1$ represents $P_1$ (i = 1, 2), for every $w_1, w_2, c_1, t_2 \in X$, if

$$w_1 \in P_1 \ast w_2 \quad \text{and} \quad t_1 \in P_1 \ast t_2 \quad (i = 1, 2) \quad \text{then} \quad w_1 \in QT_1 \quad \text{iff} \quad w_2 \in QT_2.$$  

Using this argument inductively, for every $w_1, w_2, t_1, t_2 \in X$, and every $k, h \in Z$, if

$$w_1 \in (P_1 \ast P_2)^{\ast} w_2 \quad \text{and} \quad t_1 \in (P_1 \ast P_2)^{\ast} t_2 \quad \text{then} \quad w_1 \in QT_1 \quad \text{iff} \quad w_2 \in QT_2.$$  

Consider $k = g - 1$. There are $k$ and $h$ such that $k = k \ast h = -1$. For $y \in V(k)$ and $z \in V(g)$ choose $w \in V(g-1)$. Then

$$y(P_1 \ast P_2)^{\ast} h \quad \text{and} \quad z(P_1 \ast P_2)^{\ast} h$$

$\equiv z$. Hence, $y \equiv z$ iff $z \equiv z$. It turns out that one of the following is true:

(i) for every $k, g, y \in V(k)$ and $z \in V(g)$ with $k > g$ implies $y \equiv z$.

(ii) for every $k, g, y \in V(k)$ and $z \in V(g)$ with $k > g$ implies $z \equiv y$.

(iii) for every $k, g, y \in V(k)$ and $z \in V(g)$ with $k > g$ implies $y \equiv z$. 


One only needs to know that for \( k = t \) and \( g = 0 \) \( y \in V(k) \) and \( z \in V(0) \) satisfy \( yQz \) to conclude that (1) is the case. //

At this point one can define \( u \) on \( \bigcup_{k \in \mathbb{Z}} V(k) \) by \( u(y) = k \) for \( y \in V(k) \). It is obvious that \( u \) satisfies (3) for \( x, y \in \bigcup_{k \in \mathbb{Z}} V(k) \).

Next, choose \( x_i \in V(1) \). Denote \( I = \{ x \mid x \in X \cup Q \} \). For every \( x \in I \) define \( u(x) = u_i(x) / u_i(1) \). For every \( k \in \mathbb{Z} \) define a set \( V(x, k) \) as \( V(k) \) was defined for \( x = x_0 \). Note that for \( y \in V(x, k) \) and \( z \in V(k) \), \( w \in V(k+1) \) we have \( wQyQz \). Furthermore, for every \( y \in X \) there are \( x \in I \) and \( k \in \mathbb{Z} \) such that \( y \in V(x, k) \). Hence, we define \( u(y) = z + u(x) \).

It is easy to see that for every \( x, y \in X \) and \( 1 = 1, 2 \) \( xP_y \) \( \iff \) \( u(x) - u(y) = 1 \).

We now turn to the induction step. Suppose \( n > 2 \). We already know that for \( P_1, ..., P_n \) there is a function \( u \) and positive rational numbers \( \delta_1, ..., \delta_n \) such that \( xP_i \) \( \iff \) \( u(x) - u(y) = \delta_i \) and \( xQy \) \( \iff \) \( u(x) > u(y) \). Without loss of generality assume that \( \delta_i \in \mathbb{N} \). Define \( P' \) by \( xP' \) \( \iff \) \( u(x) - u(y) > 1 \), so that \( P_i = (P')^\perp \) for \( 1 \leq i \leq n-1 \). Let \( u', \delta' \) and \( \delta'_u \) represent \( P \) and \( P' \), namely, \( u'(x) - u'(y) > \delta \) \( \iff \) \( xP' \); \n\( u'(x) - u'(y) > \delta_u \) \( \iff \) \( xP_{u'} \); \nand \( u'(x) - u'(y) > 0 \) \( \iff \) \( xQy \).

for every \( x, y \in X \) (The existence of these is guaranteed by the proof for the case \( n = 2 \)). Furthermore, \( \delta' \) and \( \delta'_u \) may be assumed to be integer \( w \).\( \circ \).

