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A GAME-THEORETIC APPROACH TO THE BINARY STOCHASTIC CHOICE PROBLEM

by

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Abstract

We provide an equivalence theorem for the binary stochastic choice problem, which may be thought of as an implicit characterization of binary choice probabilities which are consistent with a probability over linear orderings.

In some cases this implicit characterization is very useful in derivation of explicit necessary conditions. In particular, we present a new set of conditions which generalizes both Cohen-Falmagne's and Fishburn's conditions.

1. Introduction

The binary stochastic choice problem is the following: given a set N of alternatives and numbers, $\{p_{ij}\}_{i,j\in N; i\neq j}$, interpreted as "the probability that i will be preferred to j," when is there a probability distribution Pr on the |N|! linear orderings of N such that

$$p_{ij} = \Sigma_{\{R | iRj\}} \Pr(R)?$$

(In this case, $\{p_{ij}\}$ will be called consistent.)

We will not expatiate here on the motivation, background, or history of the problem. We refer the reader to Block and Marschak (1960), McFadden and Richter (1970), Falmagne (1978), Cohen and Falmagne (1978), Dridi (1980), Souza (1983), McLennan (1984), Barbera (1985), Barbera and Pattanaik (1986), Fishburn (1988), Fishburn and Falmagne (1988), Marley (1989), and Gilboa (1989).

While it is known that for every n = |N| there are finitely many linear inequalities in $\{p_{ij}\}_{ij}$ which fully characterize the consistent binary choice probabilities, there is no set of explicit conditions which are necessary and sufficient for all n. Several necessary conditions are, however, known, and they are quoted in Section 2.

Following Monderer (1989), which uses a game-theoretic approach to derive Block-Marschak conditions for the (non-binary) stochastic choice problem, this paper uses a similar approach to derive an equivalence theorem which may be considered as an implicit characterization. Although it falls short of an explicit one, i.e., it does not provide explicit formulae for finitely many linear inequalities, it may be used to derive necessary conditions. In Section 3 we state and prove the equivalence theorem, and Section 4 shows how it can be used to derive some known conditions.

These proofs show that in some cases the equivalence theorem is a useful tool in proving necessity of conditions, which is more technical and requires less imagination than direct combinatorial proofs. Indeed, the proofs in Section 4 suggested natural generalizations, and in Section 5 we present a new set of necessary conditions, which unifies and generalizes the Cohen-Falmagne conditions on one hand and Fishburn's on the other.

However, we also have some bad news. Trying to obtain the diagonal inequality, which is a generalization of the Cohen-Falmagne conditions, we were not very successful at utilizing the equivalence theorem. It seems that in this case the direct combinatorial proof is significantly simpler than the one using the theorem. In Section 6 we discuss the diagonal inequality, provide a new combinatorial proof of it, which may be insightful by itself, as well as the more complicated proof using the equivalence theorem.

Finally, Section 7 concludes this paper with a remark on the insufficiency of the known conditions.

2. Known Necessary Conditions

2.1 The Triangle Inequality

This condition, which is to be found in Block and Marschak (1960), is a direct implication of transitivity. It says that for every $i,j,k \in N$

$$p_{ij} + p_{jk} + p_{ki} \leq 2.$$

2.1 Cohen-Falmagne's Inequality

In Cohen-Falmagne (1978) we find the following condition: for every two sequences, $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$, where $\{a_i\}_{i=1}^k \cap \{b_i\}_{i=1}^k = \emptyset$,

$$\Sigma_{\{(i,j) \mid 1 \le i \ne j \le k\}} p_{a_i b_j} - \Sigma_{i=1}^k p_{a_i b_i} \le k(k - 2) + 1.$$

(Here and in the sequel, a "sequence" refers to a sequence of distinct elements of N.)

