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CAPACITY-CONSTRAINED PRICE COMPETITION WHEN UNIT COSTS DIFFER

by

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"Capacity-Constrained Price Competition when Unit Costs Differ"

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Abstract

This paper characterizes the set of Nash equilibria in a price setting duopoly in which firms have limited capacity, and in which unit costs of production up to capacity may differ. Assuming concave revenue and efficient rationing, we show that the case of different unit costs involves a tractable generalization of the methods used to analyze the case of identical costs. However, the supports of the two firms' equilibrium price distributions need no longer be connected and need not coincide. In addition, the supports of the equilibrium price distributions need no longer be continuous in the underlying parameters of the model.

Two applications of our characterization are pursued. In the Kreps-Scheinkman model of capacity choice followed by price competition we show that, unlike in the case of identical costs, Cournot equilibrium capacity levels need not arise as subgame-perfect equilibria. The low-cost firm has greater incentive to price its rival out of the market than exists under Cournot behavior. Our second application is to the analysis of the effects of tariffs and quotas in a model in which a domestic market is supplied by a price setting duopoly consisting of a domestic and a foreign firm. We obtain a strong nonequivalence result.

I. Introduction

In recent years there has been a resurgence of interest in the Bertrand-Edgeworth model of capacity constrained price competition (see Kreps and Scheinkman (1983), Osborne and Pitchik (1986), and Allen and Hellwig (1986)). The impetus for this resurgence can be traced back to three separate sources. First, there is the realization that the Bertrand-Edgeworth model provides an attractive model of short-run competition. It models price formation directly, without making use of the (already overworked) auctioneer. And unlike in the Cournot model, capacity bottlenecks here do not directly constrain the strategic variables (Krishna, 1988). Second, the Bertrand-Edgeworth model displays some features that make it attractive for industrial organization purposes. Its (typically) mixed strategy equilibria capture the observation that many markets display price fluctuations (Sweeney and Comanor (1989)) and/or persistent price dispersion (Varian (1980)). In addition, as emphasized by Ghemawat (1988), unlike models of quantity-based competition, the Bertrand-Edgeworth model does not force equality of competitors' prices, and hence admits interesting short-run share-profitability and concentration-profits relationships. Finally, the Bertrand-Edgeworth model provides an ideal framework for investigating international trade questions: capacity constraints are then interpreted as quotas or VER's (Krishna (1988, 1989)), whereas differences in production costs may result from the imposition of tariffs.

As the last example illustrates, many applications of the Bertrand-Edgeworth model are impossible, or do not become interesting, unless differential efficiencies among players are allowed. Unfortunately, almost without exception, existing treatments have focused on the case where production costs (once capacity is installed) are identical for all participants.[1] While it has been known that different unit costs could be incorporated into these models (Osborne and Pitchik, p. 251) the description of such equilibria has been deemed sufficiently complex so as to lead researchers to ignore the theoretical analysis and any applications that might follow. This paper demonstrates that the analysis is indeed tractable. We characterize equilibrium profits and strategies for efficient rationing and a class of aggregate demand functions yielding concave revenue. This class is slightly more general than in Kreps and Scheinkman (1983), but avoids the complications that may arise when using the general demand of Osborne and Pitchik (1986).

The incorporation of different unit costs involves a straightforward generalization of the methods used to analyze the model with identical costs of production up to capacity. However, unlike the case of identical costs under the specified demand, the supports of the two firms' equilibrium price distributions need not be connected and need not coincide; they may differ by a single point. Furthermore, the supports of the equilibrium price distributions need not be continuous in the underlying unit costs and capacities.

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[1]The one exception of which we are aware is the paper by Gelman and Salop (1983). In that paper, which deals with the incentive of a cost-disadvantaged entrant to keep capacity small, a leader-follower framework is imposed at the price setting stage and the cost-advantaged firm is assumed to have enough capacity to serve the whole market. Gelman and Salop assume that it is always the cost disadvantaged firm which is the leader at the price setting stage. For an analysis of the endogenous determination of the sequencing of moves in this context see Deneckere and Kovenock (1988).

Two applications of our characterization are pursued, indicating how it provides a richer set of models with which to analyze economic phenomena. In the Kreps-Scheinkman model of simultaneous capacity choice followed by simultaneous price setting, we show that, unlike for the case of identical costs, Cournot equilibrium capacity levels (for a marginal cost equal to the sum of the marginal capacity cost and the unit cost of production up to capacity) need not arise as subgame perfect equilibria. Under Cournot, the low cost firm must assume that price will always adjust to clear all quantities supplied to the market. In contrast, in the two-stage game, when deciding on its optimal response to a given capacity of its rival, the low cost firm need not assume that price will adjust to clear all capacity available for supply to the market. In fact, the low cost firm often finds it profitable to choose a high capacity level and price its less efficient opponent out of the market. As a result, subgame perfection may require nondegenerate mixed strategies at the capacity setting stage. However, for the case where the cost of capacity is negligible, we provide a necessary and sufficient condition for Cournot to hold.

Our second application is to the analysis of effects of tariffs and quotas in an international trade model in which a domestic market is supplied by a price-setting duopoly consisting of a domestic and a foreign firm. A tariff acts to raise the foreign firm's unit costs up to capacity, and a quota acts to reduce the foreign firm's capacity. A strong nonequivalence result is derived: If a positive tariff (binding quota) is levied and the resulting equilibrium is one in which neither firm is driven entirely from the market, then there exists no binding quota (positive tariff) that generates the same equilibrium price distribution.

In Section II we present the basic model of price setting duopolists with different unit costs up to capacity. Section III provides the rule for deriving the equilibrium profits. Section IV characterizes the equilibrium price distributions, for arbitrary capacities and unit costs. Section V applies the analysis of the previous two sections to the case where there is a unique Cournot equilibrium. Section VI concludes with the applications.

II. The Model

Consider a market in which two firms produce a homogeneous good. Aggregate demand for the firms' output as a function of price is $d(p)$: $R_+ \to R_+$. We assume that $d(p)$ satisfies the following assumptions:

A1. $\exists\, p_0 > 0$ s.t. $d(p) = 0 \;\forall\, p \geq p_0$ and $d(p) > 0$ if $p < p_0$. $d(p)$ is twice continuously differentiable and strictly decreasing on $(0,p_0)$. Furthermore, $pd(p)$ is strictly concave on $[0,p_0]$, with maximizer p^m.

Each firm i ($i = 1,2$) produces the good at a constant unit cost $0 \leq c_i \leq p_0$ up to a capacity level of k_i. Note that, unlike previous treatments, we do not assume that production costs are the same for both firms. Since players compete in prices and may not be able to serve the entire market, we need to specify a rule that allocates demand in terms of the prices. Following the example of Levitan and Shubik (1972), Kreps and Scheinkman (1983), and Osborne and Pitchik (1986), we assume that demand is allocated efficiently. Thus, if $p_i < p_j$ and $d(p_i) > k_i$, the demand facing firm j is $\max(0,\, d(p_j) - k_i)$. In the case of a tie in prices, the low cost firm sells its capacity first.[2] Thus, if $p_i = p_j = p$ and $c_i < c_j$, the demand faced by firm j is $\max(0, d(p) - k_i)$. For a discussion on the merits of alternative rationing schemes, see Davidson and Deneckere (1986).

Under these assumptions the profit to firm i when it sets price p_i and firm j sets price p_j is

$$
(2.1) \qquad \pi_i(p_i,p_j) = \begin{cases} L_i(p_i) \equiv (p_i - c_i)\min(k_i, d(p_i)) & \text{if } p_i < p_j \\ T_i(p_i) \equiv (p_i - c_i)\min(k_i, \max(0, d(p_i) - I_h^i k_j)), & \text{if } p_i = p_j \\ H_i(p_i) \equiv (p_i - c_i)\min(k_i, \max(0, d(p_i) - k_j)), & \text{if } p_i > p_j \end{cases}
$$

where I_h^i is an indicator that takes on the value 1 if $c_i > c_j$, or $c_1 = c_2$ and $i = 2$, and takes on the value 0 if $c_i < c_j$, or $c_1 = c_2$ and $i = 1$. Here $L_i(p_i)$ refers to the profit from being the low priced seller at p_i, $H_i(p_i)$ the profit from being the high priced seller at p_i, and $T_i(p_i)$ the profit from tying at p_i. Note that depending on the value of I_h^i, the function $T_i(p_i)$ will coincide with either L_i or H_i. The functions L_i and H_i are illustrated in Figure 1.

Let $S_i = [c_i, p_0]$ denote the pure strategy set of firm i, and Σ_i the corresponding set of mixed strategies (cumulative distribution functions on S_i). Note that we have ruled out strategies for firm i which involve pricing below its unit cost.[3] The domain of firm i's profit function can be extended in a natural way to $\Sigma_i \times \Sigma_j$. For $F_j \in \Sigma_j$ define

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[2]To break ties when $c_1 = c_2$, we arbitrarily let firm 1 sell its capacity first.

[3]Although any such price is weakly dominated by setting c_i this assumption is not completely innocuous. See section III for details.

$$(2.2) \qquad \pi_i(p,F_j) = H_i(p)(F_j(p)-\alpha_j(p)) + T_i(p)\alpha_j(p) + L_i(p)(1 - F_j(p))$$

where $\alpha_j(p)$ is the size of the masspoint in F_j at p (if one is present). Finally let $\Pi_i(F_i, F_j) = \int_{c_i}^{p_0} \pi_i(p,F_j)dF_i(p)$.

For any quadruple (k_1,k_2,c_1,c_2) we will now analyze the normal form game $G(k_1,k_2,c_1,c_2)$ with strategy sets Σ_i and payoff functions $\Pi_i(F_i, F_j)$.

III. Equilibrium Profits in $G(k_1, k_2, c_1, c_2)$

In this section we characterize the Nash equilibrium profits of the game $G(k_1,k_2,c_1,c_2)$ for all vectors (k_1,k_2,c_1,c_2) and show that they are uniquely determined. We first appeal to the well-known results of Dasgupta and Maskin (1986) to guarantee existence.

Proposition 1: For any vector of capacities and costs (k_1,k_2,c_1,c_2) there exists a mixed-strategy Nash equilibrium.

Proof: It is easily seen that Theorem 5 of Dasgupta and Maskin (henceforth D-M) applies. The only potentially problematic condition of this theorem, the upper semi-continuity of $\pi_1(p_1,p_2) + \pi_2(p_1,p_2)$ in (p_1,p_2), holds because the sum is continuous at all off-diagonal points and because along the diagonal our sharing rule minimizes the total cost of providing the good. Since revenue is continuous, this means that in approaching a point on the diagonal total profit cannot jump down. #

It is interesting to note that without the sharing rule which allows the low-cost firm to sell its capacity first in the event that $p_1 = p_2$, neither the Dasgupta and Maskin (1986) existence theorems nor the existence results of Simon (1987) apply. In particular, with the sharing rule which divides the market in proportion to capacities when $p_1 = p_2$, the sum of the two firms' profits will generally not satisfy the D-M requirement of upper semi-continuity in (p_1, p_2). The expected profit functions of the two firms will also generally not satisfy Simon's (1987) "complementary discontinuity" property on $\Sigma_1 \times \Sigma_2$.[4,5] In Section IV we will show that, despite the fact that these existence results do not apply to the game with a proportional sharing rule, for any equilibrium of G there is a payoff equivalent equilibrium in the game with a proportional sharing rule, and vice versa.

Before proceeding to characterize the equilibria of the game

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[4]To see this suppose that $d(p) = \max(0, 1 - p)$, $k_1 = 3/4$ and $k_2 = 1/4$. Suppose firm 2 sets $p = 1/4$ with probability 1/2 and $p = 3/8$ with probability 1/2. Look at a sequence of price distributions for firm 1 of the following form: For each n, with probability 1/2 firm 1 sets $p_n = 1/4 - \varepsilon_n$ and with probability 1/2 firm 1 sets $p_n = 3/8 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to$ infinity. The difference between firm 1's expected profit at the limit and the limit of its expected profits has the same sign as $c_1 - 1/8$. The difference between firm 2's expected profit at the limit and the limit of its expected profits has the same sign as $1/8 - c_2$. Thus, for c_1 slightly less than 1/8 and c_2 slightly greater than 1/8 both firms' profits jump down in the limit. It should be emphasized that while expected payoffs do satisfy the complementary discontinuity property on $S_1 \times S_2$ they do not on $\Sigma_1 \times \Sigma_2$, as required by Simon.

[5]D-M's Theorem 5b proves existence using a similar complementary discontinuity assumption on $S_1 \times S_2$. However, the theorem requires the strategy spaces to coincide making it inapplicable in our context. Enlarging the high cost firm's strategy set to equate it with the low cost's causes the complementary discontinuity property to be violated.

$G(k_1,k_2,c_1,c_2)$ we need to establish some notation. Let P_i^L be the set of prices that maximize $L_i(p_i)$ and P_i^H be the set of prices that maximize $H_i(p_i)$. Given our assumptions on demand there exists a unique $p_i^L \in P_i^L$ and $L_i(p_i)$ is continuous and strictly increasing in p_i for $p_i < p_i^L$. Let $H_i^* = H_i(P_i^H)$. If H_i^* is nonzero there is a unique element p_i^H in P_i^H. Furthermore, $H_i(p_i)$ is continuous and strictly increasing in p_i for $p_i < p_i^H$. If $H_i^* = 0$ define $p_i^H \equiv c_i$. Let

$$\underline{p}_i = \min\{p_i\colon L_i(p_i) = H_i^*\}, \quad i = 1,2.$$

Note that $\underline{p}_i \le p_i^H < p_i^L$, $i = 1,2$. In the analysis that follows we shall sometimes write p_i^L, p_i^H, and $\underline{p}_i$ as functions of (k_1,k_2,c_1,c_2) to indicate the dependence of these prices on unit costs and capacities.

We will often make use of the inverse demand function $P(q)$ defined by: $P(q) = d^{-1}(q)$ for $0 < q \le d(0)$, $P(0) = p_0$, and $P(q) = 0$ for $q > d(0)$. Another important expression in the characterization of equilibria is $r_i(k) \equiv \text{argmax}_x\ (P(x + k) - c_i)x$. This is firm i's Cournot best response when its rival puts output k on the market and firm i has a constant unit cost of production of c_i. Note that for $k \ge d(c_i)$, $r_i(k) = 0$, while for $k < d(c_i)$, $r_i(k) > 0$.

Let $(F_1(p),F_2(p))$ be a pair of equilibrium price distributions. Let $\bar{s}_i = \inf\{p\colon F_i(p) = 1\}$ and $\underline{s}_i = \sup\{p\colon F_i(p) = 0\}$ be the bounds of the support of F_i, $i = 1,2$. From our restriction on the strategy spaces, $c_i \le \underline{s}_i \le \bar{s}_i \le p_0$, $i = 1,2$. Without loss of generality suppose c_1 and c_2 are such that firm 1 sells its capacity first in the event that both firms set the same price.

Lemma 1: $\underline{s}_i \geq \mathrm{Max}(c_i, P(k_1 + k_2))$

Proof: From the restriction on S_i, if $P(k_1 + k_2) \leq c_i$ we are finished. Suppose $P(k_1 + k_2) > c_i$. By naming a price p less than $P(k_1 + k_2)$ firm i obtains $(p - c_i)k_i$, which is increasing on $[c_i, P(k_1 + k_2)]$. #

Lemma 2: Suppose $\bar{s}_1 = \bar{s}_2 = \bar{s}$ and each is named with positive probability. Then one of the following conditions holds:

(a) $\underline{s}_i = \bar{s}_i = P(k_1 + k_2)$ and $k_i \leq r_i(k_j)$, $i = 1,2$, $j \neq i$;

(b) there exists an $i \in \{1,2\}$ such that $\bar{s} = c_i \geq c_j$, and for every i such that $\bar{s} = c_i \geq c_j$, $\bar{s} \geq P(k_j)$.

