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CHEAP TALK AND COOPERATION IN THE SOCIETY*

by

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Abstract

This paper considers a society which consists of many individuals. They are divided into two types, and two individuals of different types are randomly matched to play a pure coordination game with cheap talk; that is, in the first stage, each individual announces one of his/her own actions simultaneously, and in the second stage, knowing the announcement of the opponents, they actually play a one-shot pure coordination game. We apply a noncooperative solution concept called cyclically stable sets to this society. The basic concept is accessibility which is defined, roughly speaking, as follows: a strategy profile g is accessible from another strategy profile f if there is a path from f to g where the direction of the path at each point on it is a best response to that point. A <u>cyclically stable set</u> is a set of strategy profiles which is closed under accessibility and for which any two members are accessible from each other. It is shown that cyclically stable sets always exist and that any cyclically stable set contains only Pareto optimal outcomes. In attaining a Pareto optimal outcome, cheap talk plays an important role.

INTRODUCTION

In the field of noncooperative game theory, an equilibrium is defined as a strategy profile from which no one has an incentive to make a unilateral deviation. Many solution concepts require that players make rational choices, i.e., determine their strategies so as to maximize their expected payoffs calculated based on their beliefs and that those beliefs must satisfy a certain consistency requirement. The criteria of rational choices and consistency of beliefs vary from one concept to another. For example, Nash equilibrium (Nash(1951)) requires the rationality (expected payoff maximization) of strategies and the consistency of beliefs with the equilibrium strategies (i.e., calculated according to Bayes' rule) at any information set on equilibrium paths, that is to say, those which are reached with positive probability under equilibrium strategy profile. On the other hand, sequential equilibrium (Kreps and Wilson(1982)) require, in addition to those conditions required in Nash equilibrium, a certain consistency of beliefs with the equilibrium strategies at any information set both on and off equilibrium paths. Regarding normal form games, players' beliefs must be consistent with each other as well as with the equilibrium strategy profile when we use Nash equilibrium or its refined concepts; on the other hand, rationalizability (Bernheim(1984) and Pearce(1984)) requires rational choices but only internal consistency of beliefs with rational choices, allowing two players' beliefs to be inconsistent.

In spite of such a widespread spectrum, what is common to these concepts is that players' systems of beliefs and strategies do not change throughout the entire game. The game is played exactly once (if it is a repeated game, the repetition occurs once), and if they change their beliefs or strategies, the changes are incorporated in larger systems of beliefs and strategies. In this sense, each player is treated as if he/she had a complete contingent plan of beliefs and strategies. The stability of strategies discussed in this context is called strategic stability.

In many daily life situations, on the other hand, people do not know and/or do not care about the entire structure of a game. Nevertheless, they behave so as to maximize their payoffs. In order to behave optimally, they do not necessarily have to know the entire structure of the game. What they have to know are their own payoffs and the opponents' strategies, or more extremely, they only have to know their expected payoffs from each of the actions available to them. One of various plausible stories of how they learn to behave optimally is that the game is repeated many times, and people use trials and errors in determining their actions. In this process, since the behavior pattern necessarily changes as time goes on, the belief system changes as well. Social stability refers to the stability of the stationary point in this repeated situation. Gilboa and Matsui (1989) suggest a new solution concept called cyclically stable set on the basis of this way of viewing the world.[1/] Cyclically stable sets are applied to the general class of normal form games with finite number of types each of which consists of many individuals who are matched randomly to play a single game. In the course of long time repetition, a behavior pattern of the society changes gradually to a certain class of strategy profiles. A cyclically stable set is a set of strategy profiles of the society such that once an once an actual behavior pattern falls in the set, it never leaves the set, and any strategy profile in the set may always be realized. The purpose of this paper is to apply this solution concept to a pure coordination game with cheap talk. The main result is that cheap talk forces players to cooperate to attain Pareto optimal outcomes.

We say that a game is of <u>pure coordination</u> if the participants of the game always have the same payoff. In a game of pure coordination, it has been thought intuitive that cooperation is likely to be an outcome if there is a cheap talk before the game begins. Consider the following two-person two stage game. In the first stage, players announce either L or R. In the second stage they choose either L or R after observing the first stage announcements. Payoffs are determined by the actions they take in the second stage, which are shown in the payoff matrix of Figure 1. This two stage game is called a game with cheap talk in the sense that the actions taken in the first stage do not directly affect the payoffs nor the actions available to players in the second stage of the game. The game which consists only of the second stage of the original game is called the game without cheap talk.

Figure 1

Even in this class of games, which involves no conflict of interests between players, no rigorous noncooperative theory predicts that cheap talk has ([L],[L]) chosen as a unique equilibrium outcome of the second stage.

Indeed, from the viewpoint of strategic stability, ([R],[R]) is an equally good equilibrium outcome of the second stage even in the game with cheap talk since players strictly prefer R to L if they believe that the opponents take R no matter what announcements are made in the first stage of the game.

