Discussion Paper No. 855

# QUASI-VALUES ON SUBSPACES

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September 1989

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## Abstract

Quasi-values are operators satisfying all axioms of the Shapley value with the possible exception of symmetry. We introduce the characterization and extendability problems for quasi-values on linear subspaces of games, provide equivalence theorems for these problems, and show that a quasi-value on a subspace Q is extendable to the space of all games iff it is extendable to Q + Sp{u} for every game u.

Finally, we characterize restrictable subspaces and solve the characterization problem for those which are also monotone.

## 1. Introduction

Quasi-values are solutions of TU cooperative games that satisfy all the axioms of a value, except possibly the symmetry axiom. A complete characterization of quasi-values was given in Weber (1988): "$\psi$ is a quasi-value iff there exists a probability distribution over the set of orders of the players such that $\psi v(i)$ is the expected marginal distribution of player i to its predecessors."

Here we deal with quasi-values defined on subspaces of games. Two main questions are discussed:

(i) The characterization problem. Given a subspace of games Q, find a finite number of linear conditions on operators $\psi$: $Q \to \mathbb{R}^n$ (where n is the number of players) that are necessary and sufficient to ensure that $\psi$ is a quasi-value.

(ii) The extension problem. Given a quasi-value on a subspace of games, find a finite number of linear conditions that are necessary and sufficient for its extendability to a quasi-value on the space of all games.

This work was motivated by our attempt to solve the binary stochastic choice problem, which has drawn the attention of many economics and psychologists in the last five decades. For a description of this problem, many of its interpretations, and additional references, the reader is referred to Fishburn (1988). The technical description of the problem is the following.

Find a finite number of linear conditions on vectors $\beta = (p^i_j)_{i \neq j}$ that are necessary and sufficient for the existence of a probability distribution Pr on the set of all orders of $N = \{1,2,\ldots,n\}$ such that

$$p_j^i = Pr(i \succ j) \;\forall\; i \neq j,$$

where $\succ$ denotes a generic order on N.

Monderer (1989) pointed out that the binary stochastic choice problem is equivalent to the extension problem for quasi-values on $Q_2$, where $Q_2$ is the space spanned by all unanimity games on at most two-element sets. He also showed that the characterization problem for quasi-values on the space of all games turns out to be equivalent to another problem arising in social choice and psychology, the stochastic choice problem. This problem was formulated by Block and Marschak (1960), and was solved by Falmagne (1978). Monderer (1989) gave an independent game theory-based proof utilizing Weber's characterization mentioned above. Lately, Gilboa and Monderer (1989) used the game theoretic approach to obtain additional partial progress on the binary stochastic choice problem.

Our main results are the following: we formulate a condition c(u) for every game u. It turns out that a linear operator $\psi$ on a subspace Q is a quasi-value iff it satisfies c(u) for all $u \in Q$. Furthermore, every quasi-value $\psi$ is extendable to a quasi-value over the space of all games iff it satisfies c(u) for all $u \notin Q$.

Using the above, we also prove that a quasi-value on Q can be extended to a quasi-value on the space of all games iff it can be extended to a quasi-value on Q + u for all $u \notin Q$, where Q + u is the linear space spanned by Q and u. These results are to be found in Section 2.

Next we solve the characterization problem for monotone restrictable subspaces. (Restrictable subspaces were studied by Neyman (1989) and Gilboa

(1989); monotonicity of a restrictable subspace is defined in Section 3. (The definition makes sense only in view of Lemma 3.1, which characterizes restrictable subspaces.)

## 2. General Subspaces

We start with some preliminaries. Let $N = \{1,2,\ldots,n\}$ be the set of players. The set of all TU cooperative games on N will be denoted by G. The subspace of all additive games will be denoted by A. A will be identified with the Euclidean space $\mathbb{R}^n$ in the usual way. For $x,y \in \mathbb{R}^n$ we will write $x \geq y$, if $x_i \geq y_i \ \forall\ i \in N$.

The set of all one-to-one functions $\theta$: $N \to N$ (i.e., the set of all permutations) will be denoted by R.