2.3 Fishburn's Condition

Fishburn (1988) provides the following necessary conditions: for every two sequences, $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$, with $\{a_i\}_{i=1}^k \cap \{b_i\}_{i=1}^k = \emptyset$ and $k = 2\ell - 1$,

$$\Sigma_{i=1}^k p_{a_i b_i} + \Sigma_{i=1}^k p_{a_i b_{i+1}} - \Sigma_{i=1}^k p_{a_i b_{i+\ell}} \le 3\ell - 2$$

where the addition operation on indices is done mod k.

It should be mentioned at this point that McLennan's paper includes two conditions which are very similar to Cohen-Falmagne's and to Fishburn's condition, respectively.

2.4 The Diagonal Inequality

Gilboa (1989) proved the following to be a necessary condition: for any two sequences, $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$ (not necessarily disjoint), and every $1 \le r \le k - 1$,

$$\Sigma_{\{(i,j)|1\leq i\neq j\leq k\}} p_{a_i b_j} - r\Sigma_{i=1}^{k} p_{a_i b_i} \leq k(k-1) - rk + r(r+1)/2.$$

This condition is identical to Cohen-Falmagne's in the case $r = 1$ and disjoint sequences. With $A = (i,j)$ and $B = (j,k)$ it is equivalent to the triangle inequality (assuming, w.l.o.g. (without loss of generality) that $p_{ij} + p_{ji} = 1$.)

3. <u>The Equivalence Theorem</u>

We first introduce some game-theoretic definitions. Given a finite and nonempty set (of <u>players</u>) N, we define a <u>game</u> v on N to be a set function $v: 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. (Subsets of N are interpreted as <u>coalitions</u>.) For any $\emptyset \neq T \subseteq N$ we define v_T to be the <u>unanimity game</u> on T by

$$v_T(S) = \begin{cases} 1 & S \supseteq T \\ 0 & \text{otherwise} \end{cases}$$

It is well known that $\{v_T\}_{T\subseteq N, T\neq\emptyset}$ is a basis for the linear space of games on N (endowed with the natural linear operations).

Given a game v, and player i, we define i's <u>maximal marginal contribution</u> in v to be

$$v_i^* = \max\{v(S \cup \{i\}) - v(S) | S \subseteq N, i \notin S\}.$$

A convenient abuse of notation is to identify a player i with the singleton {i}, and we will enthusiastically do so whenever possible.

We can now formulate:

<u>The Equivalence Theorem:</u> Given a finite and nonempty set N, and numbers $\{p_{ij}\}_{i,j\in N; i\neq j}$, the following are equivalent:

(i) $\{p_{ij}\}$ are consistent;

(ii) For every $\{\alpha_{ij}\}_{i,j\in N; i\neq j}$ and every game u,

$$\Sigma_{i,j\in N; i\neq j}\ \alpha_{ij}p_{ij} \leq \Sigma_{i\in N}\ (v^i - u)^*_i + u(N)$$

where

$$v^i = \Sigma_{\{j|j\neq i\}}\ \alpha_{ij}v_{\{i,j\}}.$$

This theorem is, in fact, a special case of Theorems A and B in Gilboa-Monderer (1989). However, for the sake of completeness we also provide here a proof:

a. <u>(i) => (ii)</u>

Suppose $\{\alpha_{ij}\}$ and u are given. It suffices to show that for every given linear ordering R on N the probabilities $\{p^R_{ij}\}$ defined by

$$p^R_{ij} = \begin{cases} 1 & iRj \\ 0 & \text{otherwise} \end{cases}$$

satisfy (ii). W.l.o.g., assume that $N = \{1,2,\ldots,n\}$ and that R is the natural ordering. We therefore need to show that

$$\Sigma_{i>j}\ \alpha_{ij} \leq \Sigma_i\ (v^i - u)^*_i + u(N).$$

For each i, let $S^i = \{j | j < i\}$. By definition of the * operation,

$$(v^i - u)^*_i \geq (v^i - u)(S^i \cup i) - (v^i - u)(S^i) =$$
$$= [v^i(S^i \cup i) - v^i(S^i)] - [u(S^i \cup i) - u(S^i)]$$

