Discussion Paper No. 859

MONOTONIC PREFERENCES AND CORE EQUIVALENCE

by

Alejandro M. Manelli*

Managerial Economics and Decision Sciences

Kellogg Graduate School of Management
Northwestern University
Evanston, IL 60208

Revised 9/89

ABSTRACT

Examples of well behaved sequences of economies, without monotonic preferences, are constructed. These economies have core allocations that cannot be decentralized by prices, even in a weak sense. Relaxing the monotonicity assumption results in core allocations that are not uniformly integrable, breaking the connection between the continuum and the large finite model. If in addition preferences are nonconvex, even replica sequences of economies with core allocations satisfying the equal treatment property may fail to exhibit equivalence properties. Sufficient conditions to restore convergence are provided.
Keywords: Core convergence, monotonic preferences.

---

*This paper is based on part of my Ph.D. dissertation filed at the University of California, Berkeley. I am grateful to my adviser Professor Robert M. Anderson for his assistance, which went far beyond the call of duty, and to Professor Gerard Debreu. This research originated in a seminar given by Professor Donald Brown. I also benefited from discussions with Andrew Foster, William Geller, Karl Iorio, Kathy Weinberger and from comments by three anonymous referees. Support from an Alfred P. Sloan Doctoral Dissertation Fellowship is acknowledged.

1. INTRODUCTION

Monotonicity of preferences is a common assumption in the theory of the core of an economy. It implies that any increase in consumption will be welcomed by a consumer, independent of the reference consumption bundle. Although it seems to be an innocuous assumption, there are several important instances in which monotonicity is not satisfied. The simplest one embraces commodities or services that some economic agents dislike. A second example of failure of monotonicity is given by satiation points. Many goods may increase the consumer's welfare up to a point, but still become a burden if they are consumed in excess. Goods that must be consumed in fixed proportions constitute yet another exception to the monotonicity assumption. Coffee and cream and cars and tires are typical examples.

In this paper we shall study the relationship between core allocations and the set of competitive equilibria when preferences are not monotonic.

While considerable attention has been paid to the situation with nonconvex preferences, much less attention has been paid to weakening the monotonicity assumption.[2] This is perhaps due to the fact that, in the replica setting of Debreu and Scarf (1963), and in the nonatomic setting of Aumann (1964), monotonicity plays an unimportant role. In fact Debreu and Scarf proved that, under local non-satiation and strong convexity, the allocations that belong to the core of every replica can be supported as quasi-equilibria. Aumann in the continuum model assumed monotonicity. It is immediate from his proof, however, that local non-satiation suffices to show that core allocations are quasi-equilibria.[3]

Since attempts to prove core convergence theorems without convexity,

---

[2]For instance Anderson (1978) (1985a) (1985b), Arrow and Hahn (1971), Brown and Robinson (1974), Cheng (1983), Dierker (1976), Hildenbrand (1974) and Kahn (1974) consider nonconvex preferences.

[3]See Hildenbrand (1982).

which plays a distinct role in finite economies as opposed to nonatomic ones, were relatively successful, it stood to reason that weakening monotonicity which seemed to play the same role in finite and continuum economies, would create no difficulties. Furthermore, given the close relationship between the proof in Anderson (1978) and the proof of Aumann's theorem in Hildenbrand (1974), it was natural to suppose that it would be an easy matter to modify Anderson's proof to accommodate a weakening of monotonicity. By contrast, it had been known from the outset that eliminating the convexity assumption significantly complicated the proofs of core convergence theorems. In this regard, an example of Anderson and Mas-Colell (1986) shows that the stronger convergence theorems may fail when preferences are nonconvex.[4] Thus, it had been clear that eliminating the convexity assumption raised subtle issues that did not appear to be present when consideration turned to monotonicity.

We show that this complacency concerning the role of monotonicity in core convergence is misguided. We construct examples of well behaved sequences of convex economies where the core allocations do not approach the set of competitive equilibria, even for a weak notion of approximation. Weakening the monotonicity assumption may result on core allocations that are not uniformly integrable. Thus the continuum model fails, in the absence of monotonicity, to adequately reflect the behavior of large finite economies. If in addition preferences are nonconvex, even uniformly bounded core allocations may fail to converge in a weak sense.

We introduce a condition on preferences and core allocations which suffices to prove that core allocations are close to being demand-like for some price. Essentially, we require that neither the core bundle nor the nonconvexities associated with any single individual increase too rapidly. It

[4]This example appears as an appendix in Anderson (1988).

is not possible in general to dispense with this condition as it is illustrated by several examples. We prove that replica sequences of economies with convex (not necessarily strictly convex) preferences satisfy the mentioned condition.

Our work is closely related to that of Anderson (1978), Arrow and Hahn (1971) and Dierker (1975). They obtain a bound on the sum of the measure of non-competitiveness of core allocations which is independent of the number of agents in the economy. Anderson's result admits nonconvexities but requires weakly monotone preferences. Vind (1965) shows that the number of agents that violate certain competitiveness condition is independent of the size of the economy. Vind did not assume convexity or monotonicity.

Our first example is a sequence of economies (with complete, continuous, convex and transitive preferences) which converges in distribution to a limit economy, but for which no sequence of prices approximately decentralizes the core allocations.[5] In this example preferences are proper.[6] Properness restricts the guaranteed direction of preference to increases in certain composite bundles of commodities. Only changes in consumption that belong to a given convex cone in the positive orthant are required to improve the consumer's well-being. When the cone is the entire strictly positive orthant, preferences are monotonic.

In the example there is a large coalition for which the joint core assignment is strictly less than their joint endowments. In turn the excess of goods is distributed among few individuals. In the limit there is an agent (measure zero) with a non-finite core bundle. Although the core allocations of the limit economy can be supported as quasi-equilibria, the core

[5]See Hildenbrand (1974) for the notion of limit economy.

[6]Proper preferences are used by Grodal, Trockel and Weber (1984). Also see Mas-Colell (1986).

allocations of large finite economies are far from being demand-like. The difference arises because core allocations need not be uniformly integrable when preferences are not monotonic. The lack of uniform integrability however, is not the only obstacle. Although an assumption of the sort is enough to assure that the average budget deviation is small for some price, it is not enough to restore core convergence when preferences are nonconvex.

A second example presents a sequence of economies where only one agent has nonconvex preferences. Core allocations as well as endowments are uniformly bounded. The sequence converges in distribution to a continuum economy and the supports of the distributions converge with respect to the topology of closed convergence. At any price for which the budget deviation condition is satisfied it is possible to find consumption bundles which are preferred to the core allocation and in average, strictly less expensive than the total endowments. In this example the nonconvexities increase, in a certain sense, with the number of agents in the economy. Some bound on nonconvexities must be imposed if this kind of example is to be eliminated.

Gabszewicz and Mertens (1971) provide conditions for economies with a continuum of agents and small atoms to exhibit core equivalence properties.[7] Theorem A in Shitovitz (1973) and Theorem 1 in Kahn (1976) show, for the continuum and large finite setting respectively, that core equivalence is violated in general, when large traders are present. Our examples do not have significant traders, either in the sense of having large endowments or in the sense of having large weight as in Shitovitz and Kahn.