Proof: Suppose the hypothesis of the lemma holds and that $\bar{s} > P(k_1 + k_2)$. Then there exists a firm, say j, which sells less than its capacity with certainty when it sets price $\bar{s}$. There are two cases to be examined. If $\bar{s} > \max(c_1,c_2)$ then j can improve payoffs by reducing price slightly and avoiding the positive probability of a tie at $\bar{s}$. Thus, for $\bar{s} > \max(c_1,c_2)$, $\bar{s}$ must equal $P(k_1 + k_2)$ which, together with Lemma 1, implies that $\underline{s}_i = \bar{s}_i = P(k_1 + k_2)$. Suppose that the condition $k_i \leq r_i(k_j)$, $i = 1,2$, $j \neq i$, did not hold in this case: say, $k_1 > r_1(k_2)$. By naming a price higher than $P(k_1 + k_2)$, firm 1 would obtain a return of $(D(p) - k_2)(p - c_1)$ or, letting $x = D(p) - k_2$, a return of $x(P(x + k_2) - c_1)$. This expression is maximized at $x = r_1(k_2)$, so if $r_1(k_2) < k_1$ setting $P(r_1(k_2) + k_2)$ would dominate setting $P(k_1 + k_2)$. This contradicts equilibrium.

If $\bar{s} \leq \max(c_1,c_2)$ then $\bar{s} = \max(c_1,c_2)$ from the restriction on the strategy sets. Suppose that the second condition in (b) does not hold. Then there exists an i for which $c_i = \max(c_1, c_2) = \bar{s}$ and $P(k_j) > \bar{s}$. But then firm i, which is making zero profit by charging $\bar{s}$, can strictly increase profit by raising price a small amount above $\bar{s}$, since for prices between $\bar{s}$ and $P(k_j)$ i has positive demand, even when it is undercut. #

Lemma 2 implies that if $\bar{s}_1 = \bar{s}_2 = \bar{s}$ and each firm names $\bar{s}$ with positive probability then either $\Pi_i(F_1, F_2) = L_i(\bar{s}) = T_i(\bar{s}) = H_i(\bar{s}) = H_i^*$, i = 1,2 or there exists a firm i such that $\Pi_i(F_1,F_2) = H_i^* = 0$. This allows us to make a first step in determining the equilibrium payoffs:

<u>Lemma 3</u>: $\Pi_i(F_1,F_2) \geq H_i^*$, i = 1,2. If $\bar{s}_i > \bar{s}_j$ then $\Pi_i(F_1,F_2) = H_i(\bar{s}_i) = H_i^*$. If $\bar{s}_1 = \bar{s}_2 = \bar{s}$ then there exists a firm i such that $\Pi_i(F_1,F_2) = H_i(\bar{s}_i) = H_i^*$.

<u>Proof</u>: Since $0 \leq H_i(p) \leq \Pi_i(p,F_j) \leq \Pi_i(F_1,F_2)$ for all $p \in S_i$ we know $\Pi_i(F_i,F_j) \geq H_i^*$, i = 1,2. Suppose $\bar{s}_i > \bar{s}_j$. Since $F_j(\bar{s}_j) = 1$, $\Pi_i(\bar{s}_i, F_j) = H_i(\bar{s}_i) \leq H_i^*$. But $\Pi_i(\bar{s}_i, F_j) = \Pi_i(F_i,F_j) \geq H_i^*$ since (F_1,F_2) is a pair of equilibrium strategies. Thus $\Pi_i(F_i,F_j) = H_i(\bar{s}_i) = H_i^*$. Suppose $\bar{s}_1 = \bar{s}_2 = \bar{s}$. From Lemma 2 if both firms have mass points at $\bar{s}$ then there exists a firm i such that $\Pi_i(F_1,F_2) = H_i(\bar{s}_i) = H_i^*$. So, suppose there is at least one firm, say firm j, with no mass point at $\bar{s}$. Since j has no mass point at $\bar{s}$, $\Pi_i(\bar{s}_i, F_j) = H_i(\bar{s}_i) \leq H_i^*$. But, the previous argument again implies that $\Pi_i(\bar{s}_i, F_j) = \Pi_i(F_1,F_2) \geq H_i^*$, so that $\Pi_i(F_1,F_2) = H_i(\bar{s}) = H_i^*$. #

In equilibrium at least one of the firms, say firm i, must earn a profit equal to H_i^*. The other firm, firm j, earns at least H_j^*. In order to determine the equilibrium payoffs and distributions we will now pin down the identity of a firm making H_i^* (both may do so in equilibrium). Let

$$\underline{p} = \max(\underline{p}_1, \underline{p}_2)$$

<u>Lemma 4</u>: Suppose $\underline{p}_i = \underline{p}$. Then $\Pi_i(F_1,F_2) = H_i^*$.

<u>Proof</u>: By Lemma 3 we know that $\Pi_i(F_1,F_2) \geq H_i^*$. Suppose a strict inequality holds. Then from Lemma 3, $\Pi_j(F_1,F_2) = H_j^*$. Since $L_i(p)$ is strictly increasing and continuous on $[c_i, p_i^L)$ and $p_i^L > \underline{p}_i$, there exists a price $\hat{p}_i > \underline{p}_i$ such that $\Pi_i(F_1,F_2) = L_i(\hat{p}_i) > L_i(\underline{p}_i) = H_i^*$ and such that $L_i(p) < \Pi_i(F_1,F_2)$ for all $p < \hat{p}_i$. Firm i's equilibrium strategy must put no mass below $\hat{p}_i$. But $\underline{p}_i = \underline{p}$ implies $\underline{p}_j \leq \underline{p}_i < \hat{p}_i$. Since $L_j(p)$ is strictly increasing on $[c_j, p_j^L)$ and $p_j^L > \underline{p}_j$, there exists an $\varepsilon > 0$ such that $L_j(\underline{p}_j + \varepsilon) > L_j(\underline{p}_j) = H_j^*$ and such that $\underline{p}_j + \varepsilon < \hat{p}_i$. But then H_j^* cannot be firm j's equilibrium payoff, in contradiction to the hypothesis. #

<u>Lemma 5</u>: Suppose $\underline{p}_j < \underline{p}_i$. Then $\Pi_j(F_1,F_2) > H_j^*$.

<u>Proof</u>: Since $\underline{p}_i = \underline{p}$, Lemma 4 implies that $\Pi_i(F_1,F_2) = H_i^*$. Thus, $\underline{s}_i \geq \underline{p}_i$. But then $\exists\ \delta > 0$ such that $F_i(\underline{p}_j + \delta) = 0$ and $\Pi_j(\underline{p}_j + \delta, F_i) = L_j(\underline{p}_j + \delta) > H_j^*$. Thus, since (F_1,F_2) is an equilibrium pair $\Pi_j(F_1,F_2) > H_j^*$. #

Lemmas 4 and 5 indicate the starting point in determining equilibrium profits and distributions. In the remainder of this section we divide the analysis into two cases, depending on the value of H_i^* for the set of firms for which $\underline{p}_i = \underline{p}$. Lemma 6 deals with the case where for some such firm $H_i^* = 0$. Lemmas 7 through 10 treat the case where $H_i^* > 0$ for any such firm.

<u>Lemma 6</u>: Suppose $\underline{p}_i = \underline{p}$ and $H_i^* = 0$. Then $\Pi_i(F_1,F_2) = 0$ and $\Pi_j(F_1,F_2) = \max_{p\in[c_j,c_i]} L_j(p)$. Furthermore, $\underline{s}_j = \bar{s}_j = \operatorname{argmax}_{p\in[c_j,c_i]} L_j(p)$.

<u>Proof</u>: Since $H_i^* = 0$, $P(k_j) \le c_i$, $\underline{p}_i = c_i$ and, from Lemma 4, $\Pi_i(F_1,F_2) = 0$. But this implies that $F_j(c_i) = 1$ and $c_j \le c_i$. Since firm i's strategy set is $[c_i,p_0]$, firm j must place all mass on $\operatorname{argmax}_{p\in[c_j,c_i]} L_j(p)$. #

<u>Lemma 7</u>: Suppose $\underline{p}_i = \underline{p}$ and $H_i^* > 0$ for $i = 1,2$. Then $\underline{s}_i = \underline{s}_j = \underline{p}$ and if $p_i^H \le p_j^H$ then $\bar{s}_j \le \bar{s}_i = p_i^H$.

<u>Proof</u>: From Lemma 4 we know $\Pi_i(F_1,F_2)=H_i^*$, $i = 1,2$. Thus, it must be the case that $\underline{s}_i \ge \underline{p}$, $i = 1,2$. From Lemma 3 there exists an i such that $\bar{s}_i = p_i^H$ (since $H_i^* > 0$ implies that there is a unique maximizer of H_i, $i = 1,2$). We claim that $p_i^H = \min_\ell p_\ell^H$. For suppose not. Then $\bar{s}_i = p_i^H > p_j^H$, which implies that $\Pi_j(F_1,F_2) > H_j^*$, a contradiction. Thus, $\bar{s}_i = p_i^H \le p_j^H$. It is also clear that $\bar{s}_j \le \bar{s}_i$ for, otherwise, $\Pi_i(\bar{s}_i,F_j) > H_i^*$. To complete the proof we need to show that $\underline{s}_i = \underline{s}_j = \underline{p}$. Suppose there exists an ℓ for which $\underline{s}_\ell > \underline{p}$. We know that $\underline{s}_i,\underline{s}_j \le p_i^H \le p_j^H$. If $p_i^H = \underline{p}$ we have a contradiction. If $p_i^H > \underline{p}$, then firm k, $k \ne \ell$, can set a price slightly above $\underline{p}$ and earn $\Pi_k(p,F_\ell) > H_k^*$, also a contradiction. Thus, $\underline{s}_i = \underline{s}_j = \underline{p}$. #

<u>Lemma 8</u>: If $H_i^* > 0$ and $\underline{p}_i > \underline{p}_j$, $\bar{s}_i = p_i^H$ and $\bar{s}_j \le p_i^H$.

<u>Proof</u>: From Lemmas 4 and 5, $\Pi_i(F_1,F_2) = H_i^*$ and $\Pi_j(F_1,F_2) > H_j^*$. From Lemma 3 it immediately follows that $\bar{s}_i \geq \bar{s}_j$. If a strict inequality holds, then again from Lemma 3, $H_i(\bar{s}_i) = H_i^*$, so $\bar{s}_i = p_i^H$. If $\bar{s}_i = \bar{s}_j$, then since $\Pi_j(F_1,F_2) > H_j^*$, it must be that $\Pi_i(F_1,F_2) = H_i(\bar{s}_i)$, so $\bar{s}_i = p_i^H$. #

<u>Lemma 9</u>: If $H_i^* > 0$ and $\underline{p}_i > \underline{p}_j$ then $\underline{s}_i = \underline{s}_j = \underline{p}_i$.

<u>Proof</u>: Since $H_i^* > 0$, $P(k_j) > c_i$. For $p \geq P(k_j)$, $H_i(p) = 0$, which implies that $p_i^H < P(k_j)$. Furthermore, $p_j^L \geq P(k_j)$ since $L_j(p)$ is increasing on $[c_j,P(k_j)]$ if the interval is nondegenerate, and the inequality clearly holds otherwise. Thus, $p_i^H < p_j^L$ which, together with the fact that $\underline{p}_i \leq p_i^H$ and the assumption that $\underline{p}_i > \underline{p}_j$, implies that $L_j(p)$ is strictly increasing on $[\underline{p}_j,p_i^H]$. Clearly, $\underline{s}_i \geq \underline{p}_i$, since any price named below $\underline{p}_i$ earns a profit for i that is strictly less than H_i^*. Also, $\underline{s}_j \leq \underline{p}_i$, for if the inequality did not hold, firm i could set a price p between $\underline{p}_i$ and $\underline{s}_j$ and earn $L_i(p) > H_i^*$. Finally, $\underline{s}_j \geq \underline{s}_i$, for otherwise there would exist a price p between $\underline{s}_j$ and $\underline{s}_i$ such that $\Pi_j(p,F_i) = L_j(p) > L_j(\underline{s}_j) = \Pi_j(F_1,F_2)$. Combining these inequalities gives us the claim. #

<u>Lemma 10</u>: If $H_i^* > 0$ and $\underline{p}_i > \underline{p}_j$ then $\Pi_j(F_1,F_2) = L_j(\underline{p}_i)$.

<u>Proof</u>: From Lemma 9, $\underline{s}_i = \underline{s}_j = \underline{p}_i$. Since $L_j(\underline{p}_i) > 0$, $\Pi_j(F_1,F_2) \leq L_j(\underline{p}_i)$. Suppose that a strict inequality holds. Then since $L_j(p)$ is continuous on $[\underline{p}_j,p_i^H]$ and $\underline{p}_j < \underline{p}_i$, there exists an $\varepsilon > 0$ such that $L_j(\underline{p}_i - \varepsilon) > \Pi_j(F_1,F_2)$, a contradiction to equilibrium. Thus, $\Pi_j(F_1,F_2) = L_j(\underline{p}_i)$. #

We have now established how to determine the equilibrium profits of the two firms. First, we calculate $\underline{p}_1$ and $\underline{p}_2$. If $\underline{p}_1 = \underline{p}_2$, Lemma 4 tells us that $\Pi_i(F_1,F_2) = H_i^*$, $i = 1,2$. If $\underline{p}_i > \underline{p}_j$, Lemma 4 tells us that $\Pi_i(F_1,F_2) = H_i^*$ and, if $H_i^* > 0$, Lemma 10 tells us that $\Pi_j(F_1,F_2) = L_j(\underline{p}_i)$; if $H_i^* = 0$, then Lemma 6 tells us that $\Pi_j(F_1,F_2) = \text{Max}_{p \le c_i} L_j(p)$. This also establishes that equilibrium profits are unique. Note, however, that when $\underline{p}_i > \underline{p}_j$ and $H_i^* = 0$, our restriction that Σ_i include only prices at or above c_i is essential in pinning down $\Pi_j(F_1,F_2)$. If firm i could set prices below c_i, and the conditions $P(k_j) < c_i$ and $P(k_j) < p_j^L$ hold, there would exist a continuum of equilibria in which firm j receives an equilibrium profit less than $\text{Max}_{p \le c_i} L_j(p)$. Any price p_j of firm j between $\max(P(k_j), \underline{p}_j)$ and $\min(p_j^L, c_i)$ could be supported as a pure strategy equilibrium price for firm j if i put enough mass at, or in every neighborhood above, p_j.[6] Since i plays a weakly dominated strategy in these equilibria we rule them out.

IV. Equilibrium Distributions

We are now ready to characterize the equilibrium strategies in G.

Proposition 2: A pure strategy equilibrium exists if and only if one of the following two conditions holds:

(a) $k_i \le r_i(k_j)$ $\qquad i = 1,2.$

(b) $c_j \ge P(k_i)$ and $c_j \ge \underline{p}_i$ for some i, $j \ne i$.

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[6] In an earlier draft of this paper, Deneckere and Kovenock (1987), we explicitly calculate the implied bounds on equilibrium profits for the case of linear demand.

In case (a) the equilibrium is unique and $p_i = p_j = P(k_1+k_2)$ with probability one. In case (b), if $p_i^L \le c_j$ then firm i sets p_i^L with probability one and firm j uses any strategy placing all mass at or above c_j. If $p_i^L > c_j$ firm i sets $p_i = c_j$ and firm j sets any strategy which deters firm i from raising price; one such strategy is $p_j = c_j$ with probability 1.