As $v^i(T) = \Sigma_{\{j | j \neq i; j \in T\}} \alpha_{ij}$ if $i \in T$, and $v^i(T) = 0$ if $i \notin T$, we obtain

$$v^i(S^i \cup i) = \Sigma_{\{j | j < i\}} \alpha_{ij}$$

and

$$v^i(S^i) = 0.$$

Hence,

$$\Sigma_i (v^i - u)^*_i + u(N) \geq \Sigma_i \Sigma_{\{j | j < i\}} \alpha_{ij} - \Sigma_i [u(S^i \cup i) - u(S^i)] + u(N)$$
$$= \Sigma_{j<i} \alpha_{ij}.$$

We now wish to show the converse.

(b) <u>(ii) => (i)</u>:

We have to show that every linear inequality

(*) $$\Sigma_{i \neq j} \alpha_{ij} p_{ij} \leq \beta$$

which is satisfied by $\{p^R_{ij}\}$ for every linear ordering R is also satisfied by every $\{p_{ij}\}$ satisfying (ii). (Thus, $\{p_{ij}\}$ satisfying (ii) are proved to be in the convex hull of $\{p^R_{ij}\}_R$.)

It suffices to show that (*) holds for the minimal β which is

$$\beta = \max_R \Sigma_{iRj} \alpha_{ij}.$$

Define a game u by

$$u(S) = \max_R \Sigma_{iRj;\, i,j\in S} \alpha_{ij}.$$

Condition (ii) implies that

$$\Sigma_{i\neq j} \alpha_{ij} p_{ij} \leq \Sigma_i (v^i - u)^*_i + u(N)$$

where $v^i = \Sigma_{\{j|j\neq i\}} \alpha_{ij} v_{\{i,j\}}$ and $u(N) = \beta$. Hence, all we need to show is that

$$(v^i - u)^*_i = 0$$

for all $i \in N$. However, $v^i(i) = u(i) = 0$, so that $(v^i - u)^*_i \geq 0$ is obvious. To show the converse inequality we have to convince ourselves (and the reader) that for every $S \subseteq N$ and every $i \notin S$

$$v^i(S \cup i) - v^i(S) \leq u(S \cup i) - u(S).$$

Note that

$$v^i(S \cup i) - v^i(S) = \Sigma_{j\in S} \alpha_{ij}.$$

Let R_S be a linear ordering such that

$$u(S) = \Sigma_{\{k,j\in S;kR_Sj\}} \alpha_{kj}.$$

Let R'_S be a linear ordering which agrees with R_S on S and satisfies iR'_Sj for $j \in S$. Then

$$u(S \cup i) \geq \Sigma_{\{i,j\in S\cup i;iR'_Sj\}} \alpha_{ij} = u(S) + \Sigma_{j\in S} \alpha_{ij}$$

which completes the proof. //

4. Derivation of Known Conditions

In order to get used to the game-theoretic machinery and illustrate its applicability, we devote this section to the derivation of some known conditions.

4.1 The Trivial Conditions

It is usually assumed that the binary probabilities $\{p_{ij}\}$ satisfy $p_{ij} + p_{ji} = 1$ and $p_{ij} \geq 0$. Since this was not explicitly assumed in the equivalence theorem, we conclude that these linear conditions are also derived from it. Indeed, let us first choose for some $i,j \in N$ $(i \neq j)$ $\alpha_{ij} = 1$ and $\alpha_{k\ell} = 0$ for $(k,\ell) \neq (i,j)$. Letting $u = 0$ we obtain

$$v^i = v_{\{i,j\}} \qquad v^k = 0 \qquad \forall\ k \neq i$$

and

$$p_{ij} = \Sigma_{i \neq j}\ \alpha_{ij} p_{ij} \leq \Sigma_i (v^i)^*_i = 1.$$

Next let us take $\alpha_{ij} = \alpha_{ji} = 1$ and $\alpha_{k\ell} = 0$ for $\{k,\ell\} \neq \{i,j\}$, with $u = v_{\{i,j\}}$. Then