2. DEFINITIONS AND NOTATION:

For any x in $\mathbb{R}^k$, $\|x\| = \text{Max}\,\{|x^i| : 1 \le i \le k\}$, $\|x\|_1 = \sum_{i=1}^{k}|x^i|$, where

---

[7]Also see Gabszewicz and Dreze (1971) and Gabszewicz (1985).

x^i is the i^{th} component of x, and $B(x,\varepsilon) = \{y : \|y - x\| < \varepsilon\}$. For any subset D of $\mathbb{R}^k$, $B(D,\varepsilon) = \{z : \exists y \in D \text{ with } \|y - z\| < \varepsilon\}$ and con(D) is the convex hull of the set D.

A preference relation $\mathfrak{p}$ is an element of $\mathfrak{P}$, the set of irreflexive binary relations on $\mathbb{R}^k$. A preference relation $\mathfrak{p}$ is convex if for any two commodity bundles x and x' such that $x' \;\mathfrak{p}\; x$, the bundle $(\alpha x' + (1 - \alpha)x) \;\mathfrak{p}\; x$ for all α in (0,1).

An exchange economy is a map $\mathcal{E} : A \longrightarrow P(\mathbb{R}^k) \times \mathfrak{P} \times \mathbb{R}^k$ that assigns to each agent $a \in A$, a consumption set $X(a)$, a preference relation $\mathfrak{p}_a$ and an endowment $e(a)$, where $P(\mathbb{R}^k)$ is the set of subsets of $\mathbb{R}^k$. $\mathcal{E}^n$ is a replica sequence of economies if for all n, $\mathcal{E}^n(A^n) = T$ where T is a finite set and $\forall t \in T$, $|\mathcal{E}^{n^{-1}}(t)| = n$. An allocation g is a consumption assignment, $g(a) \in X(a)$ $\forall a \in A$, that precisely exhausts the total endowments. The core of the economy $\mathcal{E}$, $\mathcal{C}(\mathcal{E})$, is the set of allocations that cannot be blocked by any coalition. A coalition $B \subseteq A$ blocks an allocation f if there is an assignment g, i. e. $g(a) \in X(a)$ $\forall a \in B$, such that $\sum_B g(a) = \sum_B e(a)$ and $g(a) \;\mathfrak{p}_a\; f(a)$ $\forall a \in B$. For a given f in $\mathcal{C}(\mathcal{E})$ and for any agent a in A, define the set of "net preferred trades" by $\phi(a) = \{x - e(a): x \in X(a),\ x \;\mathfrak{p}_a\; f(a)\} \cup \{0\}$. Prices belong to $U = \{p \in \mathbb{R}^k: \|p\| = 1\}$.

We employ a very weak notion of approximation: Given an economy $\mathcal{E}$, $a \in A$, $x \in \mathbb{R}^k$ and $p \in U$, define

$\psi(x,a) = |p \cdot [x - e(a)]| + |\inf\{p \cdot [y - e(a)]: y \in X(a),\ y \;\mathfrak{p}_a\; x\}|.$[8]

The first term is a measure of the budget deviation and the second one, of the excess expenditure. Note that if $\mathfrak{p}_a$ is continuous and $p >> 0$ then $\psi(x,a)$

---

[8]See Kahn (1974), Hildenbrand (1974), Dierker (1975) and Anderson (1978) among others.

= 0 implies that x belongs to the demand set. ψ measures the extent to which an agent's commodity bundle looks like a demand relative to a given price.

3. EXAMPLES:

This section contains two examples. The first one consists of a sequence of economies with convex and proper preferences. These economies converge in distribution to a limit economy but fail to exhibit core equivalence properties. In the second example, preferences are nonconvex and the failure persists even for uniformly bounded core allocations.[9]

The economies in the first example have two goods and three groups of identical agents. The n^{th} economy has 1 agent in the first group, one agent in the second and n agents in the third. The preferences of the first two agents change with the economy. All agents in the third group are equal and have the same characteristics over the entire sequence of economies. The positive orthant is the consumption set for all agents in all economies.

Fix an open convex cone $V = \{v \in \mathbb{R}^2_+ : v = \alpha\ (m,1) + \beta\ (1,m);\ \alpha,\beta > 0\}$ where m is any real number greater than one. Preferences in the n^{th} economy are only required to satisfy the following conditions:

For agent 1, $\left\{x:\ x \succ_1 (n,\ 0)\right\} = \left(n,\ 0\right) + V$

For agent 2, $\left\{x:\ x \succ_2 \left(0,\ \frac{n}{m}\right)\right\} = \left(0,\ \frac{n}{m}\right) + V$

For agents in group 3, $\left\{x:\ x \succ_3 \left(0,\ \frac{m-1}{m}\right)\right\} = \left(0,\ \frac{m-1}{m}\right) + V$

It is useful to think of the boundary of the cone V as the indifference curve through the consumption bundle on which the cone originates. Agents in group 3 are endowed with one unit of each good. The endowment of all other agents is zero. The core allocation f^n for the n^{th} economy assigns $(n,0)$ to

---

[9] Anderson (1978) and Dierker (1975) proved equivalence theorems for general sequences of nonconvex economies with monotonic preferences.

agent 1, $\left(0,\ \frac{n}{m}\right)$ to agent 2 and $\left(0,\ \frac{m-1}{m}\right)$ to each agent in group 3. The sets of "net preferred trades" $\phi(\cdot)$, for $m = 2$ and $n = 1$, are depicted in Figure 1.

We proceed to verify that the proposed assignment f^n is in fact a core allocation. Since $\phi(a)$ does not intersect $(-V)$ the allocation f^n is individually rational. Note that $0 \notin \{x - e(a)\colon x \succ_a f^n(a)\}$ for all a.

If the coalition formed by k $(\leq n)$ individuals of type 3 and agent 2 blocked f^n, we could find a consumption g such that $g(2) + \sum_{i=3}^{k+2} g(i) = \sum_{i=3}^{k+2} e(i)$ and $g(i) = f^n(i) + v^i$, $2 \leq i \leq k+2$. Let $f^n(3)$ be the representative commodity bundle for agents in group 3 and let $v = \sum_{i=3}^{k+2} v^i$. Then $(k,k) = \sum_{i=3}^{k+2} e(i) = g(2) + \sum_{i=3}^{k+2} g(i) = f(2) + k\,f(3) + v = k\left(0,\ \frac{m-1}{m}\right) + \left(0,\ \frac{n}{m}\right) + v$ $= \left(0,\ k\,\frac{m-1}{m} + \frac{n}{m}\right) + v$. Therefore $v = \left(k,\ k - k\,\frac{m-1}{m} - \frac{n}{m}\right) = \left(k,\ \frac{k-n}{m}\right)$, which is a contradiction since $\frac{k-n}{m} \leq 0$ and $v \in V$.

A similar argument shows that the coalition formed by k $(\leq n)$ agents of type 3 and agent 1 cannot block f^n: $(k,k) = k\,f(3) + f(1) + v = k\left(0,\ \frac{m-1}{m}\right)$ $+ \left(n,\ 0\right) + v$ and therefore $v = \left(k - n,\ k\,\frac{m-1}{m}\right)$. Since $(k - n) \leq 0$, $v \notin V$.

Consider now the coalition formed by k $(\leq n)$ individuals of type 3: $(k,\ k) = k\,f(3) + v = \left(0,\ k\,\frac{m-1}{m}\right) + v$. Thus $v = \left(k,\ \frac{k}{m}\right)$. Since $\frac{m}{k} > 0$ and $\frac{m}{k}\,v = (1,\ m) \notin V$, then $v \notin V$.