<u>Proof</u>: The "if" part of the theorem, uniqueness in case (a) and the characterization in case (b) are straightforward. To prove the "only if" part we consider two cases. First, suppose a pure strategy equilibrium exists with $p_i < p_j$. This implies that $H_j^* = 0$. For, if $H_j^* > 0$ then, since $L_i(p)$ must be maximal at p_i and $H_j(p)$ maximal at p_j, we would have $P(k_i) \le p_i^L < p_j^H$, a contradiction. So suppose $H_j^* = 0$. If $p_j > c_j$ then $p_i^L \le c_j$, for otherwise firm j would make a positive profit by undercutting p_i^L slightly. If $p_j = c_j$ then $p_i^L < c_j$ by assumption. Combining these results, $P(k_i) \le p_i^L \le c_j$, and since $\underline{p}_i \le p_i^L$, $\underline{p}_i \le c_j$. We conclude that condition (b) holds. Furthermore, it is easily seen that whenever (b) holds and $p_i^L \le c_j$, a pure strategy equilibrium with $p_i < p_j$ exists.

Now suppose that a pure strategy equilibrium with $p_i = p_j$ exists. Then $\bar{s}_1 = \bar{s}_2 = \bar{s}$ and Lemma 2 applies. If Lemma 2(a) holds, we are done. If Lemma 2(b) holds, then it is easily seen that $p_i = p_j = c_j$ and that immunity to deviations by firm i requires $\underline{p}_i \le c_j$ and $p_i^L \ge c_j$. #

Proposition 2 delineates the situations in which pure strategy equilibria arise. In case (a) there exists a firm j such that $p_j^H = \underline{p}_j \ge \underline{p}_i$ and $H_j^* > 0$. (These conditions guarantee that $\underline{p}_i = p_i^H$). In case (b) there exists a firm j such that $p_j^H = \underline{p}_j \geq \underline{p}_i$ and $H_j^* = 0$. The situations which remain to be covered are when $p_j^H > \underline{p}_j$ whenever $\underline{p}_j \geq \underline{p}_i$, $j = 1,2$, $i \neq j$. Equilibria then involve nondegenerate mixed strategies (the uniqueness of which will be proven in Appendix C).

First, we will examine the case where $\underline{p}_j > \underline{p}_i$. Lemmas 8 and 9 tell us that $\underline{s}_i = \underline{s}_j = \underline{p}_j$, $\bar{s}_j = p_j^H$, and $\bar{s}_i \leq p_j^H$. Thus, both firms' equilibrium price distributions have supports contained in $[\underline{p}_j, p_j^H]$. Unlike the case where $c_1 = c_2$, we will demonstrate that when $c_1 \neq c_2$ the equilibrium price distributions, F_1 and F_2, need not have the same support. They may differ by one point, an atom at p_j^H. The equilibrium supports need also not be connected, and both firms may have atoms in their equilibrium distributions (though each firm has at most one). We provide some examples in Appendix A.

From Lemma 10, $\Pi_i(F_1, F_2) = L_i(\underline{p}_j)$. Let

$$Q_j(p;\Pi_i^*) = (L_i(p) - \Pi_i^*)/(L_i(p) - H_i(p)),$$

where Π_i^* is the equilibrium profit of firm i. For $p \in [\underline{p}_j, p_j^H]$, $Q_j(p;\Pi_i^*)$ is well-defined and nonnegative. This follows from the fact that the stated assumptions give us $c_i \leq \underline{p}_i < \underline{p}_j < p_j^H < P(k_i) \leq p_i^L$, which means that over the nondegenerate interval $[\underline{p}_j, p_j^H]$, $L_i(p) > H_i(p)$ and $L_i(p) \geq L_i(\underline{p}_j) = \Pi_i^*$. For every price p, $\Pi_i(p, F_j)$ is given by (2.2). For all p in the support of F_i except possibly a set of F_i - measure zero, N_i, we have $\Pi_i(p, F_j) = \Pi_i^*$. Thus, if p is not a mass point of F_j, then $p \in (\text{supp } F_i)\backslash N_i$ implies that $F_j(p) = Q_j(p;\Pi_i^*)$. For $p \notin (\text{supp } F_i)\backslash N_i$, $F_j(p) \geq Q_j(p;\Pi_i^*)$. Setting

$$F_j(p) = \begin{cases} 0 & p < \underline{p}_j \\ Q_j(p;\, \Pi_i^*) & \underline{p}_j \le p < p_j^H \\ 1 & p \ge p_j^H \end{cases}$$

would yield an equilibrium strategy for firm j if, given firm i's equilibrium strategy, it received an expected profit of Π_j^* at all points in the interval $[\underline{p}_j,\ p_j^H]$, and if $Q_j(p;\, \Pi_i^*)$ were nondecreasing in p. Similarly, $F_i(p)$ defined in an analogous fashion would yield an equilibrium strategy for firm i if it received an expected profit of Π_i^* at all points in $[\underline{p}_j,\ p_j^H)$ given firm j's equilibrium strategy, and if $Q_i(p;\, \Pi_j^*)$ were nondecreasing in p. Unfortunately, while it is easily shown that $Q_i(p;\, \Pi_j^*)$ must be increasing on $[\underline{p}_j,\ p_j^H]$, with $Q_i(\underline{p}_j;\Pi_j^*) = 0$ and $Q_i(p_j^H;\Pi_j^*) = 1$, it is not always the case that $Q_j(p;\, \Pi_i^*)$ is nondecreasing. In Appendix A we provide an example where $Q_j(p;\, \Pi_i^*)$ decreases on a subinterval of $[\underline{p}_j, p_j^H]$. The following Lemma provides some characteristics of Q_i and Q_j that will be useful in constructing a pair of equilibrium strategies. In this lemma, we fix $\Pi_i = \Pi_i^*$ and $\Pi_j = \Pi_j^*$ and view Q_i and Q_j as functions of p only.

<u>Lemma 11</u>: Suppose $p_j^H > \underline{p}_j > \underline{p}_i$ and $H_j^* > 0$. Then $Q_i(p)$ satisfies the following properties:

(a) $Q_i(\underline{p}_j) = 0,\quad Q_i(p_j^H) = 1.$

(b) Q_i is differentiable at every point in $(\underline{p}_j,\ p_j^H)$, except at $P(k_j)$, when $P(k_j) \in (\underline{p}_j, p_j^H)$.

(c) Q_i is strictly increasing on $[\underline{p}_j,\ p_j^H)$.

(d) Q_i is concave on the interval $[\underline{p}_j,\ p_j^H]$ and is twice continuously differentiable except at $P(k_j)$, when $P(k_j) \in (\underline{p}_j, p_j^H)$. In that case, $\lim_{p\uparrow P(k_j)} dQ_i/dp > \lim_{p\downarrow P(k_j)} dQ_i/dp$.

The function $Q_j(p)$ satisfies the following properties:

(e) $Q_j(\underline{p}_j) = 0$, $Q_j(p_j^H) < 1$.

(f) Q_j is differentiable at every point in $(\underline{p}_j, p_j^H)$, except at $P(k_j)$ when $P(k_j) \in (\underline{p}_j, p_j^H)$. In that case, $\lim_{p\downarrow P(k_j)} dQ_j/dp > 0$ and $\lim_{p\uparrow P(k_j)} dQ_j/dp < \lim_{p\downarrow P(k_j)} dQ_j/dp$.

(g) Q_j is strictly increasing on the interval $[\underline{p}_j, p_j^H]$ except possibly on a single subinterval of the form $[a,b]$ where $a > \underline{p}_j$ and $b = \min[P(k_j), p_j^H]$. If $P(k_j) \le \underline{p}_j$, Q_j is increasing everywhere on $[\underline{p}_j, p_j^H]$.

(h) Q_j is locally concave where it is nondecreasing and differentiable.

(i) A necessary but not sufficient condition for Q_j to decrease in the interval is that $c_j < c_i$.

<u>Proof</u>: See Appendix B. #

Given Lemma 11 it is now easy to construct a pair of equilibrium strategies. Following Osborne and Pitchik (1986), let

$$IQ_j(p;\Pi_i^*) = \max_{\underline{p}_j \le x \le p} Q_j(x;\Pi_i^*)$$

be the nondecreasing cover of Q_j on $[\underline{p}_j, p_j^H]$. Note that from Lemma 11, IQ_j equals Q_j except possibly on some interval contained in $(\underline{p}_j, p_j^H]$. Then the strategy

$$F_j(p) = \begin{cases} 0 & p < \underline{p}_j \\ IQ_j(p;\, \Pi_i^*) & \underline{p}_j \le p < p_j^H \\ 1 & p \ge p_j^H \end{cases}$$

is an equilibrium strategy for firm j. To understand why $F_j(p)$ is an equilibrium strategy note first that $F_j(p)$ is nondecreasing, nonnegative, right-continuous, and is less than or equal to one for all p. It is therefore a strategy. When firm i sets a price $p \in [\underline{p}_j,\, p_j^H]$ for which $Q_j(p;\, \Pi_i^*) = IQ_j(p;\, \Pi_i^*)$ it earns its equilibrium profit. If firm i sets a price $p \in [\underline{p}_j,\, p_j^H]$ for which $Q_j(p;\, \Pi_i^*) < IQ_j(p;\, \Pi_i^*)$ it earns strictly less than its equilibrium profit. Thus, no such price will be set by firm i. From Lemma 11 we already know that $Q_i(p;\, \Pi_j^*)$ is increasing over $[\underline{p}_j, p_j^H]$. Given F_j, if firm i were indifferent between all prices in the interval, Q_i would be an equilibrium strategy, since it makes j indifferent between all prices in the interval, and earn a strictly lower profit elsewhere. However, since $F_j(p)$ may be strictly greater than $Q_j(p;\, \Pi_i^*)$ for $p \in [\underline{p}_j,\, p_j^H]$ firm i may not be indifferent; it will attach zero measure to the set of prices for which a strict inequality holds. Since firm j also attaches zero probability to intervals where the strict inequality holds (except at p_j^H, which has a mass point) we know that $F_i(p) \ge Q_i(p;\, \Pi_j^*)$ over these intervals. Since firm i must set $F_i(p) = Q_i(p;\, \Pi_j^*)$ at points in the support of $F_j(p)$, in order to remain an admissible strategy $F_i(p)$ must place a mass point at the start of any interval for which $F_j(p) > Q_j(p;\, \Pi_i^*)$, the size of which equals the increase in $Q_i(p;\, \Pi_j^*)$ over that interval. Formally, let I be the left hand closure of the subinterval of $[\underline{p}_j,\, p_j^H)$ on which $IQ_j(p;\, \Pi_i^*) > Q_j(p;\, \Pi_i^*)$ and define $\mathcal{A} = [\underline{p}_j,\, p_j^H]\backslash I$. Then

$$F_i(p) = \begin{cases} 0 & p < \underline{p}_j \\ Q_i(p;\Pi_j^*) & p \in \mathcal{A} \\ \inf_{x>p,\, x\in\mathcal{A}} Q_i(x;\Pi_j^*) & p \in I \\ 1 & p > p_j^H \end{cases}$$

is an equilibrium strategy for firm i.

The last case to be examined is when $p_j^H > \underline{p}_j = \underline{p}_i$ and $H_j^* > 0$ for some j, $i \neq j$. Without loss of generality, we may assume that $H_i^* > 0$, since if $H_i^* = 0$ we have $\underline{p}_i = p_i^H$ and the problem falls under the analysis of case (b) of Proposition 2. But $H_i^* > 0$ implies that $p_i^H > \underline{p}_i$, for otherwise $\underline{p}_i = p_i^H = P(k_1 + k_2)$ and we would have $\underline{p}_j = p_j^H = P(k_1 + k_2)$ as well. From Lemma 4 we know that $\Pi_k(F_1, F_2) = H_k^*$, $k = 1,2$. Let $p^H \equiv \min(p_1^H, p_2^H)$. Suppose that firm i is a firm for which $p_i^H = p^H$. From Lemma 7 we know that $\underline{s}_i = \underline{s}_j = \underline{p}_i = \underline{p}_j$ and $\bar{s}_j \leq \bar{s}_i = p_i^H$. Thus the two firms' supports are contained in $[\underline{p}_i, p_i^H]$. To derive the equilibrium strategies note that for $k = 1,2$, $H_k(p)$ and $L_k(p)$ are increasing over this interval, which implies that $Q_k(p)$ is then increasing as well. If for firm i, $p_i^H = p^H$ then $Q_j(p^H) = 1$, $j \neq i$. If $p_j^H > p^H$ then $Q_i(p^H) < 1$, so i must have a mass point at p^H. The characterization of the equilibrium proceeds as in the previous case with Q_i and Q_j determining the equilibrium pair of strategies. Since both of these functions are increasing, there are no "gaps" in the supports of the equilibrium strategies.

This completes our characterization of the equilibrium distributions. In Proposition 2 and Appendix C we demonstrate that uniqueness obtains, except possibly for the case where $\underline{p}_i \geq \underline{p}_j$ and $H_i^* = 0$ for some i, $j \neq i$.

One consequence of the above analysis is that when $\underline{p}_i = \underline{p}_j$ then $\bar{s}_i = \bar{s}_j = p^H$. Coupled with our derived rule for determining the equilibrium price distributions this implies that the supports of the equilibrium price distributions need not be continuous in the underlying parameters (k_1,k_2,c_1,c_2). For suppose that $p_j^H > p_i^H = p^H$. A small change in any one of these parameters making $\underline{p}_j > \underline{p}_i$ (say, for instance a slight increase in c_j) would lead firm j to not only place mass in some neighborhood above p_i^H but, also, to place a mass point (albeit a small one) at p_j^H. Thus, there is a discrete jump in the upper bound of the support of firm j's distribution.

In concluding our characterization of the Nash equilibria of the game $G(k_1,k_2,c_1,c_2)$ it is important to note that equilibrium payoffs are invariant with respect to the choice of a sharing rule in the event that $p_1 = p_2$. Equilibrium strategies are also invariant except at values of (k_1,k_2,c_1,c_2) for which the game is as in case (b) of Proposition 2 and satisfies the added restriction that $p_i^L \geq c_j$.

Let us compare the equilibria with our sharing rule (SR1) to the equilibria with some alternative sharing rule (SR2).[7] First, we claim that for every equilibrium derived under SR1 there is a corresponding payoff equivalent equilibrium using SR2. To see this, suppose (F_1,F_2) is an equilibrium with SR1. If the sets of jump-points of the two distributions, $J(F_i)$, i = 1,2, have an empty intersection, then the same strategies yield the same expected payoffs under SR2. Furthermore, any firm i could improve its payoff only by deviating to a masspoint in the distribution of its opponent. Let p* be such a point. Suppose $\Pi_i^{SR2}(p^*,F_j) > \Pi_i^{SR1}(F_1,F_2)$. Then, except for the case when $p^* = c_i$, we can define a sequence $p_n \uparrow p^*$,

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[7] For example, $T_i(p_i) = (p_i - c_i)\min(k_i,\ k_i d(p)/(k_1 + k_2))$.

$p_n \in S_i$, such that $\lim_{p_n \uparrow p^*} \Pi_i^{SR1}(p_n,F_j) \geq \Pi_i^{SR2}(p^*,F_j) > \Pi_i^{SR1}(F_1,F_2)$; for $p^* = c_i$, $\Pi_i^{SR2}(p^*,F_j) = \Pi_i^{SR1}(p^*,F_j) = 0$. Thus, F_i was not a best response to F_j under SR1, contradicting equilibrium.

Now suppose that the equilibrium strategies under SR1 do have common masspoints. As we saw in the construction above, if $p \in J(F_1) \cap J(F_2)$, then $p = P(k_1 + k_2)$ or $p = \max(c_1,c_2)$. In the first case it is clear that a change in the sharing rule has no effect on the equilibrium strategy since it has no effect on the payoffs at p. In the second case, switching from SR1 to SR2 alters the set of equilibrium strategies if for some j and $i \neq j$, $c_j < c_i$, since under SR2 it is not generally the case that $L_j(p) = T_j(p)$ at $p = c_i > c_j$. Observe, however, that under SR1, $\Pi_i(F_1,F_2) = 0$ so that there exists a continuum of payoff-equivalent equilibria in which firm i distributes all of its mass in the half-open interval above c_i in such a way as to yield $p_j = c_i$ a best response. These equilibria obviously remain equilibria under SR2. We conclude that changing the sharing rule leaves equilibrium profits invariant, and alters equilibrium strategies only in the case where SR1 yields a zero profit to the high cost firm.