$$v^i = v^j = v_{\{i,j\}} \qquad v^k = 0 \qquad \forall\ k \neq i,j$$

and

$$(v^k - u)^*_k = 0 \qquad \forall\ k \in N.$$

Whence one obtains

$$p_{ij} + p_{ji} \leq 1.$$

Similarly, $\alpha_{ij} = \alpha_{ji} = -1$ with $u = -v_{\{i,j\}}$ yields $p_{ij} + p_{ji} \geq 1$ which implies $p_{ij} + p_{ji} = 1$ and also $p_{ij} \geq 0$.

4.2 <u>The Triangle Inequality</u>

For given $i,j,k \in N$ (assumed distinct) we have to show that

$$p_{ij} + p_{jk} + p_{ki} \leq 2.$$

This formulation naturally suggests

$$\alpha_{ij} = \alpha_{jk} = \alpha_{ki} = 1.$$

(Here and in the sequel, coefficients which are not specifically mentioned should be taken to be zero.)

These coefficients, in turn, define

$$v^i = v_{\{i,j\}} \qquad v^j = v_{\{j,k\}} \qquad v^k = v_{\{i,k\}}.$$

It only remains to choose

$$u = v_{\{i,j\}} + v_{\{j,k\}} + v_{\{i,k\}} - v_{\{i,j,k\}}.$$

It is easy to verify that

$$(v^\ell - u)^*_\ell = 0 \qquad \forall\ \ell \in N$$

and

$$u(N) = 2$$

whence the triangle inequality follows.

4.3 Cohen-Falmagne's Condition

Let there be given two disjoint sequences A $(a_1,\ldots,a_k)$ and B = $(b_1,\ldots,b_k)$. We wish to prove that

$$\Sigma_{\{(i,j)\mid 1\le i\ne j\le k\}}\ p_{a_i b_j} - \Sigma_{i=1}^{k}\ p_{a_i b_i} \le k(k-2)+1.$$

Define $\alpha_{a_i b_j} = 1$ for $1 \le i \ne j \le k$ and $\alpha_{a_i b_i} = -1$ for $1 \le i \le k$. Correspondingly

$$v^{a_i} = \Sigma_{\{j|j\neq i;1\leq j\leq k\}} v_{\{a_i,b_j\}} - v_{\{a_i,b_i\}}$$

(and $v^{b_i} = 0$ for all i).

Let us define u as follows

$$u = \Sigma_{i=1}^{k} v_{\{a_i,b_1,b_2,\ldots,b_{i-1},b_{i+1},\ldots,b_k\}}$$

$$- \Sigma_{i=1}^{k} v_{\{a_i,b_1,b_2,\ldots,b_{i-1},b_i,b_{i+1},\ldots,b_k\}}$$

$$+ v_{\{b_1,\ldots,b_k\}}$$

One may verify that

$$(v^{a_i} - u)^*_{a_i} = k - 2 \qquad \text{for } 1 \leq i \leq k$$

and

$$(v^{\ell} - u)^*_{\ell} = 0 \qquad \text{for } \ell \notin \{a_i\}_{i=1}^{k}.$$

Noting that $u(N) = 1$ we obtain the desired result.

4.4 Fishburn's Condition

Given two disjoint sequences $A = (a_1,a_2,\ldots,a_k)$ and $B = (b_1,b_2,\ldots,b_k)$ with $k = 2\ell - 1$ ($\ell \in N$), we wish to show that

$$\Sigma_{i=1}^{k} p_{a_i b_i} + \Sigma_{i=1}^{k} p_{a_i b_{i+1}} - \Sigma_{i=1}^{k} p_{a_i b_{i+\ell}} \leq 3\ell - 2.$$

Although the equivalence theorem guarantees the existence of a game u which attains the exact bound on the right side, this game is quite complicated to compute. Instead, we will use a simple game which will bring us close enough.

