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"DECENTRALIZED DISEQUILIBRIUM TRADING AND PRICE FORMATION"

by

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0. INTRODUCTION

In this paper I present a generalization of the non-tatonnement model considered, among others, by Arrow and Hahn and by Hahn and Negishi. The generalization explicitly introduces a communication process into the analysis, in the spirit of Hurwicz (1960), which allows for a form of decentralized price formation and decentralized trading. Two specific examples are considered in which the messages of consumers are prices they report they are willing to pay. Market prices are viewed as an average, over all consumers, of their individually reported prices.

In the first example, it is assumed that consumers adjust their reported prices in proportion to their own excess demands for commodities evaluated at market prices. While it is not clear why consumers should behave in this manner, if they do then market prices (the average prices) will imitate the behavior of a Walrasian auctioneer's prices. That is, they will look like they adjust proportionately to aggregate excess demand even though there is no auctioneer. Utilizing this mechanism we point out a difficulty with the general model of Arrow and Hahn (p. 326). In particular we show that even if market prices converge to equilibrium values and remain at these values, it is still possible that equilibrium allocations are never attained. In the second example we consider, the reported prices of the consumers are their true maximum buying-minimum selling prices. These prices are,in effect, the consumer's marginal rates of substitutions given his current contractual obligations. In some respects, this model is similar to that of Dreze and Vallee Poussin. Given this behavior on the part of consumers(which contains an element of incentive compatibility) it can be shown that the resulting non-tatonnement process is globally (quasi) stable, Pareto-satisfactory, and informationally decentralized for a large class of exchange economies. In particular, this is true whenever preferences are representable by utility functions with continuous second derivatives, and when there exists a commodity which is always desired by all consumers. This is a larger class of economies than those for which other decentralized non-tatonnement adjust processes have been proven to possess global (quasi) stability and to be Pareto-satisfactory.

1. THE GENERAL STRUCTURE OF ADJUSTMENT

We assume there are N consumers, indexed by $i = 1,\ldots,N$, and K commodities, indexed by $k = 1,\ldots,K$. We let $x^i = (x_1^i,\ldots,x_K^i)$ be the consumption of i and we let $\{x^i \in R^K \mid x^i \geq 0\}$ be the admissible consumption set for i. Each i is presumed to own initial endowments $\omega^i = (\omega_1^i,\ldots,\omega_K^i)$ such that $\omega_k^i > 0$ for all i and k. Consumption will occur by combining initial endowments with net trades. Let $d^i \in R^K$ be a vector of net trades, then $x^i = \omega^i + d^i$. Finally, each i is assumed to have a utility function, $U^i(x^i)$. We will assume that $U^i \in C^2$ (that is, it has continuous second derivatives) although many of our results hold under weaker conditions.

Our notion of a non-tatonnement adjustment mechanism is a generalization of that contained, for example, in Arrow and Hahn. In particular, we make the process of communication explicit in much the same way that Hurwicz (1960) does. Each consumer will send a message concerning the trades he is willing to make. This message may be coded in the sense that he might send the maximum prices he is willing to buy at and/or the minimum prices he is willing to sell at rather than the explicit trades he is willing to agree to. After these messages are sent and received, the process transforms these into trade agreements which are then formalized into contracts. The process then repeats itself. For ease of exposition we will think of this sequence of events as occuring continuously through time.

We let M be a "language" or a set of messages. $m^i \in M$ will be the message of consumer i. We then formalize a non-tatonnement process by

$$\text{(1a)} \quad \dot{m}^i = f(m,d^i;\epsilon^i) \qquad \forall\, i \in I$$

$$\text{(1b)} \quad \dot{d}^i = g^i(m,d) \qquad \forall\, i$$

where ϵ^i is the characteristic of Mr. i, (that is, $\epsilon^i = (\omega^i, U^i)$), and $m = (m^1, \ldots, m^n)$. We will assume throughout that at $t = 0$, $d^i = 0$ and we will require that the process be consistent. That is,

$$\sum_{i=1}^{N} \dot{d}^i \equiv 0.$$

That this structure represents a generalization of the model of Arrow and Hahn is realtively easy to show. Let $M = R^K$ and let $\Pi(m) = \frac{1}{N} \sum_{i=1}^{N} m^i$. Let $e(\Pi, d^i; \epsilon^i)$ be the excess demand of i at the "prices" Π given the current contractural agreements d^i. That is $e(\Pi, d^i; \epsilon^i) = a^i$ iff a^i solves