For each $v \in G$ we define two vectors, $v_*$ and $V^*$ in $\mathbb{R}^n$ as follows:

$$v_*(i) = \min_{s \subseteq N \setminus i} (v(s \cup i) - v(s)),\ \forall\ i \in N,$$

and

$$v^*(i) = \max_{s \subseteq N \setminus i} (v(s \cup i) - v(s)),\ \forall\ i \in N.$$

Throughout this paper, Q will denote a linear subspace of G. An operator, $\psi$: $Q \to A$ is called a <u>Milnor operator</u> if

$$v_* \leq \psi v \leq v^*,\ \forall\ v \in Q.$$

A <u>quasi-value</u> on Q is a linear efficient Milnor operator $\psi$: $Q \to A$. If Q contains the additive games, then by Monderer (1988), a linear operator $\psi$: $Q \to A$ is a Milnor operator iff it is a positive projection. So, for subspaces $Q \supseteq A$, and in particular for $Q = G$, our definition coincides with the usual definition of quasi-values.[1]

Let Pr be a probability distribution over the set, R, of permutations. Following Weber (1988), define the random-order value $\psi_{Pr}$ on G as follows:

$$\psi_{Pr}v(i) = \Sigma_{\theta \in R} [v(S_\theta^i \cup i) - v(S_\theta^i)]Pr(\theta),$$

where

$$S_\theta^i = \{j \in N: \theta j < \theta i\}.$$

Weber (1988) proved that $\psi$: $G \to A$ is a quasi-value iff it is a random-order value.

Given a function $\psi$: $Q \to A$ and $u \in G$, we say that the condition c(u) is satisfied by $\psi$ if

$$c(u): \Sigma_{i=1}^n \psi v^i(i) \le \Sigma_{i=1}^n (v^i - u)^*(i) + u(N),\ \forall\, v^1,\ldots,v^n \in G.$$

We now state the following:

<u>Theorem A</u>: Let $\psi$: $Q \to A$ be a linear operator. Then $\psi$ is a quasi-value iff c(u) holds for all $u \in Q$. //

<u>Theorem B</u>: Let $\psi$: $Q \to A$ be a quasi-value. Then $\psi$ can be extended to a quasi-value on G iff c(u) holds for all $u \notin Q$. //

<u>Theorem C</u>: Let $\psi$: $Q \to A$ be a quasi-value. Then $\psi$ can be extended to a

[1] Quasi-values are called also nonsymmetric values.

quasi-value on G iff $\psi$ can be extended to a quasi-value on $Q + u$ for all $u \notin Q$. //

Note that Theorem A cannot be considered as a solution to our characterization problem because c(u) involves infinitely many linear inequalities. Similarly, Theorem B does not solve the extension problem.

Proof of Theorem A: Let $\psi$: $Q \rightarrow A$ be a quasi-value, and let $u \in Q$. As $\psi$ is a Milnor operator, we have:

$$(2.1) \qquad \psi(v^i - u)(i) \leq (v^i - u)^*(i) \ \forall \ i \in N.$$

Summation of (2.1) over $i \in N$ yields:

$$(2.2) \qquad \Sigma_{i=1}^{n} \psi v^i(i) \leq \Sigma_{i=1}^{n} (v^i - u)^*(i) + \Sigma_{i=1}^{n} \psi u(i).$$

By the efficiency axiom,

$$\Sigma_{i=1}^{n} \psi u(i) = \psi u(N) = u(N),$$

and whence c(u) holds.

As for the converse, let $\psi$: $Q \rightarrow A$ be a linear operator s.t. c(u) holds for all $u \in Q$. We have to show that $\psi$ is a Milnor and efficient operator. Substitute in c(u): $v^i = v$, $v^j = 0 \ \forall \ j \neq i$, and $u = 0$. This gives us:

$$\psi v(i) \leq v^*(i) \ \forall \ v \in Q \text{ and } \forall \ i \in N.$$

Applying the same arguments to $-v$ instead of $v$, while observing that $(-v)^* = -v_*$, yields $\psi v(i) \geq v_*(i)\ \forall\ v \in Q$ and $\forall\ i \in N$. Therefore, $\psi$ is a Milnor operator. To establish efficiency, substitute in $c(u)$: $v^i = v$ $\forall\ i \in N$, and $u = v$. This yields:

$$\psi v(N) = \Sigma_{i=1}^{n}\ \psi v(i) \leq v(N).$$

Applying the same arguments to $-v$ instead of $v$ proves that $\psi$ is efficient, which completes the proof of Theorem A.