Naturally, we have

$$\alpha_{a_i b_i} = \alpha_{a_i b_{i+1}} = 1, \ \alpha_{a_i b_{i+\ell}} = -1, \ \forall\ 1 \le i \le k$$

and

$$v^{a_i} = v_{\{a_i, b_i\}} + v_{\{a_i, b_{i+1}\}} - v_{\{a_i, b_{i+\ell}\}}$$

for $1 \le i \le k$. ($v^j = 0$ for $j \notin \{a_i\}_{i=1}^k$.)

Define

$$u = \Sigma_{i=1}^k v_{\{a_i, b_i, b_{i+1}\}}$$

$$- \Sigma_{i=1}^k v_{\{a_i, b_i, b_{i+1}, b_{i+k}\}}$$

$$+ (1/2)\ \Sigma_{i=1}^k v_{\{b_i, b_{i+\ell}\}}.$$

It is readily seen that $(v^{a_i} - u)^*_{a_i} = 1$ (for all $1 \le i \le k$). The equality $(-u)^*_{b_\ell} = 0$ (for $1 \le i \le k$) is slightly trickier but still correct. Thus one gets

$$\Sigma_{i=1}^k p_{a_i b_i} + \Sigma_{i=1}^k p_{a_i b_{i+1}} - \Sigma_{i=1}^k p_{a_i b_{i+\ell}} \le (2\ell - 1) + u(N)$$

$$= (2\ell - 1) + (\ell - 1/2) = 3\ell - 3/2.$$

Now we have to resort to extraneous argument to complete the proof: since all the $|N|!$ extreme points of consistent $\{p_{ij}\}$, which are integer-valued, satisfy this inequality, they also satisfy it with $(3\ell - 2)$ in the right side. Hence this is also true of every $\{p_{ij}\}$ in their convex hull and this yields Fishburn's inequality.

Derivation of New Conditions

The analysis presented above suggests additional necessary conditions. However, the "new" conditions one obtains may well be derived from known ones. For instance, should we try to develop an analog of Fishburn's condition for the case $k = 2\ell$, one may obtain the inequality

$$\Sigma_{i=1}^{k} p_{a_i b_i} + \Sigma_{i=1}^{k} p_{a_i b_{i+1}} - \Sigma_{i=1}^{k} p_{a_i b_{i+\ell}} \leq 3\ell$$

which trivially follows from

$$\Sigma_{i=1}^{k} p_{a_i b_i} - \Sigma_{i=1}^{k} p_{a_i b_{i+\ell}} \leq \ell$$

which, in turn, follows from Cohen-Falmagne's condition for $k = 2$. (And the latter, involving only four alternatives, is also derivable from the triangle inequality.)

Even if the new conditions are independent of known ones, there is still no guarantee that they are the best one may obtain, namely, that one has the smallest β for a given set of coefficients $\{\alpha_{ij}\}$.

Nevertheless, we find the inequalities we are about to present interesting even if they are not necessarily the best ones for the same sets of coefficients: these conditions will unify and generalize the Cohen-Falmagne condition and that of Fishburn. Let us first have another look at these.

Renumbering the sequence A, one may present Fishburn's condition as

$$\Sigma_{i=1}^{k} (p_{a_i b_i} - p_{a_i b_{i+\ell}} + p_{a_i b_{i+2\ell}}) \le \beta$$

(for the appropriate β).

Shifting the indices of the sequence A by $\ell = \lfloor(k/2)\rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer not exceeding x), Cohen-Falmagne's condition takes the form

$$\Sigma_{i=1}^{k} (p_{a_i b_i} + p_{a_i b_{i+1}} + \ldots + p_{a_i b_{i+\ell-1}} - p_{a_i b_{i+\ell}} + p_{a_i b_{i+\ell+1}} + \ldots + p_{a_i b_{i+k-1}}) \le \beta$$

(for some β).