While we shall not prove a converse to this proposition it is clear that the equilibrium construction here and in section III could be repeated for any other sharing rule. Except for Lemma 2 part (b) and Proposition 2 part (b), which deal with pure strategy equilibria arising when one firm prices at its unit cost, the analysis would be similar. Hence we obtain:

Proposition 3: For every sharing rule when $p_1 = p_2$, $G(k_1,k_2,c_1,c_2)$ has an equilibrium in mixed strategies. Equilibrium payoffs are uniquely determined and coincide for all sharing rules. Equilibrium strategies are unique and are the same for all sharing rules except possibly in the case where one firm earns zero profit.

V. The Ranking of the $\underline{p}_i$

As shown in Sections III and IV, the equilibrium profits and distributions depend crucially on the ranking of the $\underline{p}_i$. The characterization of this ranking is greatly simplified when the quantity setting game with constant marginal costs of production c_i $(i = 1,2)$, has a unique Cournot equilibrium. Since A.1 is insufficient to guarantee this, we will henceforth slightly strengthen the concavity of revenue $pd(p)$:[8]

<u>A.2</u>: The function $d'(p) + pd''(p)$ is strictly negative on $(0,p_0)$.

Proposition 4 below characterizes the regions in (k_1,k_2) space for which $\underline{p}_1 > \underline{p}_2$ and for which $\underline{p}_2 > \underline{p}_1$. Before stating the result, we need to introduce some notation. Let $\Delta = \{(k_1,k_2)\colon k_1 \le r_1(k_2) \text{ and } k_2 \le r_2(k_1)\}$. Also let $B_i(k_j) = \max_{p \in S_i} \{(p - c_i)(d(p) - k_j)\}$ and let $b_i(k_j)$ denote the corresponding maximizer.

<u>Proposition 4</u>: Suppose (A.1) and (A.2) hold, and suppose $c_2 > c_1$. Then:

(i) If $r_1(0) < d(c_2)$, there exists a continuous function $\theta\colon [0,\infty) \to [0,d(c_1)]$ such that $\underline{p}_2 \gtrless \underline{p}_1$ whenever $k_2 \gtrless \theta(k_1)$ and $(k_1,k_2) \notin \Delta$. Furthermore, the function θ satisfies $\theta(k_1) = r_2(k_1)$

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[8]Alternative assumptions that ensure uniqueness of the Cournot equilibrium (such as strict concavity of the function $q \to qP(q)$) would do equally well here.

for $k_1 \in [0, k_1^c]$, $r_2(k_1) < \theta(k_1) < k_1$ for $k_1 \in (k_1^c, d(c_2)]$, and $\theta(k_1) = \Phi(c_1, c_2)$ for $k_1 \in [d(c_2), \infty)$. Here $\Phi(c_1, c_2)$ denotes the solution (in k_2) to the equation: $(c_2 - c_1)d(c_2) = B_1(k_2)$.

(ii) If $r_1(0) \geq d(c_2)$, then $\underline{p}_2 > \underline{p}_1$ whenever $(k_1, k_2) \notin \Delta$.

Furthermore, in each case, $(k_1, k_2) \in \Delta$ implies $\underline{p}_1 = \underline{p}_2$.

Proof: See Appendix D.

Figure 2 illustrates the case where $r_1(0|c_1) < d(c_2)$. For $(k_1, k_2) \in \Delta$, $\underline{p}_1 = \underline{p}_2$ and Proposition 2 guarantees a pure strategy solution with prices equal to $P(k_1 + k_2)$. The pair of "Cournot" capacities for constant unit costs (c_1, c_2) is given by $\underset{\sim}{k}^c$. The function $\theta(k_1)$ emanating from $\underset{\sim}{k}^c$ and taking on the constant value $\Phi(c_1, c_2)$ for $k_1 \geq d(c_2)$ divides the remainder of the capacity space into two regions: for $k_2 < \theta(k_1)$ the low cost firm determines the lower bound of the supports of the equilibrium distributions, and for $k_2 > \theta(k_1)$ the high cost firm determines the bound. Observe that since $\theta(k_1) < k_1$, the high cost firm always determines the lower end of the supports when $k_2 \geq k_1$.

From the arguments in Section III, it follows that $\Pi_i = B_i(k_j)$ and $\Pi_j = (\underline{p}_i - c_j)k_j$ whenever $\underline{p}_i > \underline{p}_j$ $(j \neq i)$. The equilibrium distributions are in nondegenerate mixed strategies, except in the northeast corner bounded below by $k_1 = d(c_2)$ and to the left by $k_2 = \Phi(c_1, c_2)$. In this last range, firm 1 always plays a pure strategy in equilibrium, while firm 2's equilibrium strategy may be pure or mixed. The functions $\Psi_i(k_j, c_i)$, $i = 1,2$, indicated in the figure are defined by $\Psi_i(k_j, c_i) = \max_q \{[P(q) - c_i]q = B_i(k_j)\}$. For $k_i \leq \Psi_i(k_j, c_i)$ firm i is capacity constrained when it sets $\underline{p}_i$ and thereby undercuts its rival. In order to facilitate comparison, it is worth noting that in the case of identical unit costs up to capacity, $c_1 = c_2 = c$, the curves $\Psi_1(k_2,c)$ and $\Psi_2(k_1,c)$ intersect along the diagonal, as do the curves $r_1(k_2|c)$ and $r_2(k_1|c)$. The curve $\Theta(k_1)$ then coincides with the diagonal for $k_1 \in [k_1^c, d(c)]$, and $\Phi(c,c)$ coincides with the vertical line $k_2 = d(c)$.

Similar analysis can be carried out for the case where $r_1(0|c_1) \geq d(c_2)$. For costs in this range, the Cournot best reply functions do not intersect in the positive quadrant (see figure 3). This greatly facilitates the analysis since then $\underline{p}_2 > \underline{p}_1$ for all $(k_1,k_2) \notin \Delta$.

Equilibrium strategies can be derived using the methods of Section IV. When $\underline{p}_2 > \underline{p}_1$, the discussion in Section IV and part (i) of Lemma 11 guarantees that $Q_i(p;\Pi_j^*)$ is nondecreasing for $i = 1,2$, so that the equilibrium strategy of firm i coincides with Q_i over the interval $[\underline{p}_2, p_2^H)$. Thus, closed-form expressions are easily derived. When $k_1 > k_2$, in certain cases where $\underline{p}_1 > \underline{p}_2$, Q_1 may be decreasing over some subinterval of $(\underline{p}_1, p_1^H]$, as indicated in Lemma 11. The derivation of equilibrium strategies is thus somewhat more complex, but is easily done on a case-by-case basis.

VI. Applications

a. Capacity Choice

Kreps and Scheinkman (1983) study the game in which firms first simultaneously choose capacities and then simultaneously select prices. Assuming that firms have identical unit cost of production (up to capacity), that inverse demand is concave,[9] and that the cost of capacity is increasing and convex, they show that the unique subgame perfect equilibrium outcome of the game[10] coincides with the Cournot outcome (where both production and capacity costs are taken into account). In this section, we analyze the two-stage game, but relax the assumption of identical unit production costs. Throughout, we also relax the demand assumptions to A.1 and A.2. These assumptions guarantee the existence of a unique Cournot equilibrium in the quantity setting game where capacity costs are taken into account. For simplicity, we assume that the capacity cost is constant per unit; we denote this constant by $r > 0$.

The requirement of subgame perfection allows us to reduce our study of equilibria of the two-stage game to those of a single-stage game, where the payoffs accruing to firms after simultaneous capacity choices correspond to the ones in G, minus capacity costs. We will refer to this game and its associated payoff functions as $\Gamma(c_1,c_2,r)$ and $\Pi_i(k_1,k_2,c_1,c_2,r)$, respectively. The existence of a Nash equilibrium (in mixed strategies) to Γ is immediate from the continuity of $\Pi_i = L_i$ (max $(\underline{p}_1,\underline{p}_2)$) and Glicksberg's (1952) theorem. Unlike in the case of identical unit production costs, the Nash equilibria of Γ need not coincide with Cournot. In fact, as the following example demonstrates, there may be no equilibrium in which firms choose determinate capacities.

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[9] Osborne and Pitchik (1986) analyze this game under more general assumptions on demand, but still with identical production and capacity costs (implying existence, but not uniqueness of Cournot).

[10] Kreps and Scheinkman actually show that the game has a unique Nash equilibrium. We will not consider imperfect equilibria in this paper.

Suppose that $d(p)=\max(0,1-p)$ and let $c_1=0$, $c_2=.3$, and $r = 0^+$.[11] Figure 4 illustrates the best response functions for this case. As can be seen in the figure, when the high cost firm's capacity is not too large, the low cost firm's best response coincides with the Cournot best response function for unit cost $c_1 + r = 0^+$. When the high cost firm's capacity reaches a critical level, k_2^* (which depends on c_1,c_2 and r), the low cost firm's best response jumps to $k_1 = 1 - c_2$. Firm 1 finds it most profitable to respond to a capacity greater than k_2^* by choosing a capacity level that would allow it to accommodate all demand when setting $p_1 = c_2$. This enables it to price its rival out of the market in the price setting subgame. In the example here, k_2^* is strictly less than k_2^c, the Cournot output level for firm 2. When this occurs, no equilibrium of $\Gamma(c_1,c_2,r)$ will coincide with the Cournot equilibrium. Moreover, since the best response functions do not intersect, no pure strategy equilibrium exists.

A complete characterization of the equilibria of the two-stage game when Cournot does not result is beyond the scope of this paper. In the remainder of this section we deal with the problem of determining when the Cournot outcome obtains. The next two propositions analyze the case where the cost of capacity is negligible, $r = 0^+$. Proposition 5 provides a necessary and sufficient condition for Cournot to arise as an equilibrium in the game $\Gamma(c_1,c_2,0^+)$. With some additional restrictions on demand, Proposition 6 provides a partition of the cost space (c_1,c_2) into regions where Cournot does and does not hold. We conclude our analysis with a discussion of the case where r is not negligible. We continue to assume that $c_1 \leq c_2$; (k_1^c,k_2^c) and (Π_1^c,Π_2^c) denote the Cournot outputs and profits.

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[11] By 0^+ we mean an infinitesimally small positive number.

<u>Proposition 5</u>: A necessary and sufficient condition for (k_1^c, k_2^c) to be an equilibrium of the game $\Gamma(c_1, c_2, 0^+)$ is that:

(a) For every $k_1 > k_1^c$, $\underline{p}_1(k_1, k_2^c) \geq \underline{p}_2(k_1, k_2^c)$

(b) For every $k_2 > k_2^c$, $\underline{p}_2(k_1^c, k_2) \geq \underline{p}_1(k_1^c, k_2)$.

<u>Proof</u>: We will prove that firm 1 has no incentive to deviate from (k_1^c, k_2^c) if and only if (a) holds. The proof for firm 2 is analogous.

(Sufficiency) Suppose that (a) holds. Then, by Lemma 4, for every $k_1 > k_1^c$, $\Pi_1(k_1, k_2^c) = H_1^*$. But $H_1^* = \Pi_1^c$, since H_1^* is invariant with respect to k_1 above k_1^c.

(Necessity) Suppose $\underline{p}_1(k_1, k_2^c) < \underline{p}_2(k_1, k_2^c)$ for some $k_1 > k_1^c$. Then by Lemma 5, $\Pi_1(k_1, k_2^c) > H_1^* = \Pi_1^c$. #

In the analysis that follows, we restrict our attention to cost pairs for which $r_1(0) < d(c_2)$. If the reverse inequality holds, firm 1 may price at its monopoly level and undercut firm 2's unit cost. Therefore, in the game $\Gamma(c_1, c_2, 0^+)$, firm 1 sets its capacity equal to its monopoly output and firm 2 chooses a capacity of zero. This outcome coincides with the Cournot equilibrium when unit costs are (c_1, c_2).

Suppose now that $r_1(0) < d(c_2)$. Part (i) of Proposition 4 then implies that $\underline{p}_2(k_1^c, k_2) \geq \underline{p}_1(k_1^c, k_2)$ for all $k_2 > k_2^c$, so that firm 2 <u>never</u> has an incentive to deviate from k_2^c. It can also be seen from Figure 2 that when $k_2^c > \Phi(c_1, c_2)$, condition (a) of Proposition 5 will necessarily be violated at $k_1 = d(c_2)$, and hence Cournot will not arise as a subgame perfect equilibrium of the two-stage game. Referring back to the definition of Φ, this will happen whenever $(c_2 - c_1)d(c_2) > H_1^*(k_2^c)$: by choosing a capacity of $d(c_2)$ and charging a price of c_2 firm 1 can credibly threaten to drive firm 2 out of the market, and thereby increase its profits above the Cournot level.

As should be clear from Figure 2, it is possible that Cournot does not arise even when $(c_2 - c_1)d(c_2) \leq H_1^*(k_2^c)$. Nevertheless, as is shown in Proposition 6, there exists a restricted (but still interesting) class of demand functions for which Cournot does obtain when $(c_2 - c_1)d(c_2) \leq H_1^*(k_2^c)$:

<u>Proposition 6</u>: Suppose that $d'' \leq 0$ and $d''' \geq 0$, and suppose that $r_1(0) < d(c_2)$. A necessary and sufficient condition for Cournot to be an equilibrium of the game $\Gamma(c_1,c_2,0^+)$ is then that $H_1^*(k_2^c) \geq (c_2 - c_1)d(c_2)$.

<u>Proof</u>: See Appendix E.

Figure 5 shows the cost pairs (c_1,c_2) for which Cournot is not an equilibrium outcome of $\Gamma(c_1,c_2,0^+)$ when $d(p) = \max(0,1 - p)$. While Proposition 6 analyzes only the case in which $r = 0^+$, it is easily shown that for a range of capacity costs above zero there will still be unit cost pairs (c_1,c_2) for which the Cournot result does not hold. For these cost pairs, there will be no subgame perfect equilibrium in which the two firms use pure strategies in setting capacities. As the cost of capacity becomes larger, the range of unit costs up to capacity for which Cournot does not hold gets smaller. Computations carried out for the linear example show that when $r \geq .075$ all equilibria involve Cournot capacities.

b. Tariffs vs. Quotas

Our model provides a natural framework for examining the effects of tariffs and quotas in a duopolistic setting.[12] Suppose firm 1 is a domestic firm and firm 2 a foreign firm, each producing for the domestic market only. Let demand be given as in section II. We assume that the firms are capacity-constrained price setters with given capacities and unit costs of production up to capacity. In the absence of intervention in the market through a tariff or quota, the firms play the game $G(k_1, k_2, c_1, c_2)$.

The imposition of a tariff at a fixed level t is assumed to raise the unit cost of the foreign firm in providing the good to the domestic market to $c_2^t \equiv c_2 + t$. Thus, when a tariff is levied the firms play the game $G(k_1, k_2, c_1, c_2^t)$. The imposition of a quota at a level strictly less than the foreign firm's capacity restricts that capacity to the level of the quota. We shall refer to such a quota as a "binding quota."[13] The quota-constrained capacity level of the foreign firm will be denoted k_2^q. Thus, if a binding quota is levied on the foreign firm, firms play the game $G(k_1, k_2^q, c_1, c_2)$.