For the coalition formed by k $(\leq n)$ agents of type 3 and agents 1 and 2 the situation is as follows: $(k,k) = k\,f(3) + f(1) + f(2) + v = k\left(0,\ \frac{m-1}{m}\right)$ $+ \left(n,\ 0\right) + \left(0,\ \frac{n}{m}\right) + v$ therefore $v = \left(k - n,\ \frac{k-n}{m}\right)$ which implies $v \notin V$.

The coalition formed by agents 1 and 2 can not block either since they have no endowments and $0 \notin \{V + (0,\ n/m)\}$ and $0 \notin \{V + (n,\ 0)\}$. We conclude

f^n is a core allocation for the n^{th} economy.

We now verify that there is no price system that approximately decentralizes the core allocations. Technically we prove that $(1/n)$ $\sum_{A^n}\psi(f^n(a),a)$ is bounded away from zero for any p. First, consider the average budget deviation.

$$\frac{1}{n}\sum_{i=1}^{n+2}|p \cdot (f^n(i) - e(i))| = \frac{1}{n}\Big[n\,|p \cdot (f^n(3) - e(3))| + $$
$$+ |p \cdot (f^n(2) - e(2))| + |p \cdot (f^n(1) - e(1))|\Big] =$$
$$= \left|-\left(p^1 + p^2\frac{1}{m}\right)\right| + \left|p^2\frac{1}{m}\right| + |p^1|,$$

expression which does not depend on n and therefore does not converge to zero. Second, consider the average value of the least expensive commodity bundle preferred to the core allocation.

$$\frac{1}{n}\sum_{i=1}^{n+2}\left|\inf\left\{p \cdot (x - e(i)):\ x \succ_i f^n(i)\right\}\right| =$$

$$= \frac{1}{n}\sum_{i=1}^{n+2}\left|\inf\left\{p \cdot (x - e(i)):\ x \in f^n(i) + V\right\}\right|. \tag{1}$$

Let $D = \{p\colon\ p\,V \geq 0\}$. If $p \notin D$ then there is v in V such that $p \cdot v < 0$, which implies $p \cdot (sv) < 0\ \ \forall s > 0$. Thus $|\inf p \cdot (f^n(i) + s\,v - e(i))|$ can be made arbitrarily large, which implies that expression (1) stays bounded away from zero. If $p \in D$ then $|\inf \{p \cdot (x - e(i)):\ x \in (f^n(i) + V)\}| = |p \cdot (f^n(i) - e(i))|$ but we have already shown that the average budget deviation does not converge to zero.

Remark 1:

There are complete, transitive, convex preferences, which are also representable by a concave utility function and satisfy the requirements of the example. To see this, let the boundary of the cone V be the indifference curve through the core bundle (on which the cone originates), and let a family of nested cones (with parallel sides) be the family of indifference curves describing the preferences. Also note that the example holds for any cone V strictly included in the positive orthant (i.e. any m > 1) and therefore for preferences that are almost monotonic. It is a simple matter to modify the example so that the preferences of agents 1 and 2 do not change with the economy. Preferences can also be made strictly convex.

The sequence of economies presented converges in distribution to a continuum economy. A continuum or atomless economy is a measurable mapping from a nonatomic measure space of agents $(A, \mathcal{A}, \upsilon)$ into the space $\mathfrak{P} \times \mathbb{R}^k_+$ of preferences cross endowments, with the property that the mean endowment is finite. The continuum of agents makes precise the notion of a perfectly competitive economy, that is an economy where the influence of every individual agent is negligible. In the example the continuum model fails to capture the properties of large finite economies. The limit economy has an agent whose core allocation is not finite. Although this is not crucial for the continuum model (any single agent has measure zero), it is essential in the large finite case. Example 1 suggests that when core allocations do not increase too rapidly with the size of the economy, the conflict between the large finite and the continuum approach is resolved. Unlike the monotonic case, this is not so when preferences are nonconvex.

The second example introduces a sequence of economies where core allocations and endowments are uniformly bounded and where only one agent has nonconvex preferences. The expenditure-minimizing condition, however, is not satisfied.

The n^{th} economy $\mathcal{E}^n$ has $(n - 1)$ agents in group 1, n agents in group 2 and one agent in group 3. All agents in all economies have endowment $e = (1,$

1). Agents in the same group are identical and receive the same allocation. All agents have convex preferences except the agent in group 3. Fix $m = 2$ and let V be given as in example 1. *Let f be* the allocation $f(1) = (0, 0.5)$, $f(2) = (2, 1.5)$, $f(3) = (0, 0.5)$. Any preferences that satisfy the following conditions will convert f into a core allocation.

For group 1, $\{x: x \succ_1 (0, 0.5)\} = (0, 0.5) + V$

For group 2, $\{x: x \succ_2 (2, 1.5)\} = (2, 1.5) + V$

For group 3, $\{x: x \succ_3 (0, 0.5)\} = \{(0, 0.5) + V\} \cup \{(n + 1, 1) + \mathbb{R}^2_{++}\}$

Note that only the preferences of the type 3 agent change with the economy. As in the first example, it is easy to verify that f is a core allocation for all $\mathcal{E}^n$. The sets of "net preferred trades" $\phi(\cdot)$, for $n = 1$, are depicted in Figure 2. Note that the origin does not belong to any of the net preferred sets.

Despite the fact that for a price $p = (-0.5, 1)$ the value of the deviation of core allocations from endowments is zero, the net value of the least expensive consumption bundle (preferred to the core allocation) does not go to zero in average.

We emphasize that the sequence of economies is well behaved. Since endowments are constant and agents in the first two groups have the same preferences over all economies, the sequence $\mathcal{E}^n$ converges in distribution to a continuum economy. Furthermore, the supports of the distributions converge, in the topology of closed convergence, to the support of the limit economy.[10]

Remark 2:

Example 2 can be transformed into a 2-type replica sequence of economies where all agents of the same type receive the same core bundle (equal

---

[10]For a definition of closed convergence see Hildenbrand (1974).

treatment property). We briefly indicate how to convert example 2 into a replica example. First, consider a sequence of economies with no agents of type 1, n agents of type 3 and n agents of type 2, and with endowments, preferences and core allocations as in example 2. The new sequence has n agents with nonconvex preferences. As stated, the new example is not a replica sequence because there are no single preferences for the consumers in group 3 that can account for the increasing nonconvexities as n becomes large. To see this, note that the core bundles do not change with the economy and therefore the nonconvexities would have to lie on the same indifference curve for all n. Second, alter preferences and the proposed core allocations so that the increasing nonconvexities now lie on indifference curves farther away from the origin but generated by the same preferences. For instance, let $f^n(2) = \left(1.9 + \frac{0.1}{n},\ 1.4 + \frac{0.1}{n}\right)$ and $f^n(3) = \left(0.1 - \frac{0.1}{n},\ 0.6 - \frac{0.1}{n}\right)$ and let preferences satisfy: For type 2, $\{x\colon x \succ_2 f^n(2)\} = f^n(2) + V$, and for type 3 $\{x\colon x \succ_3 f^n(3)\} = \{f^n(3) + V\} \cup \left\{\left(n + 1,\ 1.1 - \frac{0.1}{n}\right) + \mathbb{R}^2_{++}\right\}$. Note that $\{x : x \succ_3 f^n(3)\} \subseteq \{x : x \succ_3 f^k(3)\}$ for all $k < n$.

4. RESULTS

The failure of convergence in the examples seems to have two different provenances. First, the core bundle of certain individuals may increase rapidly with the size of the economy. Second, nonconvexities play an important role when preferences are not monotonic or the consumption sets are not bounded below. The purpose of this section is to elucidate the connection between monotonicity and convexity. Proposition 1 identifies conditions under which the sum of the measure of non competitiveness of core allocations is bounded, independently of the number of agents in the economy.