In the context of social stability, however, when this two stage game itself is repeated in the society, it might be the case that the norm which puts a certain meaning on cheap talk develops during the process of change in behavior patterns, which leads to the Pareto optimal outcome. In the example of Figure 1, cooperation emerges as in the following story.

Suppose that the initial behavior pattern is such that all the players announce R in the first stage and take R in the second stage no matter what is announced in the first stage. Note that this strategy is a best response to itself. Through a long time of trials and errors, there develops a new strategy in which one announces R, takes R if the opponent's announcement is R, and takes L if the opponent's announcement is L. This strategy may well develop since it is also a best response to the behavior pattern. If, for instance, more than a half of the population switched to this strategy, which is likely to occur at some point in time, the best response is to announce L and take L since the one who takes this strategy gains two if he/she meets another who takes the second strategy and gains zero if he/she meets another who takes the original strategy, and then his/her expected payoff is more than one.[2/] And once this strategy prevails in the society, to take R in the second stage can never be a best response except in unreached nodes.

This verbal description essentially corresponds to a formal argument in the subsequent sections. Its logic involves three important points concerning social stability as distinguished from strategic stability (the first two points) and from evolutionary stability (the third point). First, a society we are interested in consists of many individuals, who are matched randomly to play the game with cheap talk and are never matched again in the future so that they are not involved in strategic interaction beyond a single matching situation. If the game is repeated infinitely many times between the same individuals, then it is unavoidable for the analysis to suffer from strategic interaction between different two stage games, consequently, the game should be considered as an infinitely repeated game.[3/] Our setting avoid these complicated situations. An important remark is that we do not necessarily need infinite number of individuals to cope with the problem; rather, it is enough to consider the players who ignore the small probability of rematching.

Second, we deal with the situation in which people try to figure out their opponents' strategies on the basis of what happens in the current world. If individuals examine the two stage game and try to find an optimal strategy independent of what happens in the society, the analysis is not different from the one concerning strategic stability, and there seems to be no room for cheap talk to play a role in attaining the Pareto optimal result.

The third point is that after sufficient trials and errors, only best response strategies may increase their relative frequency in the society we deal with—as distinguished from a competition among genes, in which a better-response gene, namely, better than the major existing gene, may increase its relative frequency without being a best response.

The rest of this paper is organized as follows. Section 2 presents some notations and definitions. Section 3 defines and discusses the notion of social stability. Section 4 gives the main theorems which basically state that in the pure coordination games with cheap talk only Pareto efficient outcomes are in cyclically stable sets.

2. PURE COORDINATION GAMES WITH AND WITHOUT CHEAP TALK

In a society, which is called a game, there are two types of individuals: type 1 and type 2. Some individuals, who are assumed to be anonymous, are matched randomly to take some actions. In each matching situation, one individual from each type is selected, and they are matched to play a finitely repeated game.

Let C_1 and C_2 be finite action spaces for type 1 individual and type 2 individual respectively. We assume that $C_1 \cap C_2 = \phi$ and $|C_i| \geq 2$ for $i=1,2$. A mapping $u: C = C_1 \times C_2 \to \mathfrak{R}$ is a utility function to both type individuals. We consider once or twice repetition of this game. Let K denote the number of repetitions: K=1 or 2. A pure strategy of type i individual (i=1,2) is a mapping $s: H^K \to C_i$ where $H^1 = \{e\}$ and $H^2 = \{e\} \cup C$ with e denoting the empty history. We denote by S_i the set of all pure strategies of a type i individual (i=1,2). A strategy profile for type i individuals (i=1,2) is a probability distribution over S_i. Let F_i denote the set of all strategy profile for type i, i.e.,

$$F_i = \{f_i : S_i \to \mathfrak{R}: \Sigma_{s \in S_i} f_i(s) = 1 \text{ and } f_i(s) \geq 0 \text{ for all } s \in S_i\}.$$

We write $S = S_1 \times S_2$ and $F = F_1 \times F_2$. F is called the set of strategy profiles (of the society). Given f and g in F, define $(1-\lambda)f + \lambda g$ as $h \in F$ such that

$h_i(s)=(1-\lambda)f_i(s)+\lambda g_i(s)$ for all $s\in S_i$ and $i=1,2$. Given f in F, there are two possible scenarios concerning the choice of strategies by the individuals. One is that $f_i(s)$-portion of the entire population of type i take pure strategy s for each s; and the other is that every type i individual takes the mixed strategy f_i. This distinction does not affect our analysis in the sequel. However, we find the former more appealing than the latter and prefer to keep it in mind. In considering the dynamic adjustment process, the current strategy profile will be often referred to as a behavior pattern. F is considered as a subset of a $(|S|-2)$-dimensional space on which Euclidean norm, $\|\cdot\|$, and linear operations are defined.