While there are many intriguing questions which arise in the context of this model, one topic which has received widespread attention in the

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[12] For a treatment of some of these issues in the context of a capacity constrained price game with differentiated products see Krishna (1988, 1989). Hwang and Mai (1988) examined the equivalence of tariffs and quotas in a conjectural variations model.

[13] This terminology has been used elsewhere to refer to a quota which strictly reduces the quantity sold in the market by the foreign firm at given prices. In our model, since firms may not sell all of their capacity in equilibrium, this need not be the case.

literature is whether tariffs and quotas are in any sense equivalent. The following proposition provides what we believe to be a very strong nonequivalence result:

Proposition 7: Starting from an initial position (k_1, k_2, c_1, c_2), suppose a positive tariff (binding quota) is levied such that in the resulting equilibrium neither firm is driven entirely from the market. Then there exists no binding quota (positive tariff) which generates the same equilibrium price distributions.

Proof: We prove the statement for a tariff levied. The converse will then follow immediately. A tariff transforms the game to $G(k_1, k_2, c_1, c_2^t)$ where $c_2^t > c_2$. Let $\underline{p}_i(t)$, $p_i^H(t)$, $i = 1,2$, be the critical prices of the two firms in the transformed game. In equilibrium, one of the following must hold: (i) $\underline{p}_1(t) > \underline{p}_2(t)$, (ii) $\underline{p}_2(t) > \underline{p}_1(t)$, or (iii) $\underline{p}_2(t) = \underline{p}_1(t)$.

We look first at case (i). Since, by assumption, neither firm is driven from the market, we know $H_1^* > 0$. With $\underline{p}_1(t) > \underline{p}_2(t)$, $\underline{p}_1(t) < p_1^H(t)$, and by Lemmas 8 and 9, $\underline{s}_1 = \underline{s}_2 = p_1(t)$ and $\bar{s}_2 \le \bar{s}_1 = p_1^H(t) = p_1^H(k_2, c_1) = b_1(k_2)$. By Lemma 11, firm 1 has a masspoint at $p_1^H(t)$, $\alpha_1(p_1^H(t)) > 0$.

Given any binding quota k_2^q, the resulting price distribution will be identical to that under the tariff only if these same conditions hold. Let $\underline{p}_i(q)$ and $p_i^H(q)$, $i = 1,2$, be the critical prices under the quota. Then we can obtain an identical distribution only if $\underline{p}_1(q) = \max(\underline{p}_1(q), \underline{p}_2(q))$ and $\alpha_1(p_1^H(q)) > 0$ for, otherwise, from the uniqueness of the equilibrium distribution and the characterization of equilibrium in section IV, we could not obtain a nondegenerate mixed strategy equilibrium in which, for some price p, $\bar{s}_2 \le \bar{s}_1 = p$ and $\alpha_1(p) > 0$. However, since $p_1^H(q) > p_1^H(t)$ the resulting equilibrium distributions are not identical.[14]

Case (ii) follows by a similar argument applied to $\underline{p}_2(t)$ and $p_2^H(t)$. With $\underline{p}_2(t) > \underline{p}_1(t)$ and $H_2^* > 0$, $\underline{p}_2(t) < p_2^H(t)$, and by Lemmas 8 and 9 $\underline{s}_1 = \underline{s}_2 = \underline{p}_2(t)$ and $\bar{s}_1 \le \bar{s}_2 = p_2^H(t) = p_2^H(k_1,c_2^t) = b_2(k_1,c_2^t)$. By Lemma 11, firm 2 has a masspoint $\alpha_2(p_2^H(t)) > 0$. If a quota is to duplicate this distribution it too must yield a nondegenerate mixed strategy distribution in which $p_2^H(t)$ is the upper bound of the union of the firms' supports and firm 2 has a masspoint at this price. This can only happen if $\underline{p}_2(q) = \max(\underline{p}_1,(q),\underline{p}_2(q))$ and $p_2^H(q) > \underline{p}_2(q)$. But if this holds then $p_2^H(q) < p_2^H(t)$, since the former price equals $p_2^H(k_1,c_2) = b_2(k_1,c_2)$.

Suppose now that case (iii) holds, $\underline{p}_1(t) = \underline{p}_2(t) = \underline{p}(t)$. Then from Lemma 7, $\underline{s}_1 = \underline{s}_2 = \underline{p}(t)$. We consider two subcases.

(a) Suppose $\underline{p}(t) = P(k_1 + k_2)$. Then we are in a pure strategy region and a binding quota must have $p = P(k_1 + k_2^q) > P(k_1 + k_2)$. (Note that this is the only part the proposition where a nonbinding quota will duplicate a tariff. See Footnote 14.)

(b) Suppose $\underline{p}(t) > P(k_1 + k_2)$. If $p_i^H(t) < p_j^H(t)$, from the analysis of section IV, $\bar{s}_i = \bar{s}_j = p_i^H(t)$ and firm i has a masspoint at $p_i^H(t)$. If $i = 1$ then the result follows from an argument similar to that of case (i) (the case of a nonbinding quota is covered by an argument similar to that in

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[14]In this case a nonbinding quota will also not duplicate the equilibrium distributions under a tariff since it can be shown that, with $p_1^H(t) > \underline{p}_1(t) > \underline{p}_2(t)$, $Q_1^0(p;L_2(\underline{p}_1(t))) < Q_1^t(p;L_2(\underline{p}_1(t)))$, $\forall\, p \in (\underline{p}_1(t),p_1^H(t))$, where the superscripts 0 and t denote the function Q_1 calculated for the pair (k_2,c_2) and (k_2,c_2^t), respectively.

footnote 14). If $i = 2$, then the result follows from an argument similar to that in case (ii). Finally, suppose $p_1^H(t) = p_2^H(t) = p^H(t)$. Then neither firm has a masspoint at the upper bound of the equilibrium supports, $\bar{s}_1 = \bar{s}_2 = p^H(t)$. In order to duplicate the equilibrium distributions the critical prices under a quota must satisfy $\underline{p}_1(q) = \underline{p}_2(q) = \underline{p}(t)$ and $p_1^H(q) = p_2^H(q) = p^H(t)$ (otherwise, there would exist a masspoint). But $p_1^H(t) < p_1^H(q)$ implies that this cannot hold. #

It should be noted that in Proposition 7 the one case where the restriction of levying only binding quotas is of importance in establishing nonequivalence is the case where $G(k_1,k_2,c_1,c_2^t)$ is such that $k_1 \leq r_1(k_2,c_1)$ and $k_2 \leq r_2(k_1,c_2^t)$. When capacities are below the Cournot best response functions given the tariff levied, the equilibrium price distributions under the tariff are pure strategies, with each firm charging $p = P(k_1 + k_2)$. In this case a nonbinding quota will yield the same price, but a binding quota will not. However, although the price distributions are the same under the tariff and nonbinding quota, the government obtains revenue under the tariff but does not under the quota.

It should also be noted that there are cases of nonequivalence in other ranges of the parameter space (k_1,k_2,c_1,c_2). Suppose, for instance, that we start from an initial position in which $c_1 > c_2 > 0$ and $k_1,k_2 > d(c_2)$. Since the capacity constraints will never be binding in equilibrium, this is very much like the classical Bertrand model. Now suppose a tariff is levied at a positive level t, so that $c_1 < c_2 + t < p_1^L$. In the equilibrium of the game $G(k_1,k_2,c_1,c_2 + t)$, firm 1 drives firm 2 out of the market and charges the price $p_1 = c_2 + t < p_1^L$. This result cannot be obtained with a quota.

Since $c_2 < c_1$, for firm 2 to be driven out under a quota the quota must be set at zero ($k_2^q = 0$). But in this case firm 1 charges p_1^L.

The question of the nonequivalence of tariffs and quotas is just one of many applications of our model in the trade context outlined. One interesting application, which appears immediate from our treatment of the game of capacity choice, is to compare the effects of tariffs and quotas on investment in capacity by foreign and domestic firms. Another application would involve embedding the one shot model in a supergame model of collusion allowing one to examine the effect of levying tariffs and quotas on the sustainability of collusion.[15] Yet another application would integrate the analysis of this paper with the price leadership model of Deneckere and Kovenock (1988), to analyze the effects of tariffs and quotas on the endogenous determination of a price leader.[16] These are among the topics that are the focus of our ongoing research.

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[15] See Davidson (1984) and Rotemberg and Saloner (1986).

[16] For an intriguing discussion of the trade literature utilizing price leadership see Krishna (1988).

Appendix A

In this appendix we provide an example where Q_j may decrease when $\underline{p}_j > \underline{p}_i$ and $H_j^* > 0$. Suppose demand is of the form $d(p) = \max(0, 1 - p)$, $c_1 = .3$, $c_2 = 0$, $k_1 = .1$, and $k_2 = .55$. Then

$$
\begin{aligned}
& L_1 = (p - .3)\min(.1,\ 1 - p) \\
\text{(A.1)}\qquad & H_1 = (p - .3)\min(.1,\ \max(0,\ .45 - p)) \\
& L_2 = p\ \min(.55,\ 1 - p) \\
& H_2 = p\ \min(.55,\ \max(0,\ .9 - p)).
\end{aligned}
$$

It is easily verified that $p_1^H = .375$, $p_2^H = .45$, $\underline{p}_1 = .35625$ and $\underline{p}_2 = .3681818$. Thus, $\Pi_2^* = H_2^* = .2025$ and $\Pi_1^* = L_1(\underline{p}_2) = .0068181$. Over $[\underline{p}_2, p_2^H]$ the function Q_2 is given by

$$
Q_2 = \frac{L_1(p) - \Pi_1^*}{L_1(p) - H_1(p)} = \frac{.1p - .0368181}{p^2 - .65p + .105}
$$

Differentiating with respect to p

$$
dQ_2/dp = \frac{.1(p^2 - .65p + .105) - (.1p - .0368181)(2p - .65)}{(p^2 - .65p + .105)^2}
$$

which implies that $\mathrm{sgn}(dQ_2/dp) = \mathrm{sgn}[-.1p^2 + .0736362p - .0134317]$. This is a quadratic with roots .3329675 and .4033946. Between these two values, the derivative is positive. Since $[\underline{p}_2, p_2^H] = [.3681818,\ .45]$ and $P(k_2) = 1 - k_2 = .45$, in the interval (.4033946, .45) the derivative of Q_2 is negative.

Consequently, $F_2(p) = Q_2(p)$ on $[\underline{p}_2, .4033946]$, is flat on $[.4033946, 1)$ and has a masspoint at p_2^H. It is also easily verified that $Q_1(p)$ is increasing everywhere on $[\underline{p}_2, p_2^H]$, so that supp $F_1 = [\underline{p}_2, .4033946]$, with a masspoint at the upper end of this interval.

It is easy to modify this example to obtain a case where firms have identical, but disconnected supports. Let c_1, c_2 and k_1 be as above, and let $k_2 = .6$. Some calculations yield $p_1^H = .35$, $p_2^H = .45$, $\underline{p}_1 = .325$ and $\underline{p}_2 = .3375$. Thus, $\Pi_2^* = H_2^* = .2025$ and $\Pi_1^* = L_1(\underline{p}_2) = .00375$. Since $P(k_2) = .4 \in (\underline{p}_2, p_2^H)$, $Q_2(p)$ takes on two different functional forms. On $[\underline{p}_2, P(k_2))$, $H_1(p) > 0$ so that $Q_2(p) = [.1p - .03375]/[p^2 - .6p + .09]$. On $[P(k_2), P_2^H]$, $H_1(p) = 0$ so that $Q_2(p) = [.1p - .03375]/[.1p - .03]$. It can be verified that $Q_2(p)$ is increasing on $[\underline{p}_2, .375)$, decreasing on $(.375, .4)$, and increasing again over $(.4, p_2^H]$. Also, $Q_2(.375) = Q_2(.4125)$, so that supp F_1 = supp $F_2 = [\underline{p}_2, .375] \cup [.4125, p_2^H]$. The construction of Section IV now shows that F_1 has a masspoint at .375, and that F_2 has a masspoint at p_2^H.

Appendix B

Proof of Lemma 11: By definition, $Q_i(p) = (L_j(p) - \Pi_j^*)/(L_j(p) - H_j(p))$. (a) is immediate from the definitions of $\underline{p}_j$ and p_j^H and from Lemma 4. (b) follows immediately from the fact that, throughout the interval, $H_j(p)$ is differentiable, $L_j(p)$ is differentiable at every point except $P(k_j)$, and $L_j(p) > H_j(p)$. (c) follows from the fact that, from (b),

$$\text{(B.1)} \qquad dQ_i/dp = \frac{[L_j'(p)(\Pi_j^* - H_j(p)) + H_j'(p)(L_j(p) - \Pi_j^*)]}{(L_j(p) - H_j(p))^2}.$$

With $\Pi_j^* > H_j(p)$ on $[\underline{p}_j, p_j^H)$ from Lemma 4, $L_j(p) > \Pi_j^*$ from the fact that $L_j(p)$ is strictly increasing on $[c_j, p_j^L)$ and $\Pi_j^* = L_j(\underline{p}_j)$, and both L_j' and H_j' positive on $[\underline{p}_j, p_j^H)$ where they exist, we know that (B.1) is positive where these derivatives exist. At $P(k_j)$ the function Q_i is continuous and, for $P(k_j) \in (\underline{p}_j, p_j^H)$, $\lim_{p\uparrow P(k_j)} dQ_i/dp > \lim_{p\downarrow P(k_j)} dQ_i/dp$.

The last inequality follows from the fact that for $p > P(k_j)$, $L_j'(p) = (p - c_j)d'(p) + d(p)$, while for $p < P(k_j)$, $L_j'(p) = k_j$ and the fact that $L_j'(p)$ appears in (B.1) with a positive coefficient. To prove the rest of (d), note that L_j and H_j are twice continuously differentiable except for L_j at the point $P(k_j)$. Furthermore,

$$\text{(B.2)} \qquad d^2Q_i/dp^2 = \left\{\frac{(L_j''(\Pi_j^* - H_j) + H_j''(L_j - \Pi_j^*))(L_j - H_j) - 2(L_j' - H_j')(L_j'(\Pi_j^* - H_j) + H_j'(L_j - \Pi_j^*))}{(L_j - H_j)^3}\right\}$$

It is easily verified that for $p \neq P(k_j)$, $L_j' > 0$, $H_j' > 0$, $L_j' - H_j' > 0$, $L_j'' \leq 0$ and $H_j'' < 0$. With $\Pi_j^* - H_j > 0$ and $L_j - H_j > 0$ in the open interval, the numerator in (B.2) is negative and the denominator is positive. Thus, except for $p = P(k_j)$, d^2Q_i/dp^2 is negative and continuous. This, together with the limit result at $P(k_j)$ tells us that Q_i is concave.

We now check the properties of Q_j. By definition, $Q_j(p) = (L_i(p)) - \Pi_i^*)/(L_i(p) - H_i(p))$. The claim in (e) is immediate from Lemma 10 and Lemma 5. (f) follows from the fact that, throughout the interval, $L_i(p) = (p_i - c_i)k_i$ is differentiable, $H_i(p)$ is differentiable at every point except $P(k_j)$, and $L_i(p) > H_i(p)$. With

$$\text{(B.3)} \qquad dQ_j/dp = [L_i'(\Pi_i^* - H_i) + H_i'(L_i - \Pi_i^*)]/(L_i - H_i)^2,$$

all the terms in this expression are continuous on the interval except $H_i'(p)$ at $P(k_j)$, which satisfies $\lim_{p\uparrow P(k_j)} H_i'(p) = (P(k_j) - c_i)d'(P(k_j)) < 0$, and $\lim_{p\downarrow P(k_j)} H_i'(p) = 0$ as long as $P(k_j) \in (\underline{p}_j, p_j^H)$. Substituting this into (B.3) and noting that the coefficient of $H_i'(p)$ in (B.4) is positive, we obtain $\lim_{p\downarrow P(k_j)} dQ_j/dp > \lim_{p\uparrow P(k_j)} dQ_j/dp$. Since H_i' is the only expression that can be negative in (B.3), we see that $\lim_{p\downarrow P(k_j)} dQ_j/dp > 0$, proving (f).