Given an open convex cone $V \subseteq \mathbb{R}^k_+$, $\mathfrak{P}(V)$ is the set of preferences which satisfy:

i) Properness: $\forall x \in \mathbb{R}^k_+$ $(x + V) \mathfrak{p}\, x$ and

ii) Free Disposal: if $y \mathfrak{p}\, x$ then $(y + v) \mathfrak{p}\, x$, $\forall v \in V$.

When V is the strictly positive orthant, $\mathfrak{P}(V)$ becomes the set of monotonic preferences. Free disposal here is defined within the cone V. The free disposal assumption dispenses with the need of transitivity in Proposition 1.

Let $\mathcal{E}$ be an economy with preferences in $\mathfrak{P}(V)$, and let f be in $\mathcal{C}(\mathcal{E})$. We will assume

(C1) *For every consumer a in A, there is $L_a > 0$, such that $\forall x \in con(\phi(a))$, $\exists\ z(x) \in (\phi(a) + V)$ with $\|x - z(x)\| \le L_a$.*

C1 is equivalent to requiring that $\delta(\text{con}(\phi(a)), \phi(a) + V) < L_a$, where $\delta(\cdot,\cdot)$ is the Hausdorff distance between sets.[11] To see this, note that $(\phi(a) + V) \subseteq \text{con}(\phi(a))$: Since $(\phi(a) + V) = (\phi(a) \cup V)$, it suffices to show that $V \subseteq \text{con}(\phi(a))$. Fix $x \in V$, let $B(x,\varepsilon) \subseteq V$ and let $z \in \phi(a)$. For n large, $x \in (z/n + B(x,\varepsilon))$. Thus $nx \in (z + n\, B(x,\varepsilon)) \subseteq \phi(a)$ (by properness). Since 0 and nx belong to $\phi(a)$, $x \in \text{con}(\phi(a))$. Our claim now follows from the definition of Hausdorff distance.

C1 requires that any consumption bundle x in the convex hull of $\phi(a)$ be, at most, L_a away from the set $(\phi(a) + V)$. Figures 3a and 3b clarify the meaning of C1. They picture the set $\{e(a)\} \cup \{x : x\, \mathfrak{p}_a\, f(a)\}$, which is just $\phi(a) + e(a)$. The boundaries of its convex hull are marked in both figures. C1 holds for the case graphed in Figure 3a. The vertical arrows represent a possible bound L_a. In Figure 3b the agent has preferences with increasing nonconvexities on a given indifference curve. A consumption bundle can always be chosen (from the shaded area) so that the distance to the indifference

---

[11] $\delta(C, D) = \text{Inf}\ \{\varepsilon\colon C \subseteq B(D,\varepsilon) \text{ and } D \subseteq B(C,\varepsilon)\}$. See Hildenbrand (1974).

curve is arbitrarily large. Therefore no bound L_a exists and C1 is not satisfied.

Arrow and Hahn (1971) also employ restrictions on nonconvexities, endowments and core allocations to obtain a bound on the sum of the budget deviations and on the sum of the excess expenditures.[12] Their measure of nonconvexities is more restrictive than ours. They define

$$r\left[\phi(a) + V\right] = \operatorname*{Sup}_{x \in \operatorname{con}\phi(a)} \ \operatorname{Inf} \left\{\operatorname{rad}(T)\colon\ x \in \operatorname{con}(T),\ T \subseteq \phi(a) + V\right\} \tag{2}$$

where rad(T) is the radius of the smallest ball containing T, and require that $r(\cdot)$ be bounded for all agents in all economies.[13]

Figure 3a illustrates the difference between C1 and expression (2). The commodity bundle x is a convex combination of e and y. The smallest ball containing $\{e, y\}$ has a radius of at least $\|e - y\| / 2$. Therefore it is possible to choose an element x to make (2) arbitrarily large. Any economy that verifies Arrow and Hahn's hypothesis will also satisfy our condition C1. The converse however is not true as Figure 3a shows.

Proposition 1: *Let $\mathcal{E}$ be an exchange economy with preferences in $\mathfrak{p}(V)$ and consumption sets $X(a)$ so that for any x in $X(a)$, $x + V \in X(a)$. Let $f \in \mathcal{C}(\mathcal{E})$. Choose v so that $B(v,1) \subseteq V$. Assume C1 holds and let $L = \mathrm{Max}\,\{L_a : a \in A\}$. Then, $\exists p \in U$ such that*

a) $\sum_A \left| p \cdot \left[f(a) - e(a)\right]\right| \leq 2kL\|v\|_1$

---

[12] Theorems 3, 4 and 7 in Khan (1974) also assume bounded nonconvexities.

[13] Arrow and Hahn assume local non-satiation and free disposal with respect to the positive orthant. These two assumptions combined imply monotonicity. To see this, fix x in $\mathbb{R}^k_+$ and let $z \in \mathbb{R}^k_{++}$. By local non-satiation $\forall \varepsilon > 0$, there is y with $\|y - x\| < \varepsilon$ such that $y \,\mathfrak{p}\, x$. By free disposal, $(y + \mathbb{R}^k_{++}) \,\mathfrak{p}\, x$. For ε small enough, $(x - y + z) \in \mathbb{R}^k_{++}$, or equivalently, $(x + z) \in y + \mathbb{R}^k_{++}$. Hence $(x + z) \,\mathfrak{p}\, x$. Arrow and Hahn's definition of $r(\cdot)$ is stated somewhat differently since they only consider monotonic preferences.

b) $\sum_A \left| \text{Inf} \left\{ p \cdot \left[x - e(a) \right] : \; x \in X(a), \; x \succ_a f(a) \right\} \right| \le 2kL\|v\|_1$

Proof: The proof follows closely Anderson (1978). Let $\Phi = \sum_A \phi(a)$. First we show that $\Phi \cap (-V) = \emptyset$ and then that con(Φ) does not go very far into $-V$. Finally Minkowski's Theorem establishes the existence of p.

Suppose $\Phi \cap (-V) \ne \emptyset$. Then $\exists \; G = \sum_A g(a)$, $g(a) \in \phi(a)$ and $G \in -V$. Let $B = \{a : g(a) \ne 0\}$ and let $g'(a) = g(a) + e(a) - \frac{G}{|B|}$. Therefore $\sum_B g'(a) = \sum_B g(a) + \sum_B e(a) - G = \sum_B e(a)$. Since $(-G/|B|) \in V$, $g'(a) \in X(a)$ and $g'(a) \succ_a (g(a) + e(a)) \succ_a f(a)$, which by free disposal implies $g'(a) \succ_a f(a)$.

By the Shapley-Folkman Theorem, for any $G \in$ con(Φ) we can write $G = \sum_{A \setminus K} g(a) + \sum_{i=1}^{k} g(a^i)$ where $g(a) \in \phi(a)$ for all $a \in A \setminus K$, $g(a^i) \in$ con($\phi(a^i)$), and $K = \{a^i : 1 \le i \le k\}$.