$$\underset{a^i}{\text{maximize}} \quad U^i[\omega^i + d^i + a^i]$$

$$\text{subject to} \quad \Pi \cdot d^i \leq 0.$$

Assume U^i is strictly quasi-concave, then $e(\Pi, d^i; e^i)$ is a function. We can now define an Arrow-Hahn non-tatonnement process (A-H,N-T) as:

$$\dot{m}^i = e[\Pi(m), d^i; \epsilon^i] \qquad \forall\ i \in I$$

and

$$\dot{d}^i = g^i[\Pi(m), d] \qquad \forall\ i \in I.$$

For the moment we will leave g unspecified. Since $\Pi(m) = \frac{1}{N} \Sigma_i m^i$, this system reduces to:

$$\text{(2a)} \quad \dot{\Pi} = \frac{1}{N} \sum_{i=1}^{N} e(\Pi, d^i; \epsilon^i)$$

$$\text{(2b)} \quad \dot{d} = g(\Pi(m), d)$$

which is identical to that of Arrow and Hahn (p. 326).

Define an equilibrium of (2) as $(\bar{m},\bar{d})$ such that $e(\Pi(\bar{m}),\bar{d}^i;\epsilon^i) = 0$ and $g^i[\Pi(\bar{m}), \bar{d}] = 0$ for all i. It is easy to show, under our assumptions that the allocation $\bar{x} = (\omega^1 + \bar{d}^1,\ldots,\omega^N + \bar{d}^N)$ is Pareto-optimal. Therefore, the question of interest is stability.

It is established in Arrow and Hahn (pp. 327-28) that the dynamic behavior of prices in (2) inherits its stability properties from the dynamic behavior of prices in the usual tatonnement model where

$$(3)\quad \dot{\Pi} = E(\Pi) \text{ and } E(\Pi) = \sum_{i=1}^{N} e(\Pi,\omega^i;\epsilon^i).$$

For example, they show that under the assumption of gross substitutes, if the functions g^i satisfy the property that $\Pi \cdot g^i(\Pi,d) \equiv 0$, then as $t \to \infty$ $E(\Pi,d) = \sum_{i=1}^{N} e[\Pi,d^i;\epsilon^i] \to 0$. Thus $\dot{\Pi} \to 0$ and, since $\dot{E}_k(\Pi,d) = \sum_{\ell}^{K}(\partial E_k/\partial\Pi_\ell)\,\dot{\Pi}_\ell$, then $\dot{E} \to 0$. Thus, prices converge to equilibrium prices and, once there, remain unchanged. However, it is entirely possible that $g^i(\Pi,d)$ does not converge and, therefore, that equilibrium contracts are never attained. To show this we construct an example of an (A-H, N-T).

3. A PARTICULAR ARROW-HAHN TYPE PROCESS

Let (2a) remain as above except that we will normalize prices. Let $M = \{m \in R^K \mid M_K = 1\}$. We write our system as:

$$\text{(4a)} \quad \dot{m}_k^i = e_k(\Pi, d^i, \epsilon^i) \qquad \forall\ k = 1,\ldots,K-1$$

$$\text{(4b)} \quad \dot{m}_K^i = 0 \qquad \forall\ i = 1,\ldots,N.$$

$$\text{(4c)} \quad \dot{d}_k^i = m_k^i - \Pi_k \qquad \forall\ K = 1,\ldots,K-1 \quad \forall\ i = 1,\ldots,N$$

$$\text{(4d)} \quad \dot{d}_K^i = -\sum_{k=1}^{K-1} \Pi_k(m_k^i - \Pi_k) \qquad \forall\ i = 1,\ldots,N.$$

Commodity K is a "numeraire" for which trade occurs in a residual manner. In particular, given $(\dot{d}_1^i,\ldots,\dot{d}_{K-1}^i)$, $\dot{d}_K^i$ is chosen so that $\Pi \cdot \dot{d}^i = 0$, or so that individual budgets balance. It is also easy to show that, under the rules (4c) and (4d), $\sum_{i=1}^N \dot{d}^i = 0$ along the path because $\Pi = \frac{1}{N}\sum_{i=1}^N m^i$. Thus, this process satisfies all the conditions of Arrow and Hahn.