To prove (g), note that $L_i' = k_i > 0$ and $L_i'' = 0$. Also note that for $p < P(k_j)$, $H_i' = d(p) - k_j + (p - c_i)d'(p) \gtrless 0$ and $H_i'' = 2d'(p) + (p - c_i)d''(p) < 0$; for $p > P(k_j)$, $H_i' = H_i'' = 0$. When $P(k_j) > \underline{p}_j$, for $p_i < P(k_j)$, (B.3) cannot be signed since H_i' may be negative. However, at $p = \underline{p}_j$, $L_i(\underline{p}_j) = \Pi_i^*$ from Lemma 10, so

$$\text{(B.4)} \qquad (dQ_j/dp)(\underline{p}_j) = [L_i'(\underline{p}_j)(\Pi_i^* - H_i(\underline{p}_j))]/[L_i(\underline{p}_j) - H_i(\underline{p}_j)]^2 > 0.$$

Since dQ_j/dp is continuous except at $P(k_j)$, (B.4) implies that there is some neighborhood above $\underline{p}_j$ in which Q_j is increasing. For $p > P(k_j)$, $dQ_j/dp > 0$ since $H_i' = 0$. Therefore, to prove the claim we need only to show that if Q_j turns down in the interval then it does not turn up again unless it hits $P(k_j)$. To show this note that

$$\text{(B.5)} \qquad d^2Q_j/dp^2 = \left\{\frac{(L_i''(\Pi_i^* - H_i) + H_i''(L_i - \Pi_i^*))(L_i - H_i) - 2(L_i' - H_i')(L_i'(\Pi_i^* - H_i) + H_i'(L_i - \Pi_i^*))}{(L_i - H_i)^3}\right\}$$

The sign of this expression is equal to the sign of the numerator. Suppose $dQ_j/dp = 0$. Then $L_i'(\Pi_i^* - H_i) + H_i'(L_i - \Pi_i^*) = 0$, which implies that $\mathrm{sgn}(d^2Q_j/dp^2) = \mathrm{sgn}(H_i''(L_i - \Pi_i^*)(L_i - H_i)) < 0$. (Here we make use of the fact that $L_i'' = 0$). Thus, at a critical point of Q_j (other than $P(k_j)$), the function is locally concave: once Q_j turns down it cannot turn up again until it hits $P(k_j)$. This proves (g). To prove (h) note, more generally, that since $L_i' - H_i' = k_i + k_j - d(p) - (p - c_i)d'(p) > 0$, from (B.5) we see that $\mathrm{sgn}(d^2Q_j/dp^2) < 0$ whenever $\mathrm{sgn}(dQ_j/dp) > 0$. Thus, whenever Q_j is locally nondecreasing (except at $P(k_j)$) it is locally concave. Finally, to show (i), note that from (g), Q_j can be decreasing only on a subinterval of $(\underline{p}_j, \min(P(k_j), p_j^H))$. For this range of prices it is easily verified that $Q_j = [(k_i/k_j)(p - c_j)/(p - c_i)]Q_i$, which implies that

$$\frac{dQ_j}{dp} = \frac{k_i}{k_j}\frac{(c_j - c_i)}{(p - c_i)^2}Q_i + \frac{k_i}{k_j}\frac{(p - c_j)}{(p - c_i)}\frac{dQ_i}{dp}.$$

Since the second summand is positive, if $c_j \geq c_i$, Q_j is increasing. Thus, a necessary condition for Q_j to decrease is that $c_j < c_i$. #

Appendix C

In this Appendix, we will demonstrate the uniqueness of the mixed strategy equilibrium for the case where $\underline{p} > P(k_1 + k_2)$ and $H_i^* > 0$ whenever $\underline{p}_i = \underline{p}$. Lemmas 7, 8 and 9 then imply that $\underline{s}_i = \underline{s}_j = \underline{p}$ and $\bar{s}_i \le \bar{s}_j = p_j^H$, where $j \in \{\ell\colon p_\ell = \underline{p}\}$. The first lemma, which is due to Osborne and Pitchik (1983, p.18), demonstrates that--except possibly for the single point p_j^H--the supports of the equilibrium strategies coincide.

Lemma C.1: Supp F_j = supp $F_i \cup \{p_j^H\}$.

Proof: Suppose $p \in (\underline{p}, p_j^H)$ and $p \notin$ supp F_i. Then, since L_j and H_j are increasing at p, so is $\Pi_j(p, F_i)$. Hence $p \notin$ supp F_j.

Next, suppose $p \in (\underline{p}, p_j^H)$ and $p \notin$ supp F_j. Let $x = \max\{[\underline{p}, p) \cap \text{supp } F_j\}$ and $y = \min\{(p, p_j^H] \cap \text{supp } F_j\}$. Observe that $F_j(x) < 1$ and that for $s \in (x,y)$: $\Pi_i(s, F_j) = (1 - F_j(x))L_i(s) + F_j(x)H_i(s)$. Observe also that since L_i is linear on $[\underline{p}, p_j^H]$ and since H_i is strictly concave on $[\underline{p}, P(k_j)]$ and identically zero on $[P(k_j), p_0]$, $\Pi_i(s, F_j)$ is increasing and/or strictly concave on $[\underline{p}, P(k_j)]$ and increasing on $[P(k_j), p_j^H]$. We conclude that if $p \in$ supp F_i, then p maximizes $\Pi_i(p, F_j)$ on (x,y) and $p \in J(F_i)$. Now clearly $x \notin J(F_i)$ (this is obvious if $x \in J(F_j)$; if $x \notin J(F_j)$ then $\Pi_i(x, F_j) < \Pi_i(p, F_j)$ by the maximization property of p). But then $\Pi_j(F_j, F_i) = \Pi_j(x, F_i) = F_i(x)H_j(x) + (1 - F_i(x))H_j(x)$. We then obtain an immediate contradiction to equilibrium, since on $[x,p)$ both L_j and H_j are increasing and since i puts no mass on $[x,p)$. #

Our next lemma shows that gaps in i's support can occur only on the set $\{p: Q_j(p;\Pi_i^*) < IQ_j(p;\Pi_i^*)\}$.

Lemma C.2: Let $p \in (\underline{p}, p_j^H)$ with $p \notin \operatorname{supp} F_i$. Then $Q_j(p;\Pi_i^*) < IQ_j(p;\Pi_i^*)$.

Proof: First, we deal with the case where $F_i(p) = 1$. Then $\bar{s}_i < p < p_j^H$, and so $F_j(\bar{s}_i) = \lim_{s\uparrow p_j^H} F_j(s)$. Also $\bar{s}_i \notin J(F_j)$, since L_j and H_j are increasing on $[\bar{s}_i, p_j^H]$ and since $\Pi_j(p_j^H, F_i) = \Pi_j^*$. We conclude that $H_i(\bar{s}_i)F_j(\bar{s}_i) + L_i(\bar{s}_i)(1 - F_j(\bar{s}_i)) = \Pi_i(\bar{s}_i, F_j) = \Pi_i^* = H_i(\bar{s}_i)Q_j(\bar{s}_i) + L_i(\bar{s}_i)(1 - Q_j(\bar{s}_i))$, and hence that $F_j(\bar{s}_i) = Q_j(\bar{s}_i) < 1$. Observe now that $Q_j'(\bar{s}_i) = 0$. Indeed, if $Q_j'(\bar{s}_i) < 0$, then there would exist $s < \bar{s}_i$ such that $Q_j(s) > F_j(s)$ and if $Q_j'(\bar{s}_i) > 0$, then there would exist $s > \bar{s}_i$ such that $Q_j(s) > F_j(s) = F_j(\bar{s}_i)$. In both cases, by playing s, firm i would net $F_j(s)H_i(s) + (1 - F_j(s))L_i(s) > Q_j(s)H_i(s) + (1 - Q_j(s))L_i(s) = \Pi_i^*$, a contradiction to equilibrium. Observe also that $Q_j(s) \le F_j(\bar{s}_i)$ for all $s \in (\bar{s}_i, p_j^H)$, for otherwise there would exist $s \in (\bar{s}_i, p_j^H)$ such that $Q_j(s) > F_j(s)$, yielding the same contradiction. From Lemma 11, we conclude that $Q_j(s) < IQ_j(s)$ for all $s \in (\bar{s}_i, p_j^H)$.

Next, let us suppose that $F_i(p) < 1$. Let $x = \max\{[\underline{p}, p) \cap \operatorname{supp} F_i\}$ and $y = \min\{(p, p_j^H] \cap \operatorname{supp} F_i\}$. Since $\Pi_j(\bullet, F_i)$ is increasing on $[x,y)$, it must be that $\lim_{s\uparrow y} F_j(s) = F_j(x)$. Next we claim that $y \notin J(F_j)$. Otherwise $y \notin J(F_i)$ and since $y \in \operatorname{supp} F_i$ we would have $\Pi_i^* = \Pi_i(y, F_j) = H_i(y)F_j(y) + L_i(y)(1 - F_j(y))$. This contradicts the fact that for all $s \in [x,y)$: $\Pi_i^* \ge \Pi_i(s, F_j) = H_i(s)F_j(x) + L_i(s)(1 - F_j(x))$. Suppose then that $y \notin J(F_j)$. Then $\Pi_i(y, F_j) = \Pi_i^*$ and so $F_j(x) = F_j(y) = Q_j(y;\Pi_i^*)$.

Next, observe that $x \in J(F_i)$. For if not, then since by Lemma C.1,

$x \in \text{supp } F_j$, $\Pi_j^* = \Pi_j(x,F_i)$. This would contradict equilibrium, as $\Pi_j(s,F_i)$ is increasing on (x,y). Now $x \in J(F_i)$ implies $F_j(x) = Q_j(x;\Pi_i^*)$ and so $Q_j(x;\Pi_i^*) = F_j(x) = F_j(y) = Q_j(y;\Pi_i^*)$. As in the case of $F_i(p) = 1$, we conclude that $Q_j'(x) = 0$ and that $Q_j(s) < F_j(x)$ for all $s \in (x,y)$. Hence, $Q_j(s) < IQ_j(s)$ for all $s \in (x,y)$, proving the desired result. #

The next two lemmas allow us to completely pin down the supports of the distributions.

<u>Lemma C.3</u>: Suppose $Q_j(p) < IQ_j(p;\Pi_i^*)$. Then $p \notin \text{supp } F_i$.

<u>Proof</u>: Suppose to the contrary that $p \in \text{supp } F_i$. By Lemma C.1, $p \in \text{supp } F_j$. Furthermore, $F_j(p) \geq IQ_j(p;\Pi_i^*) > Q_j(p)$. Now $p \notin J(F_i)$, since otherwise $p \notin J(F_j)$ and so $\Pi_i^* = H_i(p)F_j(p) + L_i(p)(1 - F_j(p))$, implying $F_j(p) = Q_j(p;\Pi_i^*)$. For the same reason, F_i is not right increasing at p. We conclude that F_i is left increasing at p, and so there exists $\{p_n\} \subset \text{supp } F_i$, $p_n \uparrow p$ such that $\Pi_i(p_n,F_j) = \Pi_i^*$. Hence, $F_j(p_n) = Q_j(p_n;\Pi_i^*)$. However, for large n, $Q_j(p_n;\Pi_i^*) < IQ_j(p_n;\Pi_i^*)$, yielding the contradiction $F_j(p_n) = Q_j(p_n;\Pi_i^*) < IQ_j(p_n;\Pi_i^*) \leq F_j(p_n)$. #

<u>Lemma C.4</u>: Suppose $p_j^H \in \text{supp } F_i$. Then $\exists\ \varepsilon > 0$ such that $\forall\ p \in (p_j^H - \varepsilon, p_j^H)$, $Q_j(p;\Pi_i^*) = IQ_j(p;\Pi_i^*)$.

<u>Proof</u>: Suppose not. Then from Lemma 11 $\exists\ \varepsilon > 0$ such that $\forall\ p \in (p_j^H - \varepsilon, p_j^H)$, $Q_j(p) < IQ_j(p)$. From Lemma C.3, for every p in this interval $p \notin \text{supp } F_i$. Furthermore, p_j^H cannot be a masspoint of firm i since

then $\lim_{p \uparrow p_j^H} F_i(p)H_j(p) + (1 - F_i(p))L_j(p) > H_j(p_j^H) = H_j^*$. Thus, $p_j^H \notin \operatorname{supp} F_i$. #

Let $a = \inf\{p\colon Q_j(p;\Pi_i^*) < IQ_j(p;\Pi_i^*)\}$ and $b = \sup\{p \le p_j^H\colon Q_j(p;\Pi_i^*) < IQ_j(p;\Pi_i^*)\}$. Combining Lemmas C.1-C.4 we have:

$$\operatorname{supp} F_j = [\underline{p}, p_j^H] \backslash (a,b) \quad \text{and} \quad \operatorname{supp} F_i = \operatorname{cl}(\operatorname{supp} F_j \backslash p_j^H),$$

where cl denotes closure. Finally, it is easy to argue that masspoints in the distributions can occur for i only at a and for j only at p_j^H (otherwise, there would have to be additional gaps in the support). This completely pins down the distribution functions, since if $p \in \operatorname{supp} F_k$ and $p \notin J(F_\ell)$ (for $\ell \ne k$), $\Pi_k(p, F_\ell) = \Pi_k^*$ and so $F_\ell(p) = Q_\ell(p;\Pi_k^*)$.

Appendix D

Proof of Proposition 4: First, observe that $k_i \le r_i(k_j)$ implies $\underline{p}_i = p_i^H = P(k_1 + k_2)$, and that $k_i > r_i(k_j)$ implies $P(k_1 + k_2) < \underline{p}_i < p_i^H = b_i(k_j)$. Consequently, $\underline{p}_1 = \underline{p}_2 = P(k_1 + k_2)$ for $(k_1,k_2) \in \Delta$, and $\underline{p}_i = P(k_1 + k_2) < \underline{p}_j$ for $(k_1,k_2) \notin \Delta$ with $k_i \le r_i(k_j)$. We are left with the ranking of the $\underline{p}_i$ in the region where $k_1 > r_1(k_2)$ and $k_2 > r_2(k_1)$.

It is important to first establish the locus of points where firm i is exactly capacity constrained at $\underline{p}_i$, i.e., $d(\underline{p}_i) = k_i$. Observe that since $k_i \ge r_i(k_j)$, $H_i^*(k_j) = B_i(k_j)$ and hence that k_i must satisfy $[P(k_i) - c_i]k_i = B_i(k_j)$. Our assumptions on $d(\bullet)$ imply that the function $q \to [P(q) - c_i]q$ is strictly quasiconcave, so that for each $k_j > 0$, there are exactly two solutions in $[0,d(c_i)]$ to this equation. The smallest of these solutions necessarily satisfies $k_i < r_i(k_j)$ and hence is inadmissible. Hence, for each $k_j \ge 0$ there is a $k_i = \psi_i(k_j) \equiv \max\{q: [P(q) - c_i]q = B_i(k_j)\}$ such that $d(\underline{p}_i) = k_i$. Furthermore, since $B_i(k_j)$ is decreasing in k_j, ψ_i is increasing in k_j. The functions ψ_i are illustrated in Figure 2.

Next, we claim that $k_2 \ge \psi_2(k_1)$ and $k_1 > 0$ implies $\underline{p}_2 > \underline{p}_1$. Indeed, if $\psi_2(k_1) \le k_2 < d(c_1)$ and $\underline{p}_1 \ge \underline{p}_2$, then $d(p_1^H) \le d(\underline{p}_1) \le d(\underline{p}_2) \le k_2$ so that $H_1^*(k_2) = 0$, contradicting $k_2 < d(c_1)$. If $k_2 \ge d(c_1)$, then $\underline{p}_1 = c_1 < c_2 \le \underline{p}_2$. A similar argument establishes that $\psi_1(k_2) \le k_1 < d(c_2)$ and $k_2 > 0$ implies $\underline{p}_1 > \underline{p}_2$.