By C1, there is $L_i > 0$ and $z(a^i) \in (\phi(a^i) \cup V)$ so that $\|g(a^i) - z(a^i)\| < L_i$. Therefore

$$G = \sum_{A \setminus K} g(a) + \sum_{i=1}^{k} z(a^i) + \sum_{i=1}^{k} \left[g(a^i) - z(a^i) \right],$$

where

$$\left[\sum_{A \setminus K} g(a) + \sum_{i=1}^{k} z(a^i) \right] \in (-V)^c \quad \text{and} \quad \sum_{i=1}^{k} \|g(a^i) - z(a^i)\| < kL.$$

That is, G can be written as the sum of two terms, one of them belongs to $(-V)^c$ and the other is bounded by kL. Let $r = kLv$. Since $B(v, 1) \subseteq V$ by hypothesis, $B(r, kL) + V \subseteq V$. Therefore $G \notin -(r + V)$. We conclude that con(Φ) $\cap -(r + V) = \emptyset$.

By Minkowski's Theorem, there is $p \in U$, $p \ne 0$ such that

$\text{Inf}\,\{p \cdot \Phi\} \ge \sup \left\{ p \cdot w : \; w \in -(r + V) \right\} = - p \cdot r = - kL\,(p \cdot v) \ge -kL\|v\|_1$.

By continuity $p \cdot (f(a) - e(a)) \ge \text{Inf}\; p \cdot \phi(a)$.

Let $S = \{a : \; p \cdot (f(a) - e(a)) < 0\}$. Then

$$0 \ge \sum_S p \cdot (f(a) - e(a)) \ge \sum_S \text{Inf}\,\{p \cdot \phi(a)\} \ge - kL\|v\|_1.$$

Since f is an allocation, $\sum_A p \cdot (f(a) - e(a)) = 0$ and therefore

a) $\sum_A |p \cdot (f(a) - e(a))| = 2 \sum_S |p \cdot (f(a) - e(a))| \leq 2\, kL\|v\|_1$ and

b) $\sum_A |\text{Inf}\ \{p \cdot (x - e(a)) \colon x \in X(a),\ x \succsim_a f(a)\}| \leq$

$$\leq \sum_S |\text{Inf}\ \{p \cdot \phi(a)\}| + \sum_{A \setminus S} |p \cdot (f(a) - e(a))| \leq$$

$$\leq kL\|v\|_1 + kL\|v\|_1 \leq 2kL\|v\|_1$$ —✕—

Since v is chosen so that $B(v, 1) \subseteq V$, the bound, $2kL\|v\|_1$, depends on the cone V. As V becomes "narrower", $\|v\|_1$ becomes larger.

The following Proposition shows that the sum of the budget deviations over all agents is bounded and the bound depends on the difference between the core allocation and the endowment of a single individual.

Proposition 2: *Let $\mathcal{E}$ be an exchange economy with preferences in $\mathfrak{P}(V)$ and consumption sets $X(a)$ so that for any x in $X(a)$, $x + V \in X(a)$. Let $f \in \mathcal{C}(\mathcal{E})$. Then $\exists\ p \in U$ such that*

$$\sum_A |p\ (f(a) - e(a))| \leq 2\ k\ \|v\|_1 \left[\operatorname*{Max}_A \|f(a) - e(a)\|\right].$$

Proof: The proof is an application of Proposition 1. Fix an $\varepsilon > 0$. Let $H(a) = \{\phi(a) \cap B(f(a) - e(a), \varepsilon)\} \cup \{0\}$. Then $\forall a$, $H(a) \subseteq \phi(a)$. For any given agent a and x in $\text{con}(H(a))$, let $z(x) = 0$. Then by construction $\|x - z(x)\| = \|x\| \leq \|f(a) - e(a)\| + \varepsilon$. Let $L_a = \|f(a) - e(a)\| + \varepsilon$. Thus $H(a)$ satisfies C1. Following the proof of Proposition 1 we conclude

$$\exists p : \sum_{A^n} \left| p \cdot \left[f(a) - e(a)\right] \right| \leq 2k\|v\|_1 \left[\varepsilon + \operatorname*{Max}_A \|f(a) - e(a)\|\right].$$

Since the inequality holds for all $\varepsilon > 0$, it also holds at the limit. —✕—

Proposition 1 and 2 are stated in terms of a single economy. To obtain convergence in mean, situations where the bound increases very rapidly with respect to the size of the economy have to be ruled out. This is accomplished by our assumption C2. Note that $L^n = \text{Max}\{L_a\colon a \in A^n\}$ where L_a is defined as in C1.

(C2) $$\lim_{n \to \infty} \frac{L^n}{|A^n|} = 0$$

Corollary: *Let $\mathcal{E}^n$ be a sequence of economies. Let $f^n \in \mathcal{C}(\mathcal{E}^n)$. If the hypotheses of Proposition 1 and C2 hold then $\exists\ \{p^n\} \in U$, such that*

$$\lim_{n \to \infty} \frac{1}{|A^n|} \sum_{A^n} \psi\left[f^n(a),\ a\right] = 0.$$

We now consider how C2 is simplified when additional assumptions are made. Three alternatives are examined: a) monotonic preferences and consumption sets included in the positive orthant, b) convex preferences and c) bounded nonconvexities.

a. When preferences are monotonic, and the consumption sets lie in the positive orthant C2 becomes simply

(C2') $$\lim_{n \to \infty} \frac{1}{|A^n|} \max_{a \in A^n} \|e(a)\| = 0.$$ [14]

To see this note that con $\phi(a) \subseteq -e(a) + \mathbb{R}^k_+$ (because $X(a) \subseteq \mathbb{R}^k_+$) and that $\mathbb{R}^k_+ \subseteq \phi(a) + V$ (because $0 \in \phi(a)$ and $V = \mathbb{R}^k_{++}$). Therefore, for any $x \in -e(a) + \mathbb{R}^k_+$, there is $z(x) \in (\phi(a) + V)$ so that $\|x - z(x)\| \le \|e(a)\|$. Taking $L_a = \|e(a)\|$, C1 holds.

Nonconvexities are, in a sense, bounded because it is always possible to find a consumption bundle in V within $\|e(a)\|$ of any nonconvex region. This is illustrated in Figure 4. The indifference curve through the core allocation exhibits nonconvexities which are increasing in Arrow and Hahn's sense. Since preferences are monotonic C1 is satisfied. Hence, a sequence of economies with this type of preferences and with negligible individual endowment (C2')

[14] This is Theorem 2 in Anderson (1978).

will exhibit core convergence properties.

C2' arises from a very special relationship between consumption sets and preferences. If consumption sets were not bounded below, as it is often the case with short sales in financial models, even convex and monotonic preferences may not suffice to imply convergence. When preferences are in $\mathfrak{P}(V)$, the same special relationship between consumption sets and preferences emerges, if there is a sequence of numbers $d^n > 0$ so that the consumption set of every individual $X(a)$ is included in $(-d^n) + V$ and $(d^n/|A^n|) \longrightarrow 0$ as $n \longrightarrow \infty$. If this is the case C2' implies C2.

b. When preferences are convex, any element x in $\phi(a)$ can be used as the bound L_a in C1. To see this, fix $x \in \phi(a)\backslash\{0\}$. For any $w \in \text{con}(\phi(a))$, there is α in $[0,1]$ and $y(w)$ in $\phi(a)$ such that $w = \alpha\ y(w) + (1 - \alpha)\ 0 = \alpha\ y(w)$. By convexity of preferences $[\alpha\ y(w) + (1 - \alpha)\ x] \in \phi(a)$. Then $\|w - [\alpha\ y(w) + (1 - \alpha)\ x]\| = \|-(1 - \alpha)\ x\| \leq \|x\|$. Therefore C2 becomes

(C2'') $$\lim_{n \longrightarrow \infty} \frac{1}{|A^n|} \sup_{a \in A^n} \text{Inf} \left\{\|x\|:\ x \in \phi(a)\backslash\{0\}\right\} = 0.$$

Thus, whenever there is a net consumption bundle which is preferred to the core allocation, and which does not increase too rapidly with the size of the economy, the average budget deviation and the average excess expenditure tend to zero.

c. In the nonconvex case, in addition to C2'', we need to prevent fast increases in the nonconvexities of preferences. Our second counter example illustrates this point. This is not surprising since Anderson's result as well as Arrow-Hahn's relies on the Shapley-Folkman theorem. This theorem basically states that the nonconvexities associated with a finite sum of arbitrary sets are of the same order than those associated to only one of the sets considered.