The following lemma will be used in the proof of Theorem B. It is, however, interesting in its own right.

Lemma 2.1: Let $\psi$: $Q \rightarrow A$ be a quasi-value, and let $u \notin Q$. Then $\psi$ can be extended to a quasi-value on the space $Q + u$ spanned by $Q$ and $u$ iff $c(u)$ and $c(-u)$ hold.

Proof: The "only if" part follows from Theorem A. Suppose now that $c(u)$ and $c(-u)$ hold. Define $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ as follows:

$$y_i = \sup_{v \in Q}\ (\psi v(i) - (v - u)^*(i))\ \forall\ i \in N,$$

and

$$z_i = \inf_{v \in Q}\ ((v + u)^*(i) - \psi v(i))\ \forall\ i \in N.$$

Claim: $y$ and $z$ are well defined, $y \leq z$, and

$$\Sigma_{i=1}^{n} y_i \leq u(N) \leq \Sigma_{i=1}^{n} z_i.$$

Suppose we have already proved the claim. Choose any $x \in \mathbb{R}^n$ that satisfies $y \leq x \leq z$ and $x(N) = u(N)$, and define $\psi_u$: $Q + u \rightarrow A$ by:

$$\psi_u(v + \alpha u) = \psi v + \alpha x \;\forall\; v \in Q, \text{ and } \forall \text{ real number } \alpha.$$

We will show that $\psi_u$ is a quasi-value. Obviously, $\psi_u$ is linear and efficient. To prove that $\psi_u$ is a Milnor operator we have to show that

$$(v + \alpha u)_* \leq \psi v + \alpha x \leq (v + \alpha u)^* \;\forall\; \alpha, v. \tag{2.4}$$

Obviously, for all $w \in G$: $(-w)^* = w_*$ and $(\alpha w)^* = \alpha w^*$ for $\alpha > 0$. Therefore, (2.4) is equivalent to the following two sets of inequalities:

$$\psi v - x \leq (v - u)^* \;\forall\; v \in Q, \tag{2.5}$$

and

$$\psi v + x \leq (v + u)^* \;\forall\; v \in Q. \tag{2.6}$$

Equation (2.5) holds because $x \geq y$, and (2.6) holds because $x \leq z$. Therefore, $\psi u$ is a quasi-value.

We still have to prove the claim.

Proof of Claim: For each $v \in Q$ substitute in $c(u)$: $v^i = v$, and $v^j = 0$

$\forall\ j \neq i$. Therefore, $\psi v(i) = (v - u)^*(i) \le \Sigma_{j=1;j\neq i}^{n} (-u)^*(j) + u(N)$. Hence, y is well defined. Similarly, z is proved to be well defined (using the condition c(-u)). To prove that $y \le z$ it suffices to show that for every $v^1, v^2 \in Q$

$$\psi v^1(i) - (v^1 - u)^*(i) \le (v^2 + u)^*(i) - \psi v^2(i). \tag{2.7}$$

Indeed, because $\psi$ is a Milnor operator,

$$\begin{aligned}\psi(v^1 + v^2)(i) &\le (v^1 + v^2)^*(i) \\ &= ((v^1 - u) + (v^2 + u))^*(i) \le (v^1 - u)^*(i) + (v^2 + u)^*(i),\end{aligned} \tag{2.8}$$

by the obvious property of the * operator: $(v + w)^* \le v^* + w^*\ \forall\ v,w \in G$.

Rearranging terms in (2.8) yields (2.7). Finally, substituting $v^i = v$ for all i in c(u) and c(-u) yields $\Sigma_i[\psi v(i) - (v - u)^*(i)] \le u(N) \le \Sigma_i[(v + u)^*(i) - \psi v(i)]$ for all v, whence $\Sigma_i y_i \le u(N) \le \Sigma_i z_i$ follows. This completes the proof of Lemma 2.1.