Now let us assume that we have a point (m^*, d^*) such that $\sum_{i=1}^N e(\Pi^*, d^{*i}; \epsilon^i) = 0$. Since this implies that $\dot{\Pi}^* = 0$, we can write our system as

$$\text{(5a)} \quad \dot{m}_k^i = e_k^{*i} = d_k^i \qquad k = 1,\ldots,K-1 \quad i = 1,\ldots,N$$

$$\text{(5b)} \quad \dot{d}_k^i = m_k^i - \Pi_k^* \qquad k = 1,\ldots,K-1 \quad i = 1,\ldots,N$$

$$\text{(5c)} \quad \dot{d}_K^i = -\sum_{k=1}^{K-1} \Pi_k^*(m_k^i - \Pi_k^*) \qquad i = 1,\ldots,N$$

and

$$\text{(5d)} \quad \dot{m}_K^i = 0 \qquad i = 1,\ldots,N,$$

where $e^{*i} = e(\Pi^*, d^{*i}; \epsilon^i) - d^{*i}$. We note that for each i, the rules of motion for (m^i, d^i) are independent of (m^h, d^h) for any $h \neq i$. Thus, without loss of generality, we can look at the behavior of (m^1, d^1). The behavior of d^1_K is determined entirely by the behavior of $(d^1_1, \ldots, d^1_{K-1})$. We can, therefore, look at the system

$$\text{(6a)} \quad \dot{\tilde{m}} = \tilde{d}^1 + e^{*1}$$

$$\text{(6b)} \quad \dot{\tilde{d}}^1 = \tilde{m}^1 - \Pi^*$$

where $\tilde{m}^1 = (m^1_1, \ldots, m^1_{K-1})$, and $\tilde{d}^1 = (d^1_1, \ldots, d^1_{K-1})$. The homogeneous part of this linear system is

$$\begin{bmatrix} \dot{\tilde{m}}^1 \\ \dot{\tilde{d}}^1 \end{bmatrix} = \begin{bmatrix} 0 & I_{K-1} \\ I_{K-1} & 0 \end{bmatrix} \begin{bmatrix} m^1 \\ d^1 \end{bmatrix}$$

which is not stable since it has K-1 characteristic roots equal to 1 and K-1 characteristic roots equal to -1. Thus, we cannot expect, in general, that the mechanism described in (4) will converge to equilibrium contracts <u>even if prices converge to equilibrium prices</u>. If $K = 2$, this is immediately obvious, since the solution of (6) is then

$$m^1 = c_1 e^t + c_2 e^{-t} + \Pi^*$$

$$d^1 = c_1 e^t - c_2 e^{-t} - e^{*1}$$

where $\quad c_1 = \frac{1}{2} [m^{*1} - \Pi^* + e^{*1} + d^{*1}]$

and $\quad c_2 = \frac{1}{2} [m^{*1} - \Pi^* - e^{*1} + d^{*1}]$

Thus, unless $c_1 = 0$, $m^1 \to \pm \infty$ and $d^1 \to \pm \infty$. It is easy to show that $m^{*1} - \Pi^* + e^{*1} + d^{*1}$ need not be equal to zero, even though $\Sigma^N_{i=1}(m^{*1} - \Pi^* + e^{*1} + d^{*1}) = 0$

This example also emphasizes the fact that the rules (4c-d) for altering contracts do not ensure that $x^i = \omega^i + d^i$ is always an admissible consumption vector. Our next example of a non-tatonnement process will, however, correct this fault.

In the next section, we will show that by altering the communication system, while leaving the rules for trading alone that it is possible to "create" a process which yields a globally quasi-stable system.