Hence, \( u' \) also satisfies \( u'(x) - u'(y) > \delta' \) \( \iff \) \( xP_1 \) for \( 1 \leq i \leq n \).
this completes the proof of the theorem. //

Next we note that axioms A1 and A2 are independent.

Proof. Consider the following example: \( n = 2, X = \mathbb{N}, u_1(x) = x, \)
\( u_2(x) = \begin{cases} 
2k + 5/6(x - 3k), & 3k \leq x < 3k+2, \quad k \in \mathbb{Z}; \\
2k + 5/4 + 3/4(x - 3k - 2), & 3k+2 \leq x < 3k+3, \quad k \in \mathbb{Z}. 
\end{cases} \)

Define \( xP_1y \iff u_1(x) - u_1(y) > 1 \) and \( xP_2y \iff u_2(x) - u_2(y) > 1 \).

\( P_1 \) and \( P_2 \) satisfy A2: \( (P_1)^{-1} = (P_2)^2 \) and for every \( k, l \in \mathbb{N} \) \( k \geq (3/4)l \) implies \( (P_1)^k \subseteq (P_2)^l \). However, to see that A1 fails to hold take \( x = 1, z = 3.65 \).

\( \exists P_{1,2} \) \( x \) but \( \not\exists (P_1P_2)(x) \). //


Proof. Again consider an example with \( n = 2, X = \mathbb{N}, u_1(x) = x \). For \( 0 \leq x \leq 2 \)
define
\( u_2(x) = \begin{cases} 
(1/2)x, & 0 \leq x < 0.1; \\
0.05 + 2(x - 0.1), & 0.1 \leq x < 0.2; \\
0.75 + (x - 0.2)/2, & 0.2 \leq x < 0.3; \\
x, & 0.3 \leq x < 2. 
\end{cases} \)

Extend \( u_2 \) to \( \mathbb{N} \) in the unique way that will satisfy
\( u_2(x + 1) + 1 = u_2(u_1^{-1}u_2^{-1}(x) + 1) + 1 \)
(it is easy to see that there exists a unique continuous and strongly non-decreasing \( u_2 \) which satisfies this condition).

Finally, define \( P_1 \) and \( P_2 \) by \( u_1 \) and \( u_2 \) respectively with a just-
noticeable difference of 1.

By definition, \( P_1P_2 = P_2P_1 \). However, A2 does not hold: for \( x = 0.1 \) and \( y = 1.12 \) we have \( yP_2x \) but \( \not\exists (yP_2x) \) while for \( z = 0.2 \) and \( w = 0.76 \) \( zP_2w \) holds
while \( zP_1w \) does not. //
1. Interval order is an irreflexive binary relation \( R \) such that \( xRy \) and \( zRw \) imply either \( xRw \) or \( zRy \). It is easy to see that any semi-order is an interval order but not vice versa. Viewing semi-orders as particular interval orders we may suggest another interpretation of semi-orders.

Suppose that \( X \) is a set of signals about real quality of alternatives (like test score signals about students' knowledge). When decision maker observes \( x \) he/she does not know for sure that the true quality is \( x \), but he/she may have in mind for every \( x \) a range of qualities that can generate the signal \( x \). Suppose further that a decision maker prefers \( x \) to \( y \) if and only if he prefers any true quality that can generate \( x \) to any true quality that can produce \( y \). Then if ranges above are the same for all \( x \) in \( X \) such a model induces a semi-order on \( X \).

For instance, in many cases one may assume a fixed error rate of measured quantity (with respect to the true one.) On a logarithmic scale we thus get a fixed range length.

Furthermore, in Lapson, Lugachev (1983) there are several sectors to each of which there corresponds an "error rate" induced by applied technology. Thus, it may serve as an example of "correlated semi-orders".

In these examples semi-ordered structures arise not as a result of psychological peculiarities but rather as a result of imprecise measurement or lack of information.

2. This observation is due to David Schmeidler.
REFERENCES