Let us now investigate the region where $k_1 \ge d(c_2)$, so that $\underline{p}_2 = c_2$. First, consider the case where $r_1(0) \le d(c_2)$. Then $\underline{p}_1$ satisfies the equation $L_1(\underline{p}_1) = B_1(k_2)$. For $k_2 \ge d(c_1)$, $\underline{p}_1 = c_1 < c_2 = \underline{p}_2$, and for $k_2 = 0$, $\underline{p}_1 = b_1(0) \ge c_2 = \underline{p}_2$. Since $\underline{p}_1$ is strictly decreasing in k_2 on

$[0,d(c_1)]$ there exists, for each k_1, a unique value of k_2 such that $\underline{p}_1 = c_2 = \underline{p}_2$. Since $k_1 \geq d(c_2)$, $k_2 = \theta(k_1)$ must satisfy:

$$L_1(c_2) = (c_2 - c_1)d(c_2) = B_1(k_2)$$

for all $k_1 \geq d(c_2)$. Now consider the case where $r_1(0) > d(c_2)$. For $d(c_2) \leq k_1 \leq r_1(k_2)$, we already showed that $\underline{p}_2 > \underline{p}_1$ (except at $k_1 = d(c_2)$ and $k_2 = 0$, where equality holds). A similar argument to the case $r_1(0) \leq d(c_2)$ then establishes that $\underline{p}_2 > \underline{p}_1$ for all $k_1 \geq d(c_2)$.

We are left with the region where $r_1(k_2) \leq k_1 \leq \min\{d(c_2),\Psi_1(k_2)\}$ and $r_2(k_1) \leq k_2 \leq \Psi_2(k_1)$. We will refer to this region as Ω (observe that Ω may be empty when $r_1(0) > d(c_2)$). Observe that on Ω each firm i is capacity constrained at $\underline{p}_i$, so that $\underline{p}_i = c_i + B_i(k_j)/k_i$. Let $\Phi = (\underline{p}_2 - \underline{p}_1)k_1k_2$ be viewed as a function of k_2. Then:

$$\Phi = (c_2 - c_1)k_1k_2 + k_1B_2(k_1) - k_2B_1(k_2)$$
$$\Phi' = (c_2 - c_1)k_1 + [b_1(k_2) - c_1][k_2 - r_1(k_2)]$$
$$\Phi'' = -\{2r_1(k_2) - k_2 - k_2r_1'(k_2)\}/d'(b_1(k_2)) \ .$$

For each k_1 for which there exists k_2 such that $(k_1,k_2) \in \Omega$ let $\underline{\delta}(k_1) = \min\{k_2: (k_1,k_2) \in \Omega\}$ and $\bar{\delta}(k_1) = \max\{k_2: (k_1,k_2) \in \Omega\}$. We wish to study the behavior of Φ on $[\underline{\delta},\bar{\delta}]$. Observe that for $k_2 \geq \hat{k}_2 \equiv r_1(\hat{k}_2)$, Φ' is positive. Observe also that for $k_2 \leq \hat{k}_2$, $\Phi'' > 0$. We conclude that either $\Phi'(k_2) > 0$ at $k_2 = \underline{\delta}(k_1)$ so that $\Phi' > 0$ on $[\underline{\delta}(k_1),\bar{\delta}(k_1)]$, or else there exists a uniquely defined $k_2 = \mu(k_1) \in (\underline{\delta}(k_1),\hat{k}_2)$ such that $\Phi' < 0$ on $[\underline{\delta}(k_1),\mu(k_1))$ and $\Phi' > 0$ on $(\mu(k_1),\bar{\delta}(k_1)]$.

First, suppose $k_1 > k_1^c$. Then $\Phi(\underline{\delta}(k_1)) < 0$, so that $\Phi(k_2) < 0$ on $[\underline{\delta}(k_1), \mu(k_1)]$, and $\Phi' > 0$ on $(\mu(k_1), \bar{\delta}(k_1)]$. Hence there exists at most one value of $k_2 \in (\mu(k_1), \bar{\delta}(k_1)]$ such that $\Phi(k_2) = 0$. In fact, since $\Phi(\bar{\delta}(k_1)) > 0$, the existence of such a solution is guaranteed. Denote this solution by $k_2 = \theta(k_1)$.

Finally, let $k_1 \le k_1^c$. Observe that $\Phi(k_2) \ge 0$ at $k_2 = \underline{\delta}(k_1) = r_1^{-1}(k_1)$, with strict inequality for $k_1 < k_1^c$. We now claim that $\Phi'(k_2) \ge 0$ at $k_2 = \underline{\delta}(k_1)$, so that $\Phi > 0$ on $(\underline{\delta}(k_1), \bar{\delta}(k_1)]$. Define $\omega(k_2) = \Phi'(k_2)|_{k_1 = r_1(k_2)}$. Then:

$$
\begin{aligned}
\omega(k_2) &= (c_2 - c_1) r_1(k_2) + [b_1(k_2) - c_1][k_2 - r_1(k_2)] \\
&= [b_1(k_2) - c_1] k_2 - [b_1(k_2) - c_2] r_1(k_2).
\end{aligned}
$$

Now if $r_1(0) > d(c_2)$, then $c_1 \le b_1(k_2) < b_1(0) \le c_2$, so that $\omega \ge 0$ at $k_2 = r_1^{-1}(k_1)$. If $r_1(0) \le d(c_2)$, then $\omega(k_2^c) = 0$. Observe also that $\omega'(k_2) = b_1'(k_2)[k_2 - r_1(k_2)] + [b_1(k_2) - c_1] - [b_1(k_2) - c_2] r_1'(k_2)$. Now if $k_2 \le \hat{k}_2 = r_1(\hat{k}_2)$, then $k_2 \le r_1(k_2)$ and so $\omega'(k_2) > 0$ on $[k_2^c, \hat{k}_2]$. For $k_2 \ge \hat{k}_2$, direct inspection of the expression for $\omega(k_2)$ reveals that $\omega(k_2) > 0$. Consequently, $\omega(k_2) > 0$ on $(k_2^c, r_1^{-1}(0)]$, establishing the claim. We conclude that $\underline{p}_2 > \underline{p}_1$ for $k_1 \le k_1^c$ and $k_2 \ge \underline{\delta}(k_1) = r_1^{-1}(k_1)$ (except at (k_1^c, k_2^c), where equality holds). #

Appendix E

Proof of Proposition 6: We need only show that if $H_1^*(k_2^c) \geq (c_2 - c_1)d(c_2)$ then firm 1 has no incentive to deviate from $k_1 = k_1^c$. First, observe that $\pi_1(k_1,k_2^c) \leq H_1^*(k_2^c)$ for all $k_1 \geq d(c_2)$. Indeed, if $k_1 \geq d(c_2)$ then $H_2^*(k_1) = 0$ and so $\underline{p}_2 = c_2$. If $\underline{p}_1 \leq c_2$, then $\pi_1(k_1,k_2^c) = (c_2 - c_1)d(c_2) \leq H_1^*(k_2^c)$. If $\underline{p}_1 > c_2 = \underline{p}_2$, then $\pi_1(k_1,k_2^c) = H_1^*(k_2^c)$, and so firm 1 has no incentive to deviate to $k_1 \geq d(c_2)$.

Suppose now that $k_1^c < k_1 < d(c_2)$. Then $c_2 < P(k_1) < P(k_2^c)$. The remainder of the proof is broken up in two cases, depending upon the relationship between $\underline{p}_1$ and $P(k_1)$. First, if $\underline{p}_1 \geq P(k_1)$, then $H_2(\underline{p}_1) = 0$ and so $p_2^H \leq \underline{p}_1$. This implies $\underline{p}_2 \leq \underline{p}_1$, and so $\pi_1(k_1,k_2^c) = H_1^*(k_2^c)$.

Suppose now that $\underline{p}_1 < P(k_1)$. We now claim that $\underline{p}_2 < P(k_2^c)$. Indeed, if $\underline{p}_2 \geq P(k_2^c)$, then $\underline{p}_2 \geq P(k_2^c) > P(k_1) > c_2$ and so $H_2(p_2^H) = 0$ implying $\underline{p}_2 = c_2$, a contradiction. Hence:

$$\underline{p}_1 = c_1 + H_1^*(k_2^c)/k_1$$
$$\underline{p}_2 = c_2 + H_2^*(k_1)/k_2^c.$$

Let $\Psi(k_1) = k_1k_2^c(\underline{p}_1 - \underline{p}_2) = k_1k_2^c(c_1 - c_2) + k_2^cH_1^*(k_2^c) - k_1H_2^*(k_1)$. Then $\Psi(k_1^c) = 0$ and $\Psi(d(c_2)) = k_2^c\{H_1^*(k_2^c) - (c_2 - c_1)d(c_2)\} \geq 0$. We may now calculate:

$$\Psi'(k_1) = -(c_2 - c_1)k_2^c + [b_2(k_1) - c_2][k_1 - r_2(k_1)]$$
$$\Psi''(k_1) = b_2'(k_1)[k_1 - r_2(k_1)] + [b_2(k_1) - c_2][1 - r_2'(k_1)].$$

Now, the f.o.c. for $b_2(k_1)$ implies that $b_2(k_1) - c_2 = -r_2(k_1)/d'(b_2(k_1))$. Also, since $d(b_2(k_1)) = k_1 + r_2(k_1)$, we have $d'(b_2(k_1))b_2'(k_1) = 1 + r_2'(k_1)$. Thus, we obtain:

$$\Psi''(k_1) = -\{2r_2(k_1) - k_1(1 + r_2'(k_1))\}/d'(b_2(k_1)).$$

Also observe that the sign of Ψ'' is equal to the sign of $-d'(b_2(k_1))\Psi''(k_1)$, and that the latter expression has derivative $r_2'(k_1) - 1 - k_1r_2''(k_1)$. Our assumptions on demand can be shown to imply that $r_2'' \geq 0$, so that the above derivative is negative. Also note that $\Psi'(k_1^c) = -[b_2(k_1^c) - c_1]k_2^c + [b_2(k_1^c) - c_2]k_1^c$. From the f.o.c. for $b_2(k_1)$ and $b_1(k_2)$ we see that $b_2(k_1^c) - c_2 = -r_2(k_1^c)/d'(b_2(k_1^c))$ and $b_1(k_1^c) - c_1 = -r_1(k_2^c)/d'(b_1(k_2^c))$. Substituting this into the expression for $\Psi'(k_1^c)$, and noting that $b_2(k_1^c) = b_1(k_2^c)$ then yields $\Psi'(k_1^c) = 0$.

There are now two cases to consider. First, assume that $\Psi''(k_1^c) \leq 0$. Then $\Psi''(k_1) < 0$ on $(k_1^c, d(c_2)]$, and hence $\Psi'(k_1) < 0$ on $(k_1^c, d(c_2)]$. The latter statement contradicts $\Psi(k_1^c) = 0$ and $\Psi(d(c_2)) \geq 0$. Thus it must be that $\Psi''(k_1^c) > 0$. Since $\Psi''(d(c_2)) < 0$, we know that Ψ' is increasing on $(k_1^c, \hat{k}_1)$, where $\hat{k}_1$ is the unique solution to $\Psi''(k_1) = 0$ in $(k_1^c, d(c_2))$, and decreasing thereafter. Since $\Psi(k_1^c) = 0$ and $\Psi(d(c_2)) \geq 0$, this proves $\Psi(k_1) \geq 0$ on $[k_1^c, d(c_2)]$ so that $\pi_1(k_1, k_2^c) = H_1^*(k_2^c)$. We conclude that firm 1 has no incentive to deviate to $k_1 \in (k_1^c, d(c_2)]$ either.

Finally, for $k_1 \in [0, k_1^c)$, $\pi_1(k_1, k_2^c) = k_1[P(k_1 + k_2^c) - c_1]$, an expression that is maximized as $k_1 = k_1^c$. #

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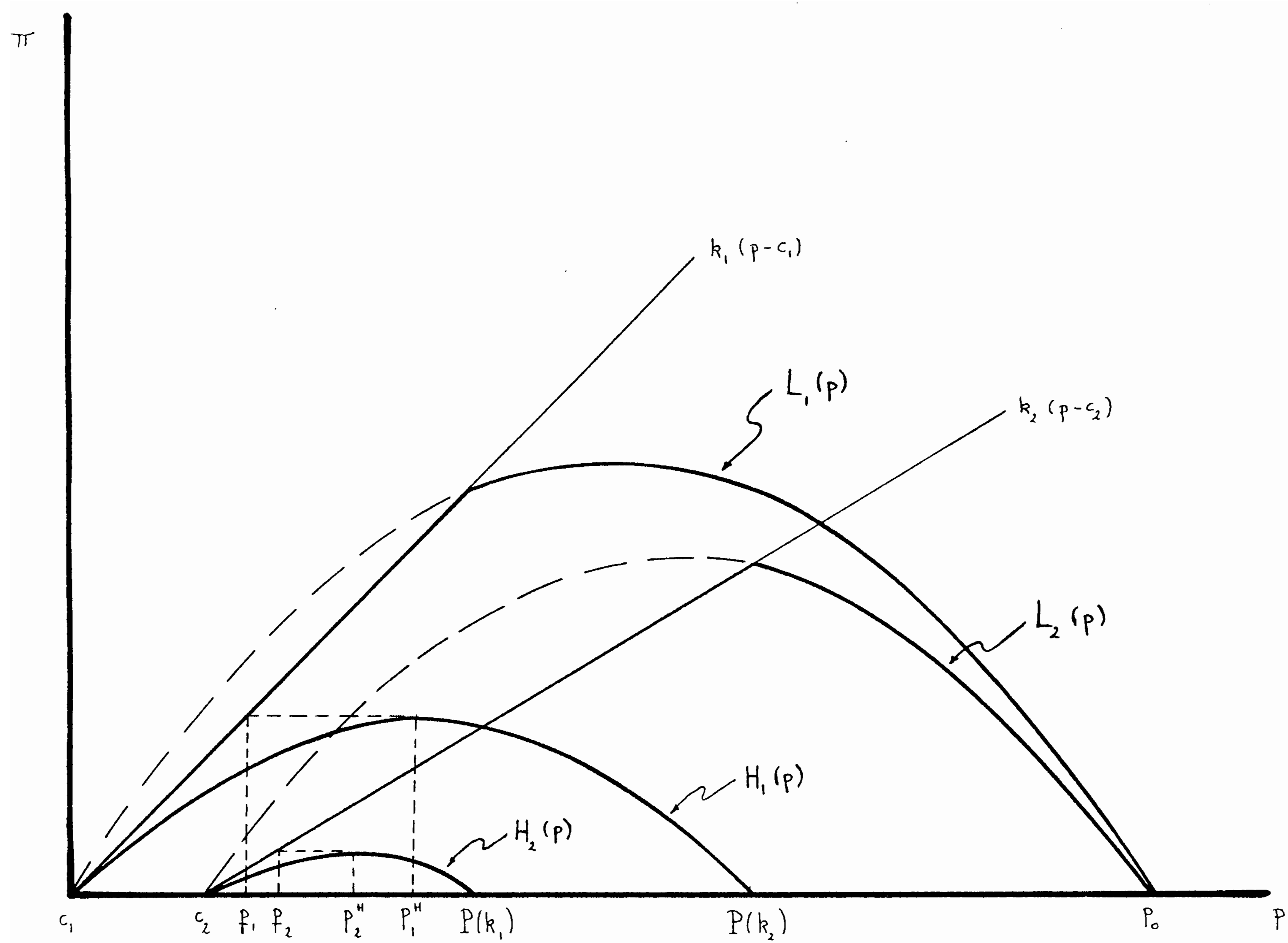


<u>Figure 1</u> : The functions L_i and H_i with the critical prices.