4. A GENERALIZED NON-TATONNEMENT SYSTEM

In this section we alter the rules by which messages are calculated. Assume that for all $\omega^i + d^i$ such that $U^i(\omega^i + d^i) \geq U^i(\omega^i)$ it is true that $\partial U^i/\partial x^i_K \geq \delta > 0$ when evaluated at $x^i = \omega^i + d^i$. For the purposes of this paper, this assumption need hold only on the set of attainable consumptions for i (i.e. the projection on $\{x^i\}$ of $\{x^1,\ldots,x^N) \mid x^i \geq 0$ and $\Sigma^N_{i=1} x^i = \Sigma^N_{i=1}\omega^i\}$). Let $MRS^i(x^i) = [U^i_1,\ldots,U^i_{K-1}]/U^i_K$ evaluated */ at x^i. MRS^i is merely a vector of marginal rates of substitution of commodity K for commodity k.

The process we wish to consider is (where $\Pi = \Sigma^N_{i=1} \frac{1}{N} m^i$):

$$\text{(7a)} \quad m^i_k = MRS^i_k[\omega^i + d^i] \qquad i = 1,\ldots,N$$

$$k = 1,\ldots,K-1$$

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*/ We use U^i_k to indicate $\partial U^i/\partial x^i_k$.

(7b) $m^i_K = 1$ $\quad i = 1,\ldots,N$

(7c) $\dot{d}^i_k = m^i_k - \Pi_k$ $\quad k = 1,\ldots,K-1$

$\quad i = 1,\ldots,N$

(7d) $\dot{d}^i_K = -\sum_{\ell=1}^{K-1} \Pi_\ell (m^i_\ell - \Pi_\ell)$ $\quad i = 1,\ldots,N$

Note that (7c-d) are identical to (4c-d). The difference between (4) and (7) occurs in the (a) equations. In (4), m^i was basically an indirect controller of the system, yielding price behavior that is inertial. In (7), m^i becomes a direct control thus eliminating the inertial nature of prices.

The mechanism described by (7) has several desirable properties. First, the utility of any individual is never decreasing along the path of the system. It is easy to show that $\dot{U}^i = \sum_{\ell=1}^{1} U^i_k \dot{d}^i_k =$ $U^i_K [\sum_{\ell=1}^{K-1} (MRS^i_\ell - \Pi_\ell)^2] \geq 0$.

We now assume that $\{x^i \geq 0 \mid U(x^i) \geq U(\omega^i)\} \subseteq \{x^i \mid x^i_k > 0 \quad \forall\ k=1,\ldots,K\}$. That is, consumptions preferred or indifferent to the initial endowment must contain positive amounts of all commodities. With this assumption and the fact that $\dot{U}^i \geq 0$ we will be assured that along the path of the process $\omega^i + d^i(t)$ will always be an admissible consumption.

Thus, A.13.1 of Arrow and Hahn (p. 328) is satisfied. Another process with this property is the Edgeworth Barter Process of Uzawa. However, that system has an undesirable property in that it requires some centralized communication through the use of a social welfare function, $S(U^1,\ldots,U^N)$. This leads to the second desirable property of (7). It is informationally decentralized in the sense of Hurwicz (1960). That is, each i needs to know (to compute his m^i and his $\dot{d}^i$) only his own characteristic, ϵ^i, and an aggregate of the others messages $\Pi = \frac{1}{N} \sum_i m^i$. In addition, each i sends a message which is dimensionally equivalent to a price vector.

One can provide the following interpretation of the rules embodied in (7). Each agent is sending a vector of individualized relative prices indicating his maximum buying (minimum selling) price of good k in terms of good K. He is then allocated commodities in proportion to the difference between the prices he is willing to pay, m^i, and the "market" or average price, Π. Whether one can actually formalize a parable in discrete time which would yield (7c and d) as the allocating mechanism, in the limit, is still an open question.*

A third property of this mechanism is that it contains incentive-compatible rules in a restricted sense. In particular, it is true that given arbitrary sequences of messages of the other agents, say $m^2(t),\ldots,m^N(t)$, the only way that Mr. 1 can ensure that $\dot{U}^1 \geq 0$ is for him to send $m^1(t) = \text{MRS}\ (\omega^i + d^i(t))$. This is easy to see since

$$\dot{U}^1 = U_K^1 \left[\sum_{\ell=1}^{K-1} \text{MRS}_\ell^1 \dot{d}_\ell^i + \dot{d}_K^i\right] =$$

$$U_K^1 \sum_{\ell=1}^{K-1} (\text{MRS}_\ell^1 - \Pi_\ell)(m_\ell^i - \Pi_\ell).$$

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* For example, in a random search model without a centralized Walrasian market, one might expect that people with radically differing MRS's might trade larger quantities with each other than with those with similar MRS's. Thus, on average, one might expect to see behavior as described in (7). It seems hardly necessary to add that this is only wild speculation on my part.