For any consumer a in A, let $S(y) = \{x : x \succ_a y\}$. Define

$$c(a) = \text{Sup}\left\{d : B(x,d) \cap S(y) = \emptyset,\ x \in \text{con } S(y),\ y \in X(a)\right\}.$$

The number $c(\cdot)$ measures nonconvexities directly on preferences independent of the particular core bundle and endowment.

(**C3**) $$\text{Lim} \frac{1}{|A^n|} \max_{a \in A^n} c(a) = 0$$

Conditions C2'' and C3 combined imply C2. Hence they guarantee that, in average, core allocations are close to be demand-like for some price.

5. APPLICATIONS

We may conclude from the previous section that, when preferences are proper, if the core allocations do not increase too rapidly with the size of the economy, and neither do the nonconvexities of the preferences, then core convergence results hold. Proposition 3 uses this point to prove that in the replica setting with convex preferences, the average budget deviation and the average excess-expenditure become small as the number of agents in the economy increases.

For the next proposition we use a weaker definition of blocking. A coalition B weakly blocks an allocation f if there is an assignment g, $g(a) \in X(a)\ \forall a \in B$, $\sum_B g(a) = \sum_B e(a)$ such that $g(a) \succ_a f(a)$ for at least one agent $a \in B$ and $f(a)$ is not preferred to $g(a)$ for any $a \in B$.

Proposition 3: *Let $\mathcal{E}^n$ be a replica sequence of economies with preferences in $\mathfrak{P}(V)$ and convex consumption sets $X(a)$ such that for any x in $X(a)$, $x + V \in X(a)$. Let $f^n \in \mathcal{C}(\mathcal{E}^n)$. If preferences are complete, continuous, convex, and transitive, then $\exists p^n \in U$ such that*

$$\lim_{n \to \infty} \frac{1}{|A^n|} \sum_{A^n} \psi\left[f^n(a),\ a\right] = 0.$$

Proof: Since preferences are convex, we only need to show that C2'' holds in order to apply Proposition 1. If C2'' does not hold then $\exists\varepsilon > 0$ and a sequence $(a^n) \in A^n$ such that $(\|f^n(a)\|/|A^n|) > \varepsilon \ \ \forall n$. Let D^n be the set of elements b in A^n such that b and a^n are of the same type.

Debreu and Scarf (1963) proved that, in the replica case, core allocations assign to agents of the same type consumption bundles which make them indifferent to the average consumption for that type. Therefore for all b in D^n, $f^n(b)$ is indifferent to $f^n(a^n)$. Thus, $\forall b \in D^n$, $\overline{(f^n(b) - e(b))}$ belongs to $\phi(a^n)\setminus \{0\}$. Since C2'' is violated then $\exists\delta > 0$ such that

$$\frac{1}{|A^n|} \operatorname*{Min}_{D^n} \|f^n(b)\| > \delta.$$

Finally, note that $(|D^n|/k) \operatorname*{Min}_{D^n} \|f^n(b)\| \leq \|\sum_{D^n} f^n(b)\|$. Therefore

$$\frac{|D^n|}{|A^n|k} \operatorname*{Min}_{D^n} \|f^n(b)\| \leq \frac{1}{|A^n|} \|\sum_{D^n} f^n(b)\| \leq \frac{1}{|A^n|} \|\sum_{A^n} e(a)\|,$$

which generates a contradiction since the first component in the inequality goes to infinity as n increases but the average endowment remains constant.

—x—

Our discussion of the second example indicates that it is not possible to dispense entirely with the convexity assumption, even in the replica setting.