It seems natural to consider a "step size" which differs from $\lfloor(k/2)\rfloor$ (as in Fishburn's condition) and from 1 (as in Cohen-Falmagne's). Let us take some integer s satisfying $1 < s < \lfloor(k/2)\rfloor$, and assume that $k = qs - r$ ($q \in N$, $0 \le r \le s - 1$). Let $t = \lfloor(q/2)\rfloor$, and let us try to estimate

$$S \equiv \Sigma_{i=1}^{k} [\Sigma_{\nu=0}^{t} p_{a_i b_{i+\nu s}} - p_{a_i b_{i+(t+1)s}} + \Sigma_{\nu=t+2}^{q-1} p_{a_i b_{i+\nu s}}]$$

for the case $d \equiv \lfloor(t/2)\rfloor > 2$.

(Note that for the case $r = 0$ the best possible bound follows from

Cohen-Falmagne's condition for $k' = q - 1$, as was the case with the analog of Fishburn's condition for $k = 2s = 2\ell$.)

So let us set

$$\alpha_{a_i b_{i+\nu s}} = 1 \qquad \text{for } 1 \le i \le k \text{ and } 1 \le \nu \le q-1,\ \nu \ne t+1$$

and

$$\alpha_{a_i b_{i+(t+1)s}} = -1 \quad \text{for } 1 \le i \le k.$$

Correspondingly, define, for $1 \le i \le k$,

$$v^{a_i} = \Sigma_{\nu=0}^{t} v_{\{a_i, b_{i+\nu s}\}} - v_{\{a_i, b_{i+(t+1)s}\}} + \Sigma_{\nu=t+2}^{q-1} v_{\{a_i, b_{i+\nu s}\}}.$$

Next, define

$$u = \Sigma_{i=1}^{k} v_{\{a_i, b_i, b_{i+s}, \ldots, b_{i+ts}, b_{i+(t+2)s}, \ldots, b_{i+(q-1)s}\}}$$

$$- \Sigma_{i=1}^{k} v_{\{a_i, b_i, b_{i+s}, \ldots, b_{i+ts}, b_{i+(t+1)s}, \ldots, b_{i+(q-1)s}\}}$$

$$+ \Sigma_{\mu=0}^{s-1} [v_{\{b_{\mu-ds}, b_{\mu-(d-1)s}, \ldots, b_{\mu}, b_{\mu+s}, \ldots, b_{\mu+ds}\}}$$

$$+ v_{\{b_{(t+1)s+\mu-ds}, b_{(t+1)s+\mu-(d-1)s}, \ldots, b_{(t+1)s+\mu}, \ldots, b_{(t+1)s+\mu+ds}\}}].$$

Again, it is not difficult to see that

$$(v^{a_i} - u)^*_{a_i} = q - 2$$

and

$$(-u)^*_{b_i} = 0$$

for all $1 \leq i \leq k$, while $u(N) = 2s$.

We therefore obtain

$$S \leq k(q - 2) + 2s$$

as a necessary condition on $\{p_{ij}\}$ to be consistent. We note that this bound does not have to be the best that one may obtain in specific cases. For instance, for $s = 1$ we obtained $(-u)^*_{b_i} = 0$ with $u(N) = 1$ (rather than $u(N) = 2$).

6. The Diagonal Inequality

As opposed to the other known conditions, which could be derived from the equivalence theorem relatively easily, and even suggested some generalizations, the diagonal inequality does not seem to be readily obtained by this method. Indeed, the sufficiency proof is constructive enough to specify a game u for any given set of coefficients $\{\alpha_{ij}\}$. However, the computation of the game u used in the proof is more complicated than a direct computation of the right side β: we know that $\beta = u(N)$, so that computing β directly is tantamount to estimating u on one (rather than all) coalitions.

However, actual computation of the game u provided by the theorem may be insightful in some cases. In particular, we would like to compute it for the diagonal inequality since the (combinatorial) method of computation may be useful by its own right.

We will restrict our attention to the case of disjoint sequences A and B, although this does not exhaust the richness of the diagonal inequality.