<u>Proof of Theorem B</u>: The "only if" part follows from Lemma 2.1. As for the converse, suppose c(u) holds for all $u \notin Q$. For $\theta \in R$, denote by $\psi_\theta$ the quasi-value defined on Q by the permutation $\theta$. that is,

$$\psi_\theta v(i) = v(\{j \in N: \theta j \le \theta i\}) - v(\{j \in N: \theta j < \theta i\}).$$

By the characterization theorem of quasi-values on G (Weber (1988)), it suffices to show that $\psi$ belongs to the convex hull of $\{\psi_\theta: \theta \in R\}$. By standard convex analysis arguments, $\psi$ belongs to the convex hull of $\{\psi_\theta\colon \theta \in R\}$ iff it satisfies every linear inequality satisfied by $\{\psi_\theta | \theta \in R\}$. These may be summarized in:

$$(2.9) \qquad \Sigma_{i=1}^{n} \psi v^i(i) \le \max_{\theta\in R} \Sigma_{i=1}^{n} \psi_\theta v^i(i) \ \forall\ v^1, v^2, \ldots, v^n \in Q.$$

Let then $v^1, v^2, \ldots, v^n \in Q$. We will show that (2.9) holds. Define $u \in G$ by:

$$u(S) = \max_{\theta\in R} \Sigma_{i\in S} \psi_\theta v_S^i(i),\ \forall\ S \subseteq N,$$

where $v_S^i(T) = v^i(S \cap T)\ \forall\ T \subseteq N$.

If $u \notin Q$, then $c(u)$ holds because of our assumption. If $u \in Q$, then $c(u)$ holds by Theorem A. Thus, we may use $c(u)$ to prove (2.9). Note that it suffices to show

$$(v^i - u)^*(i) \le 0\ \forall\ i \in N,$$

or, equivalently, to show that

$$(2.10) \qquad u(S \cup i) - u(S) \ge v^i(S \cup i) - v^i(S),\ \forall\ i \in N,\ \forall\ S \subseteq N\backslash i.$$

Let then $i \in N$ and $S \subseteq N\backslash i$. Denote $s = \#S$. Let $\theta \in R$ satisfy:

$$u(S) = \Sigma_{j\in S} \psi_\theta v_S^j(j).$$

Denote by $\tau$ the element of R for which

$$u(S \cup i) = \Sigma_{j \in S \cup i} \Psi_\tau v^i_{S \cup i}(j)$$

and let $\bar{\theta}$ be a permutation such that

$$\theta k > \bar{\theta} i, \ \forall\ k \notin S \cup i$$

$$\bar{\theta} i > \bar{\theta} j, \ \forall\ j \in S$$

and

$$\bar{\theta} j > \bar{\theta} k \iff \theta j > \theta k, \ \forall\ j,k \in S.$$

Then

$$u(S \cup i) = \Sigma_{j \in S \cup i} \Psi_\tau v^j_{S \cup i}(j) \geq \Sigma_{j \in S \cup i} \Psi_{\bar{\theta}} v^j_{S \cup i}(j)$$

$$= \Sigma_{j \in S} \Psi_{\bar{\theta}} v^j_{S \cup i}(j) + \Psi_{\bar{\theta}} v^i_{S \cup i}(i).$$

However, for $j \in S$, $\Psi_{\bar{\theta}} v^j_{S \cup i}(j) = \Psi_{\bar{\theta}} v^j_S(j)$, so that

$$\Sigma_{j \in S} \Psi_{\bar{\theta}} v^j_{S \cup i}(j) = u(S).$$

Noting that

$$\Psi_{\bar{\theta}} v^i_{S \cup i}(i) = v(S \cup i) - v(S)$$

completes the proof. //

Note that Theorem B and Lemma 2.1 prove Theorem C.

## 3. Restrictable Subspaces

Recall that for each $v \in G$ and $S \subseteq N$ the game $v_S$ is defined by $v_S(T) = V(S \cap T)\ \forall\ T \subseteq N$. Q is restrictable if $v_S \in Q\ \forall\ S \subseteq N$ whenever $v \in Q$.

Let $\{w^T\colon T \neq \emptyset\}$ be the linear base of G consisting of all unanimity games. That is:

$$w^T(S) = \begin{cases} 1, & S \supseteq T \\ 0, & \text{otherwise.} \end{cases}$$

For each $\Sigma \subseteq 2^N\backslash\{\emptyset\}$ let $Q_\Sigma$ be the linear space spanned by $\{w^T\colon T \in \Sigma\}$. Note that

$$w^T_{T_0} = \begin{cases} w^T, & \text{if } T_0 \supseteq T \\ 0, & \text{if } T_0 \not\supseteq T. \end{cases} \tag{3.1}$$

Therefore, $Q_\Sigma$ is a restrictable space.