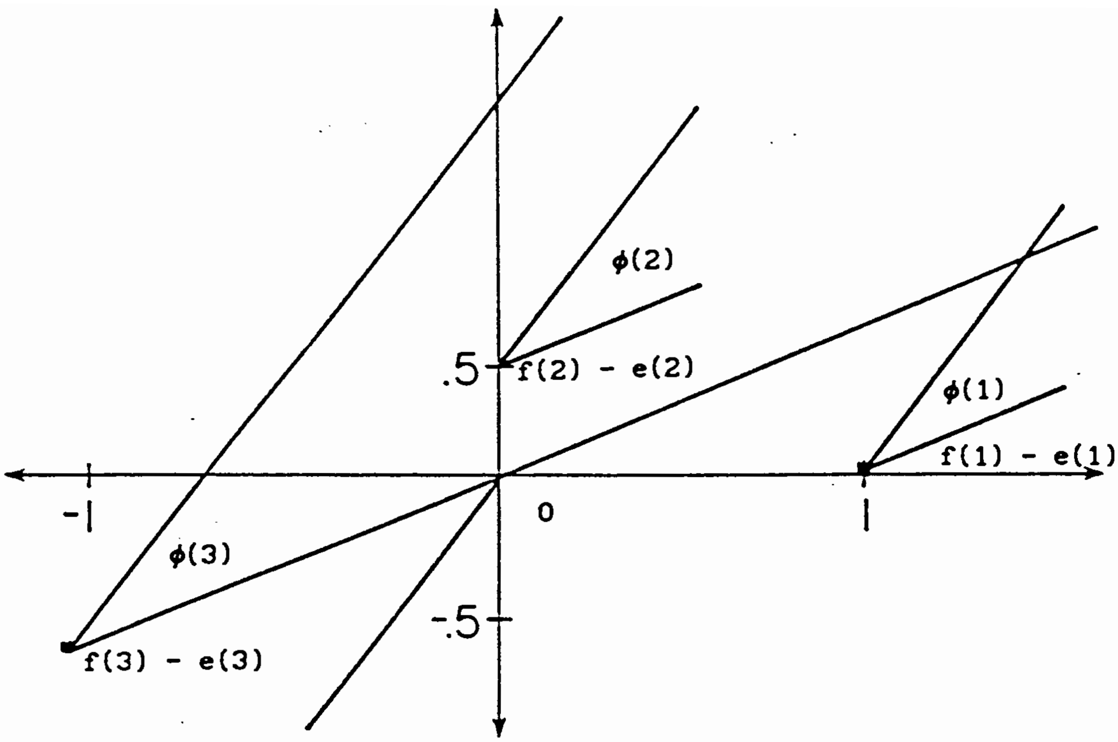

FIGURE 1

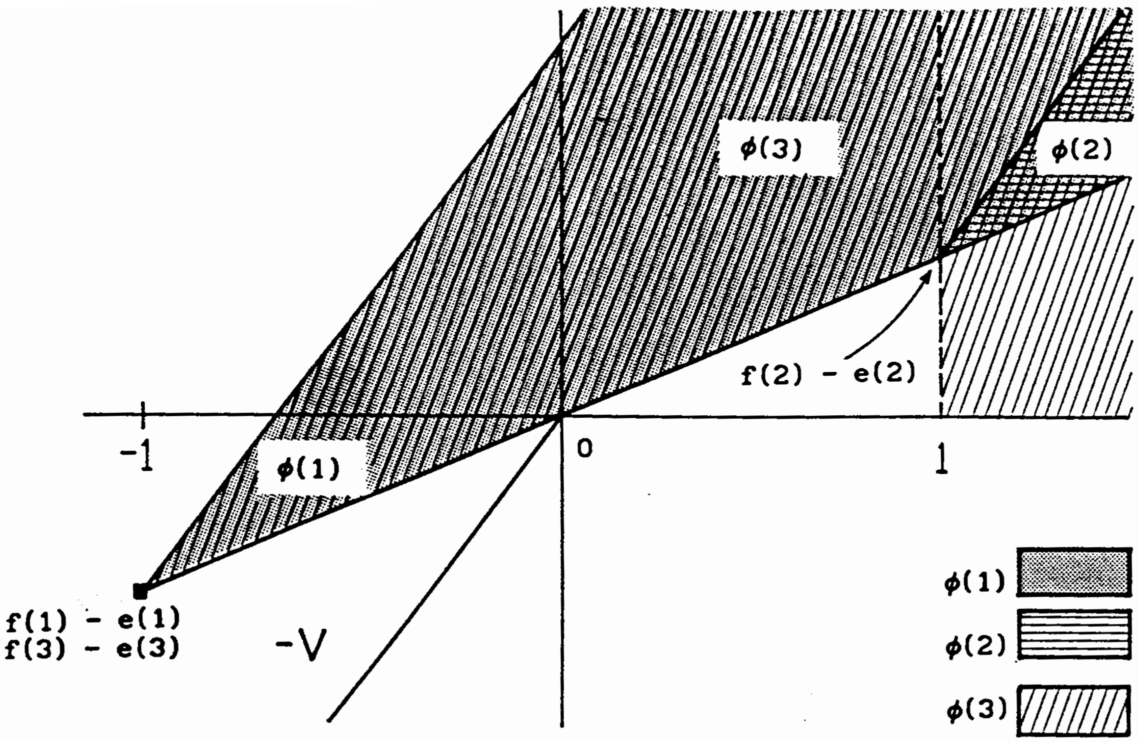

FIGURE 2

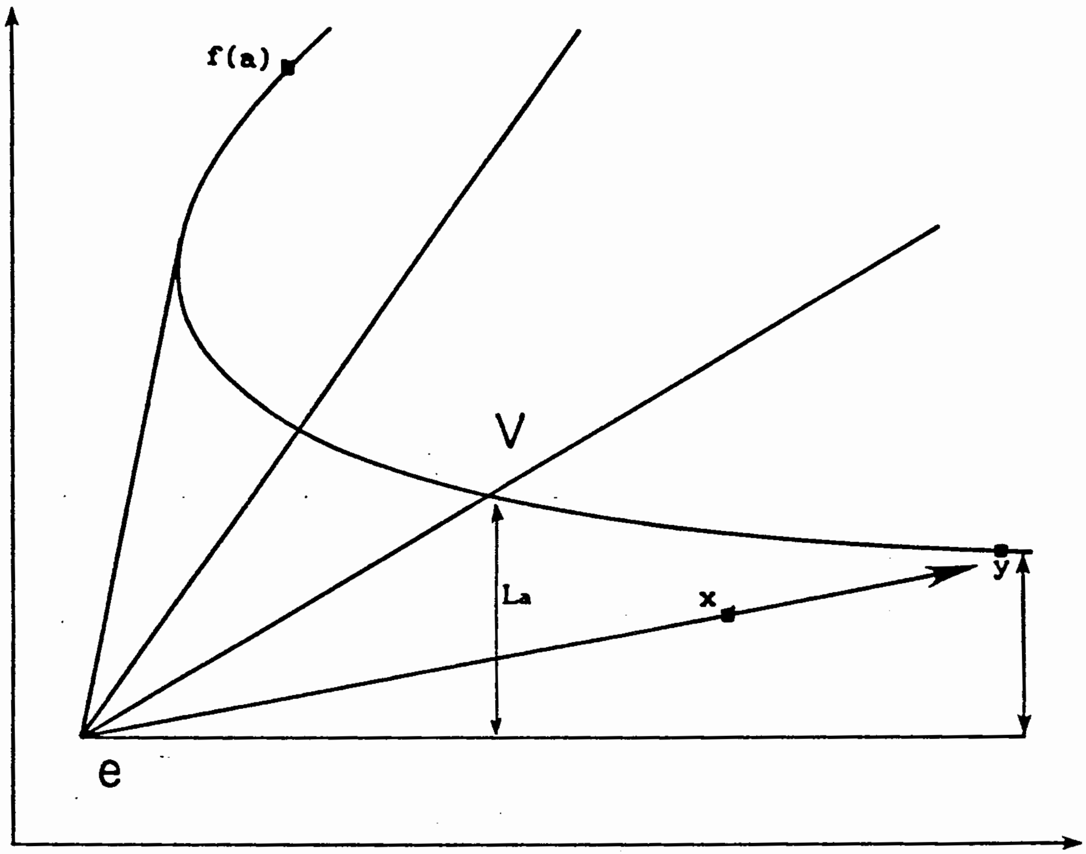

FIGURE 3a

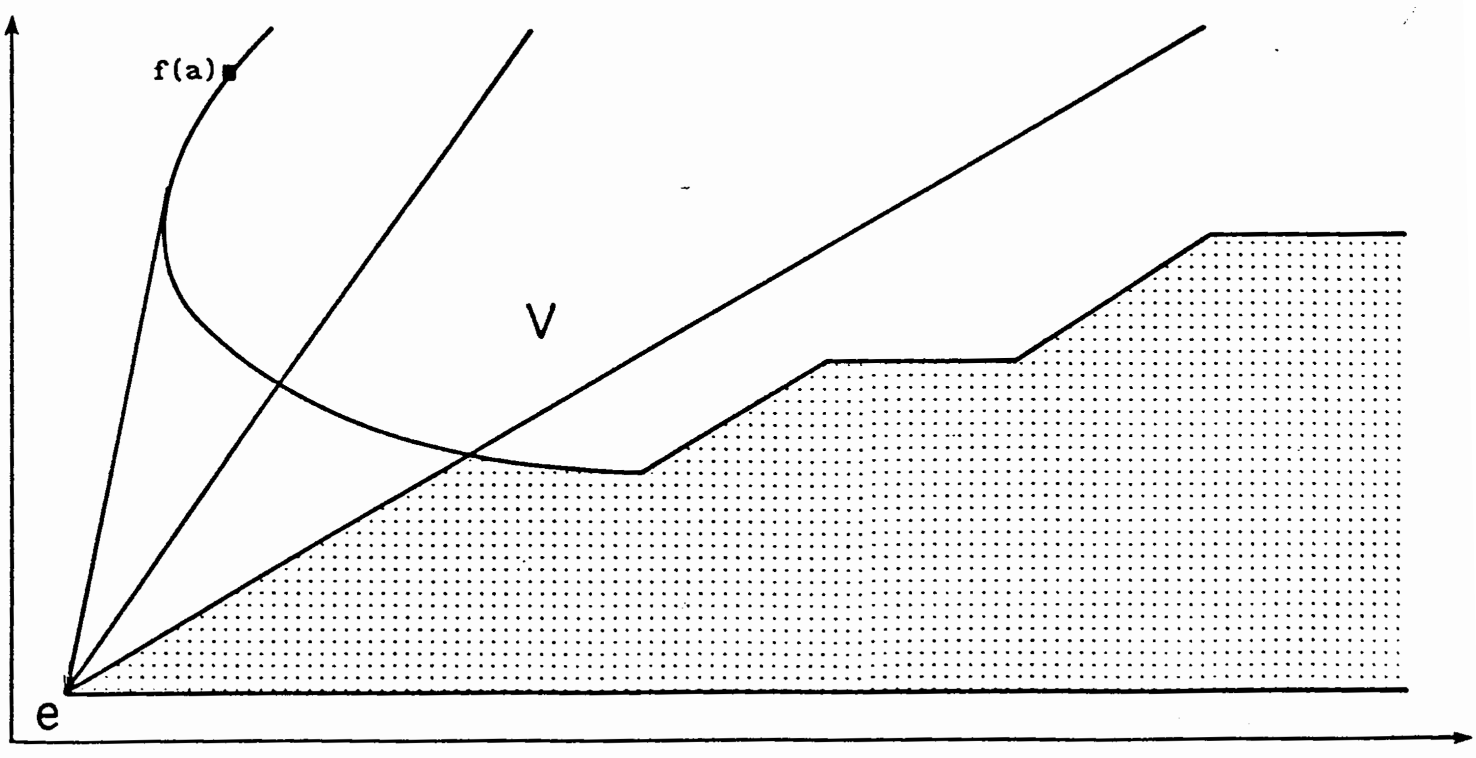

FIGURE 3b

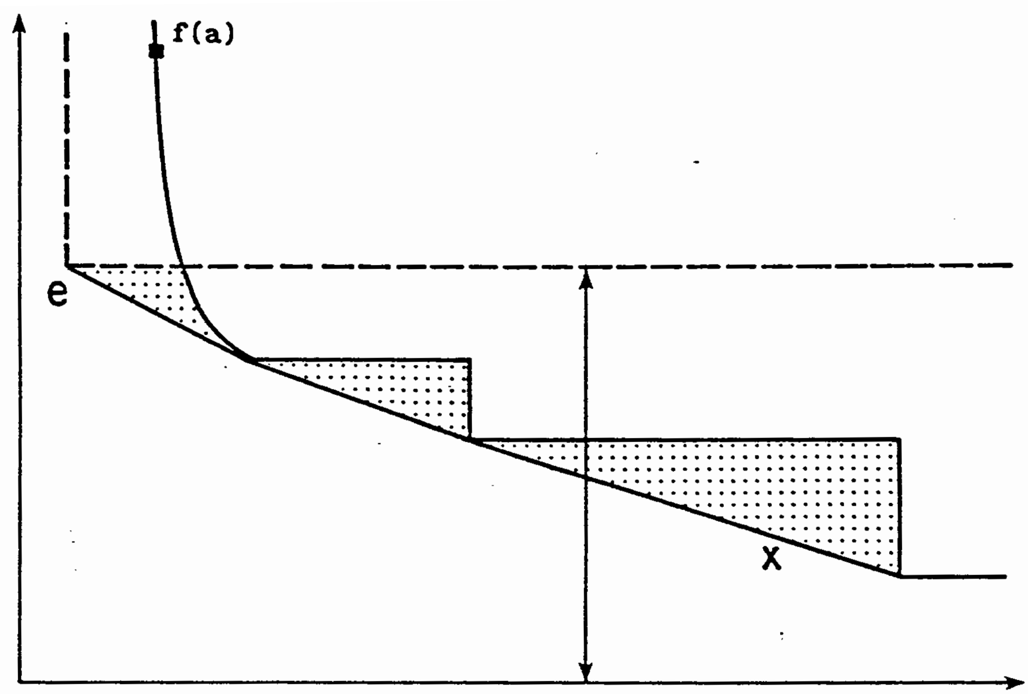

FIGURE 4

6. REFERENCES

Anderson, Robert M., 1978, An elementary core equivalence theorem, *Econometrica* 46, 1483-1487.

Anderson, Robert M., 1985a, Strong core theorems with nonconvex preferences, *Econometrica* 53, 1283-1294.

Anderson, Robert M., 1985b, Core allocations are almost utility maximal, Working paper, Center for Research in Management, U.C. Berkeley.

Anderson, Robert M., 1986, Notions of core convergence, chapter 2 in *Contributions in Mathematical Economics: In honor of Gerard Debreu*, W. Hildenbrand and A. Mas-Colell editors, North-Holland, New York.

Anderson, Robert M., 1988, The second welfare theorem with nonconvex preferences, *Econometrica* 56, 361-382.

Anderson, Robert M. and Andreu Mas-Colell, 1986, Core allocations and Pareto optima far from demand sets, Working paper M.S.R.I.

Arrow, Kenneth J. and F.H. Hahn, 1971, *General Competitive Analysis*, North-Holland, New York

Aumann, Robert J., 1964, Markets with a continuum of traders, *Econometrica* 32, 39-50.

Bewley, T., 1973, Edgeworth's conjecture, *Econometrica* 41, 415-452.

Brown, Donald J. and Abraham Robinson, 1974, The cores of large standard exchange economies, *Journal of Economic Theory* 9, 245-254.

Cheng, Hsueh-Cheng, 1983, A uniform core convergence result for nonconvex economies, *Journal of Economic Theory* 31, 269-282.

Debreu, Gérard and Herbert Scarf, 1963, A limit theorem on the core of an economy, *International Economic Review* 4, 236-246.

Dierker, Egbert, 1975, Gains and losses at core allocations, *Journal of Mathematical Economics* 2, 119-128.