Let, then, be given $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ with $\{a_i\}_{i=1}^{i} \cap \{b_i\}_{i=1}^{k} = \emptyset$, and a number $1 \le r \le k - 1$. We define

$$\alpha_{a_i b_j} = 1 \qquad 1 \le i \ne j \le k$$
$$\alpha_{a_i b_i} = -r \qquad 1 \le i \le k,$$

which define, for all $1 \le i \le k$,

$$v^{a_i} = \Sigma_{j=1}^{i-1} v_{\{a_i, b_j\}} - rv_{\{a_i, b_i\}} + \Sigma_{j=i+1}^{k} v_{\{a_i, b_j\}}$$

(and $v^{b_i} = 0$).

The associated game u is defined by

$$u(S) = \max_R \Sigma_{\{iRj; i, j \in S\}} \alpha_{ij}$$

(where the max is taken over all linear orderings on N, or, equivalently, on S).

Given a coalition S, let us assume it contains exactly m pairs (a_i, b_i), ℓ elements of A whose counterpart is in S^c and q elements of B whose counterpart is in S^c. (Thus, $|S \cap \{a_i\}_{i=1}^{k}| = m + \ell$; $|S \cap \{b_i\}_{i=1}^{i}| = m + q$.)

Let us compute a linear order R which maximizes $\Sigma_{\{iRj; i, j \in S\}} \alpha_{ij}$. consider an element a_i such that $b_i \in S^c$. Obviously, for every $b_j \in S$, $a_i R b_j$ has to hold for R to be maximal. Similarly, for $b_i \in S$ with $a_i \in S^c$ one has to have $a_j R b_i$ for all $a_j \in S$. the interesting part is, therefore, the m pairs (a_i,b_i). Assume w.l.o.g. that these are $\{(a_i,b_i)\}_{i=1}^{m}$. It is obvious that R may be defined arbitrarily over $\{b_i\}_{i=1}^{m}$. W.l.o.g. assume $b_1 R b_2 R \ldots R b_m$.

The main point in this direct computation method is the following: given the order defined over $\{b_i\}$, each a_i may be separately located in the order R so as to maximize its contribution to the expression $\Sigma_{i \neq j}\, p_{a_i b_j} - r\, \Sigma_i\, p_{a_i b_i}$.

Let us now distinguish between two cases: (i) $m \leq r$ and (ii) $m > r$. If $m \leq r$, it is quite straightforward to verify that the maximal order R has to satisfy $b_i R a_i R b_{i+1}$ (for $1 \leq i \leq m$). In this case $u(S) = \ell(m + q) + mq + m(m - 1)/2$.

As for case (ii), $b_i R a_i R b_{i+1}$ still has to hold for $1 \leq i \leq r$. However, for $r + 1 \leq i \leq m$, $a_i R b_1$ is a necessary condition for R's optimality. In this case, one obtains

$$u(S) = \ell(m + q) + (m - r)(m + q - r - 1) + r(m - r + q) + r(r - 1)/2.$$

It is also easy to verify that for $S = N$ we obtain $m = k$, $\ell = q = 0$ and

$$u(N) = k(k - 1) - rk + r(r + 1)/2.$$

(Which proves the diagonal inequality either directly or via the equivalence theorem.)

Worthy of note is the fact that this method of computation of the game u (or of u(N) directly) cannot be applied without some symmetry consideration that would allow assuming an arbitrary order over the sequence

B (or part of it). Trying to apply it to Fishburn's condition, for instance, involves combinatorial arguments which are tantamount to a direct proof.

7. A Remark Regarding Sufficiency

None of the explicit conditions mentioned in this paper--the known and the new ones--is sufficient, nor are they sufficient in conjunction. This may be proved by the same example used in Gilboa (1989) to establish the insufficiency of the diagonal inequality. This example involves probabilities $p_{ij} \in \{1/3, 2/3\}$, and it is easy to see that for such $\{p_{ij}\}$ all conditions hold. Yet, it was proved that the specific set of $\{p_{ij}\}$ given there is not consistent.

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