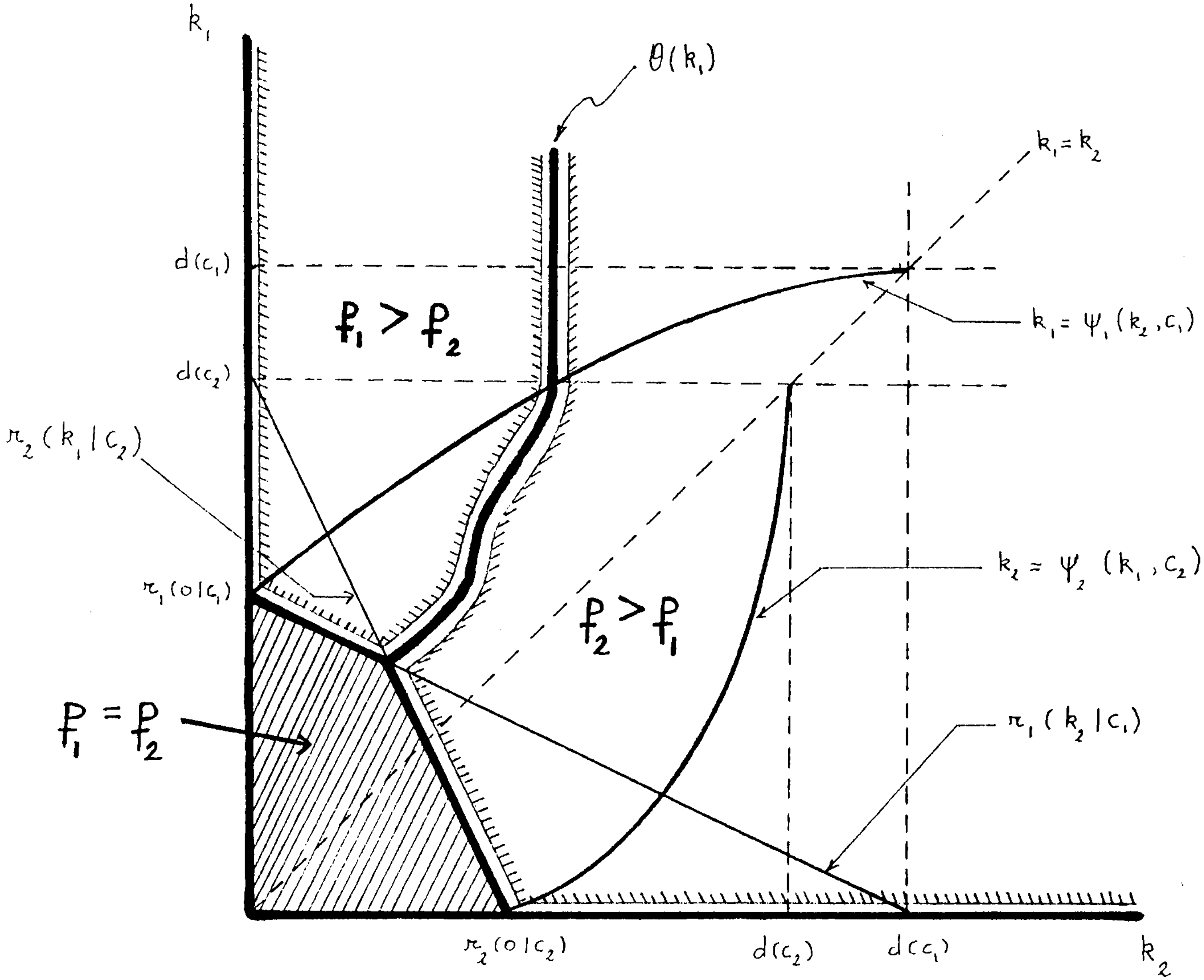


Figure 2 : The ranking of $\underline{p}_1$ and $\underline{p}_2$ when $r_1(0|c_1) < d(c_2)$.

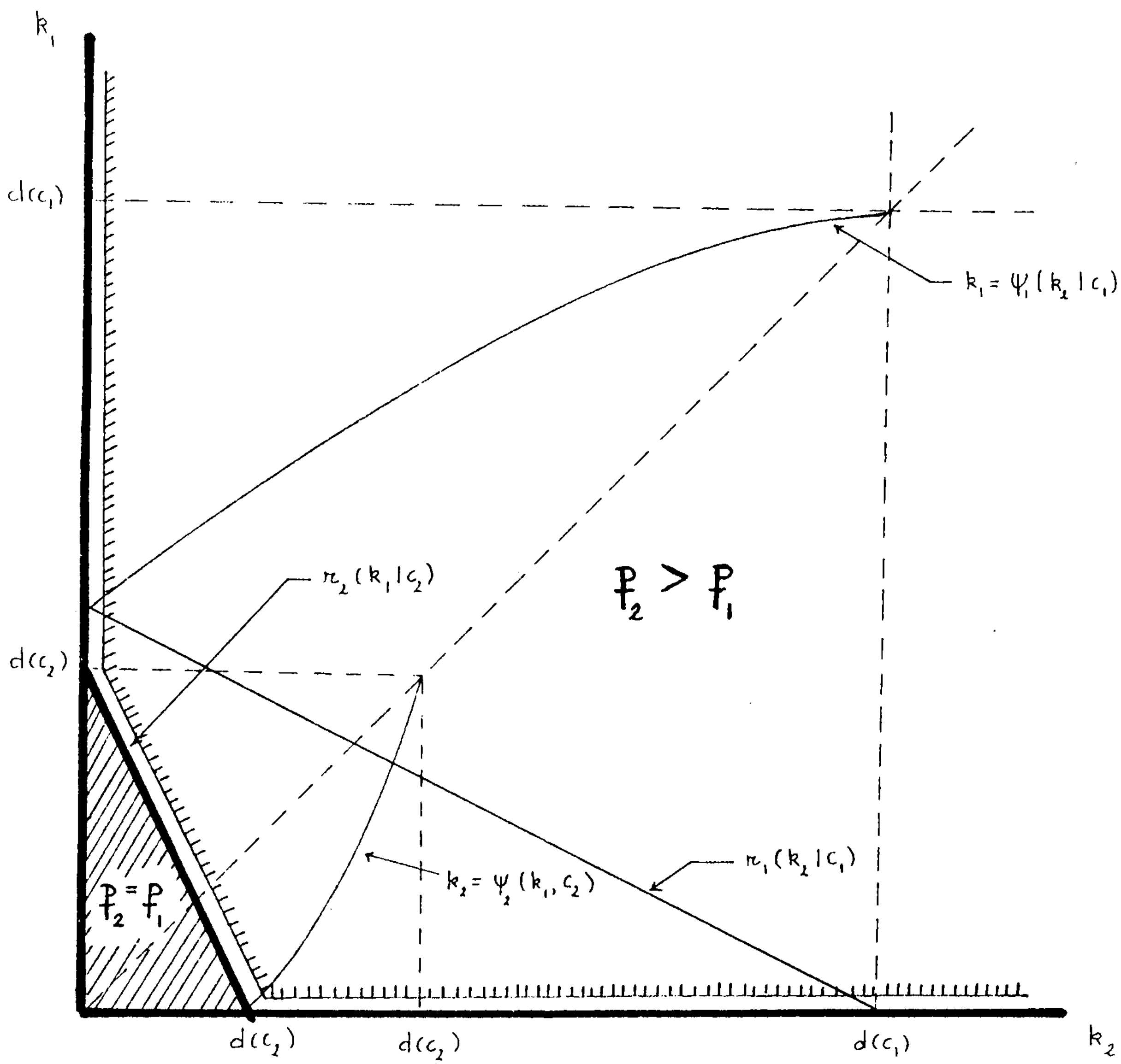


Figure 3 : The ranking of $\underline{p}_1$ and $\underline{p}_2$ when $r_1(0|c_1) \geqslant d(c_2)$.

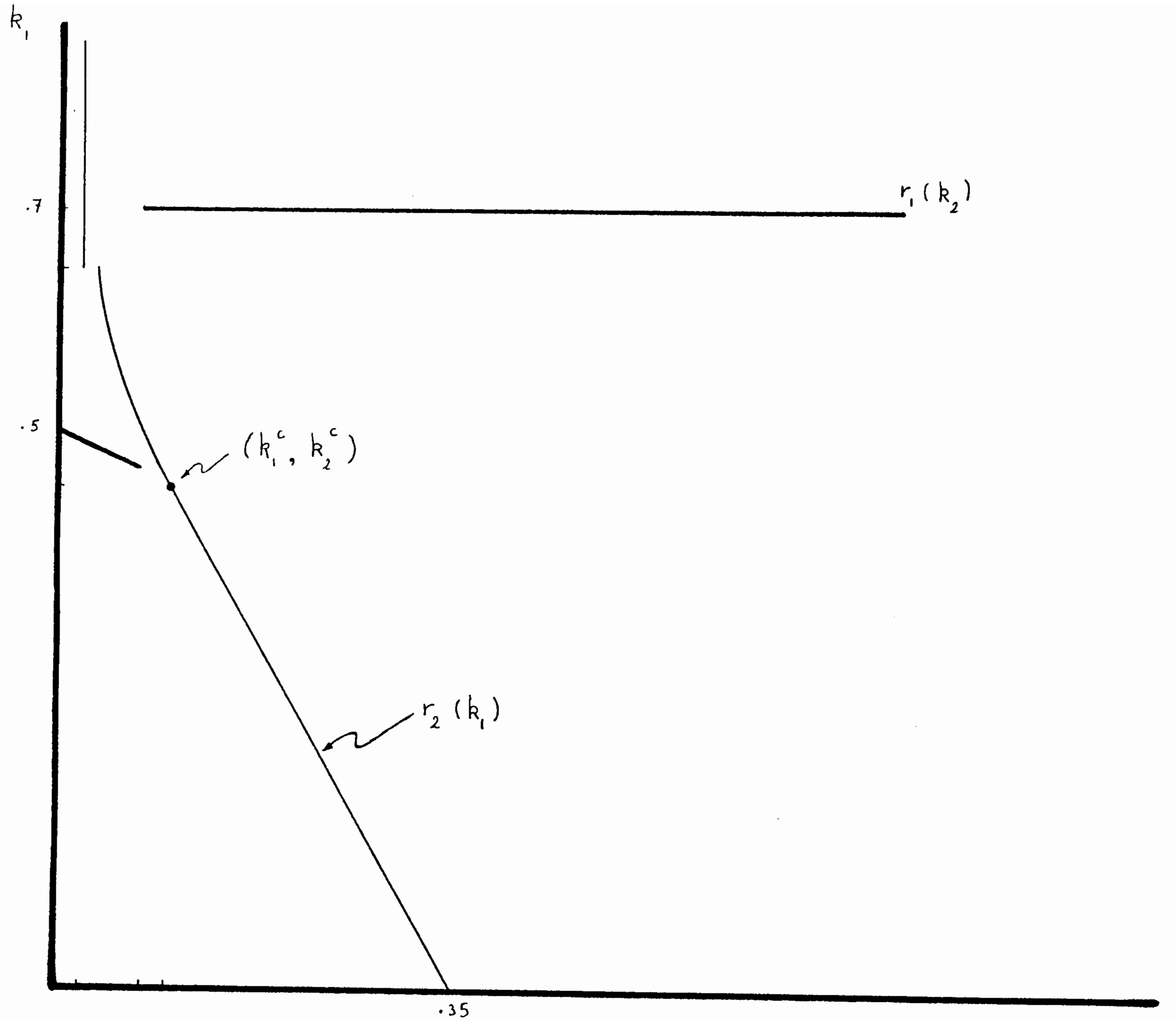
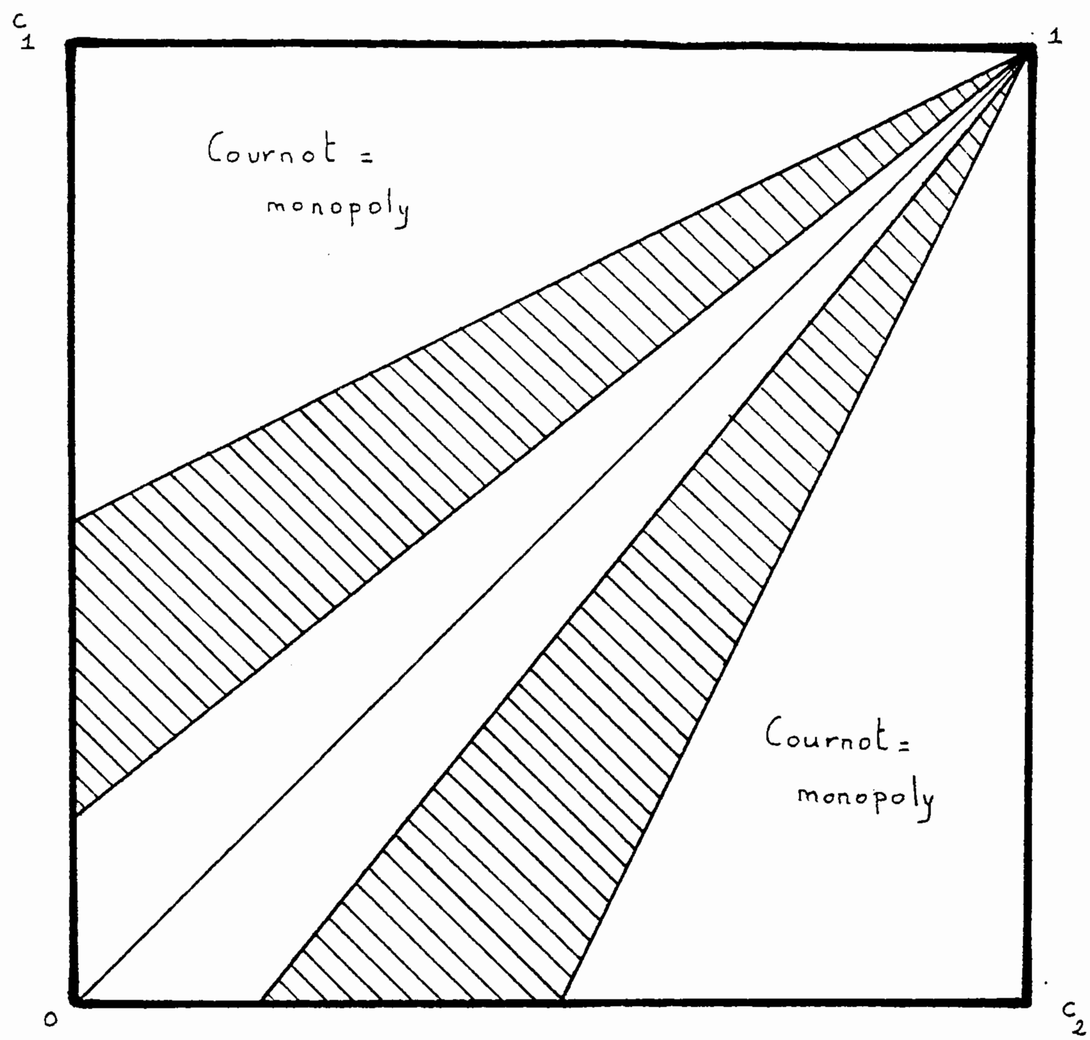


Figure 4 : Best Response functions for the game $\Gamma(0,.3,0^+)$ when $d(p) = \max(0,1-p)$.



= the two-stage game does not yield Cournot

<u>Figure 5</u> : Cost pairs for which Cournot is not an equilibrium of $\Gamma(c_1, c_2, 0^+)$ when $d(p) = \max(0, 1-p)$.