Gabszewicz, Jean and J. F. Mertens, 1971, An equivalence theorem for the core of an economy whose atoms are not too big, *Econometrica,* 39, 713-721.

Gabszewicz, Jean and J. H. Dreze, 1971, Syndicate of traders in an exchange economy, in *Differential Games and Related Topics,* edited by G. Szego, North-Holland, Amsterdam, 399-414.

Gabszewicz, Jean, 1985, Comment on imperfect competition and general equilibrium, in *Frontiers of Economics,* edited by K. J. Arrow and S. Honkapohja, Basil Blackwel, Oxford, 150-169.Grodal B., W. Trockel and S.

Weber, 1984, On approximate cores of non-convex economies, *Economics Letters,* 15, 197-202.

Hildenbrand, Werner, 1974, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, NJ.

Hildenbrand, Werner, 1982, Core of an economy, chapter 18 in *Handbook of Mathematical Economics,* edited by K.J. Arrow and M.D. Intriligator, North-Holland, New York.

Kannai, Yakar, 1970, Continuity properties of the core of a market, *Econometrica* 38, 791-815.

Khan, M. Ali, 1974, Some equivalence theorems, *Review of Economic Studies* 41, 549-565.

Khan, M. Ali, 1976, Oligopoly in markets with a continuum of traders: An asymptotic approach, *Journal of Economic Theory* 12, 273-297.

Manelli, Alejandro M., 1988, *Essays in Mathematical Economics,* Ph.D. Dissertation, University of California, Berkeley.

Mas-Colell, Andreu, 1985, *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, Cambridge.

Shitovitz, Benyamin, 1973, Oligopoly in markets with a continuum of traders, *Econometrica,* 41, 467-501.

Starr, Ross, 1969, Quasi-equilibria in markets with nonconvex preferences, *Econometrica* 37, 25-38.

Trockel, Walter, 1976, A limit theorem on the core, *Journal of Mathematical Economics,* 3, 247-264.

Vind, Karl, 1965, A theorem on the core of an economy, *Review of Economic Studies* 32, 47-48.