<u>Lemma 3.1</u>: Q is restrictable iff $Q = Q_\Sigma$ for some $\Sigma \subseteq 2^N\backslash\{\emptyset\}$.

<u>Proof</u>: The "if" part is true by (3.1). As for the converse, denote

$$\Sigma = \{T\colon w^T \in Q\}.$$

We will show that $Q = Q_\Sigma$. For this purpose, it suffices to show that for all $v \in Q$, if the representation of v in the base $\{w^T\}_{T \subseteq N}$ is $v = \sum \alpha_T w^T$, then $w^T \in Q$ whenever $\alpha_T \neq 0$.

We will prove the latter claim by an induction on the number k of sets

T for which $\alpha_T \neq 0$. For $k = 1$ the claim is trivial. Suppose the claim holds for $k < p$, and let $v = \sum \alpha_T w^T$ satisfy $v \in Q$ and $\#\{T: \alpha_T \neq 0\} = P$. Let $T_0$ be a minimal set (w.r.t. inclusion) such that $\alpha_{T_0} \neq 0$. By (3.1) $v_{T_0} = \alpha_{T_0} w^{T_0}$, and hence $w^{T_0} \in Q$. Denote $v_0 = v - \alpha_{T_0} w^{T_0}$. Then $v_0 \in Q$, $v_0 = \sum_{T \neq T_0} \alpha_T w^T$, and $\#\{T \neq T_0: \alpha_T \neq 0\} = P - 1$. Therefore, by the induction hypothesis $w^T \in Q$, $\forall\ T \neq T_0$ for which $\alpha_T \neq 0$. //

We now turn to formulate the characterization problem for restrictable spaces, and to solve it for the class of monotone restrictable spaces. Let $\Sigma \subseteq 2^N \backslash \{\emptyset\}$. Let $\beta = (\beta^i(T))_{t \in \Sigma; i \in N}$ be a vector of real numbers. Define a linear operator $\psi_\beta$: $Q_\Sigma \to A$ by defining it on the base $\{w^T: T \in \Sigma\}$ as follows:

$$\psi_\beta w^T(i) = \beta^i(T),\ \forall\ i \in N.$$

<u>The Characterization Problem on</u> $Q_\Sigma$

Find a finite number of linear inequalities on $\beta$ that are necessary and sufficient for $\psi_\beta$ to be a quasi-value on $Q_\Sigma$.

Obvious necessary conditions are:

$$(3.2) \qquad \beta^i(T) = 0, \qquad \forall\ i \notin T \in \Sigma,$$

$$(3.3) \qquad \beta^i(T) \geq 0, \qquad \forall\ i \in T \in \Sigma,$$

and

$$(3.4) \qquad \Sigma_{i \in T}\ \beta^i(T) = 1, \qquad \forall\ T \in \Sigma.$$

In some trivial cases the conditions (3.2)-(3.4) are also sufficient. E.g., if for all $S \neq T \in \Sigma$, $S \not\subseteq T$ and $T \not\subseteq S$. A class $\Sigma \subseteq 2^N \backslash \{\emptyset\}$ is <u>monotone</u> if $T \in \Sigma$ and $S \supseteq T$ imply $S \in \Sigma$. A restrictable subspace $Q_\Sigma$ is said to be <u>monotone</u> if $\Sigma$ is such. Let $\Sigma$ be a monotone class. For each $i \in T \in \Sigma$ define:

(3.5) $$c^i_\beta(T) = \Sigma^n_{k=t} (-1)^{k-t} \Sigma_{A \supseteq T; |A|=k} \beta^i(A),$$

where $t = \#T$.