Thus, if Π is unknown, a priori, $m^1 = MRS^1$ will ensure $\dot{U}^1 \geq 0$. This type of incentive compatibility is identical to that of Dreze and de la Vallee Poussin (1971). It is important to note, however, that incentive compatibility in the sense of Hurwicz (1970) does not obtain in either (7) or in Dreze-Vallee Poussin. That is, if 1 knows (or can predict from knowledge of preferences) the sequences $m^2(t),\ldots,m^N(t)$, then he can by sending $m^1(t) \neq MRS^1(\omega^i + d^i(t))$ ensure that in the limit $U^1[\omega^1 + d^1(\infty)]$ will be greater than he would get by sending $m^1(t) = MRS^1(t)$.

To see this consider the following problem: given $m^2(t),\ldots,m^N(t)$, for all $t \geq 0$, choose $\bar{m}^1(t)$ to maximize $U^1[\omega^1 + d^1(\infty)]$ where $d^1(\infty) = \lim_{t\to\infty} d^1(t)$ where $d^1(t)$ solves $\dot{d}^1_k = m^1_k(t) - \frac{1}{N}\Sigma^N_{i=1} m^i_k(t)$, for $k = 1,\ldots,K-1$ and $\dot{d}^1_K = -\Sigma^{K-1}_{k=1}(\frac{1}{N}\Sigma^N_{i=1} m^i_k(t))(m^1_k(t) - \frac{1}{N}\Sigma^N_{i=1} m^i_k(t)]$. This is an optimal control problem where d^1 is the state variable and m^1 is the control variable. It is fairly easy to show that the optimal solution to this problem is $\bar{m}^1_k(t) = \delta_k/\delta_K + (\frac{1}{N}\Sigma_{i\neq 1} m^i_k)(2 - N/N-1)$ where $\delta_k = U^i_k[\omega^1 + d^1(\infty)]$. It is highly unlikely that $\bar{m}^1_k = MRS^1_k$ along this path. [For $N = 2$ the solution is particularly simple since $\bar{m}^1_k = \delta_k/\delta_K$ is constant along the optimal path no matter what $m^2(t)$ is.] It is possible to assume that 1 knows $U^2,\ldots,U^N$ and solves the problem above subject to equations (7a) for $i = 2,\ldots,N$ and (7b-d) for $i = 1,\ldots,N$. However, this is extremely messy and really leads to no new insights.

Finally, the system described in (7) is globally quasi-stable under the assumptions we have made. This follows directly from some of the previously mentioned properties. In particular, let $W(d) = \Sigma_{i=1}^{N} \alpha_i U^i[\omega^i + d^i]$ where $\alpha_i > 0 \ \forall_i$. Let $\overline{W} = \max W(d)$ subject to $\sum_i d^i = 0$ and $\omega^i + d^i \geq 0$. $\overline{W}$ exists. Let $V(d) = \overline{W} - W(d)$. $V(d) \geq 0$. Now along the path generated by (7), $\dot{V}(d) = -\sum_i \alpha_i \dot{U}^i = -\sum_i \alpha_i U_K^i \cdot [\sum_{k \neq K} (MRS_k^i - \Pi_k)^2]$. It is easy to see that $\dot{V} < 0$ unless $\Pi_k = MRS_k^i \ \forall\ i = 1,\ldots,N$ and every $k = 1,\ldots,K-1$. It is also obvious that $\dot{d}^i = 0$ iff $MRS_k^i - \Pi_k = 0$ for all $k = 1,\ldots,K$. Thus, $\dot{V} \leq 0$ and $\dot{V} = 0$ iff $\dot{d}^i = 0$ for all i. Thus, the global quasi-stability of (7) can be established by using V as a Lyapunov function since the sequence $d(t)$ belongs to a compact set.

To summarize, the system (7) is a globally quasi-stable, Pareto-satisfactory, informationally decentralized, non-tatonnement adjustment process which possesses a limited form of incentive compatability.