<u>Theorem D</u>: Let $\Sigma$ be a monotone class, and let $\beta = (\beta^i(T))_{T\in\Sigma; i\in N}$. Then $\psi_\beta$ is a quasi-value on $Q_\Sigma$ iff $\beta$ satisfies (3.2)-(3.4), and the following conditions:

(3.6) $$c^i_\beta(T) \geq 0, \qquad \forall\ i \in T \in \Sigma,$$

and

(3.7) $$\Sigma_{\{T\in\Sigma: i\in T\}}\ c^i_\beta(T) \leq 1, \text{ for every } i \in N.$$

<u>Proof</u>

<u>Necessity</u>

Suppose $\psi_\beta$ is a quasi-value. We have to show that (3.6) and (3.7) hold.

For each $T \neq N$ define the game $v^T$ as follows:

$$v^T(S) = \begin{cases} 1, & T \subset S \\ 0, & T \subseteq S \end{cases}$$

(where $\subset$ denotes strict inclusion).

It is well known, and also easily verified, that for every $B \subset N$

(3.8) $$v^B = \Sigma_{k=t+1}^{n} (-1)^{k-t+1} \Sigma_{|A|=k;A\supseteq B} w^A.$$

Fix $T \in \Sigma$ and $i \in T$. Then by (3.8) $v^{T\setminus i} = v_1^{T,i} + v_2^{T,i}$, where

$$v_1^{T,i} = \Sigma_{k=t+1}^{n} (-1)^{k-t+1} \Sigma_{\{A||A|=k;A\supseteq T\}} w^A$$

and

$$v_2^{T,i} = \Sigma_{k=t+1}^{n} (-1)^{k-t+1} \Sigma_{\{A||A|=k;i\notin A\supseteq T\setminus i\}} w^A.$$

Note that

$$v_1^{T,i}(S \cup i) - v_1^{T,i}(S) = v^{T\setminus i}(S \cup i) - v^{T\setminus i}(S)$$

for all $S \subseteq N\setminus i$, whence $(v_1)_*(i) \geq 0$. Since $Q$ is monotone, $v_1^{T,i} \in Q$ and the Milnor condition yields $\psi_\beta v_1^{T,i}(i) \geq 0$. However, $\psi_\beta v_1^{T,i}(i) = c_\beta^i(T)$, and (3.6) follows.

To prove (3.7), let there be given $i \in N$. Denote

$$\bar{v} = \Sigma_{\{T\in\Sigma:i\in T\}} v_1^{T,i}.$$

By linearity,

(3.9) $$\psi_\beta v(i) = \Sigma_{\{T\in\Sigma: i\in T\}} c^i(T).$$

On the other hand, for each $i \in S \in \Sigma$, $v(S) - v(S\backslash i) = 1$ (because $v_1^{T,i}(S) - v_1^{T,i}(S/i) = 1$ iff $S = T$), and by monotonicity of Q for each $i \in S \notin \Sigma$, $v(S) - v(S\backslash i) = 0$. Therefore, $v^*(i) \le 1$. Combining this with (3.9) and the Milnor condition yields (3.7).

Sufficiency

$\psi$ is obviously linear and efficient. We proceed to prove that it is a Milnor operator. Fix $i \in N$. Denote

$$\Sigma_i = \{T \in \Sigma: i \in T\}$$

and

$$\bar{\Sigma}_i = \{T \in \Sigma: i \notin T\}.$$

Obviously,

$$Q_\Sigma = Q_{\Sigma_i} + Q_{\bar{\Sigma}_i}.$$

As $\psi_\beta v(i) = v^*(i) = 0$ for all $v \in Q_{\bar{\Sigma}_i}$ (by (3.2)), it suffices to show that

$$\psi_\beta v(i) \le v^*(i),\ \forall\ v \in Q_{\Sigma_i}.$$

It is easily verifiable that $\{v_1^{T,i}: i \in T \in \Sigma\}$ is a linear base for $Q_{\Sigma_i}$.

Let $v = \Sigma_{i\in T\in\Sigma}\, d_T v_1^{T,i}$. Then

$$v^*(i) = \max\{0, \max_{i \in T \in \Sigma} d_T\}.$$

On the other hand,

$$\psi_\beta v(i) = \Sigma_{i \in T \in \Sigma} d_T c^i(T) \le \max\{0, \max_{i \in T \in \Sigma} d_T\} \Sigma_{i \in T \in \Sigma} c^i(T) \le v^*(i)$$

by (3.6) and (3.7).

Therefore, $\psi$ is a Milnor operator. //

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