The payoff function in a single matching situation is a mapping $\pi:S\to\mathbb{R}$ calculated based on u. We shall consider in the sequel two cases: games with and without cheap talk. We make the following two alternative assumptions. Let $c^k(s)$ $(k=1,2)$ be a pair of actions at the k-th stage induced by $s\in S$. Then

A-1(without cheap talk): $K=1$, and

$$\pi(s)=u(c^1(s)).$$

A-2(with cheap talk): $K=2$, and

$$\pi(s)=u(c^2(s)).$$

We say a game is <u>without (with) cheap talk</u> if assumption A-1 (resp. A-2) holds. In the sequel, we assume that either A-1 or A-2 (but not both) holds.

Let c^* and $\underline{c}$ satisfy $c^*\in\text{argmax}_{c\in C}u(c)$ and $\underline{c}\in\text{argmin}_{c\in C}u(c)$. Let $\pi^*=u(c^*)$

and $\underline{\pi}=u(\underline{c})$. Note that $\pi^*=\max_{s\in S}\pi(s)$ and $\underline{\pi}=\min_{s\in S}\pi(s)$. Since we assume $C_1\cap C_2=\phi$, we may write $\pi(s_1,s_2)=\pi(s_2,s_1)$, in which case we define the payoff function as $\pi:S_1\times S_2\cup S_2\times S_1\rightarrow\Re$. We write $\pi(s\backslash s_i')=\pi(s_i',s_j)$ $(i=1,2)$. Given a strategy profile $f\in F$, the expected payoff for an individual of type i (i=1,2) if he/she takes a strategy $g_i\in F_i$ is:

$$\Pi_i(f;g_i) = \Sigma_{r_i\in S_i}\Sigma_{s_j\in S_j}g_i(r_i)f_j(s_j)\pi(r_i,s_j)$$

where j denotes not i. This is also the expected payoff of type j individual if type i takes g_i and j takes f_j. We denote $\Pi_i(f)=\Pi_i(f;f_i)$. Note that $\Pi_1(f)=\Pi_2(f)$. Therefore, we occasionally write $\Pi(\cdot)=\Pi_i(\cdot)$ (i=1,2) in the sequel. This should cause no confusion. Let $Br_i(f)$ be the set of strategy profiles for individuals of type i (i=1,2) that are best responses to f, i.e.,

$$Br_i(f)=\text{argmax}_{g_i\in F_i}\Pi_i(f;g_i).$$

We may write $Br(f)=Br_1(f)\times Br_2(f)$. Let a function $[\cdot]:S_1\cup S_2\cup S\rightarrow F_1\cup F_2\cup F$ satisfy $[s](s)=1$ for all $s\in S_1\cup S_2\cup S$. Notice that if s and s_i' are a pure strategy pair and a pure strategy respectively, then $\Pi([s])=\pi(s)$ and $\Pi([s];[s_i'])=\pi(s\backslash s_i')$. We use $\Pi(\cdot)$ and $\pi(\cdot)$ interchangeably.

3. SOCIAL STABILITY AND CYCLICALLY STABLE SET

This section defines and discusses the concept of cyclical stability. To capture the idea of social stability, we consider the following two points: (1) there are no strategic considerations between any two matchings (there may be strategic interaction within each matching situation); and (2) unlike a deviation made by a single player, a change in behavior pattern is likely to be continuous. The former reflects the fact that individuals are anonymous and are matched randomly. The latter expresses the idea that within a small time interval, only a small portion of individuals change their strategies. In order to express these points, the notion of accessibility is given, which is an unperturbed version of accessibility as defined in Gilboa and Matsui(1989).

DEFINITION: A strategy profile g is <u>accessible</u> from another strategy profile f if there exist a continuous $p:[0,1]\to F$, $h:[0,1)\to F$ which is continuous from the right, and $\alpha:[0,1)\to[0,\infty)$ which is continuous from the right such that $p(0)=f$, $p(1)=g$,

$$(d^{+}/dt)p(t)=\alpha(t)\{h(t)-p(t)\} \quad \text{for } t\in[0,1)$$

and

$$h(t)\in Br(p(t)) \quad \text{for } t\in[0,1).$$

The definition says that, in case of $\alpha>0$, a behavior pattern moves in the direction of a best response to the current behavior pattern, and it may stay at the same place only when the behavior pattern is a best response to itself. By including the case of $\alpha=0$, we assure that a strategy profile is always accessible from itself. We call the function p an accessible path from f to g. It is easy to verify that accessibility satisfies transitivity.

The interpretation of this definition is that only small and equal potions of individuals in each type realize the current behavior pattern and change their behavior pattern to another which is a best response to it. Using this notion of accessibility, we are now in a position to present the definition of cyclical stability.

DEFINITION: A nonempty subset F^* of F is an (unperturbed) cyclically stable set (CSS) if

no $g \notin F^*$ is accessible from any $f \in F^*$, and

every g in F^* is accessible from all f in F^*.

A cyclically stable set (CSS) is stable in the sense that once the actual behavior pattern falls into it, another strategy profile may be realized if and only if it is within the CSS. The interpretation of this concept is as follows: for a long time, individuals have sought better strategies. After they experience enough, a behavior pattern falls into a CSS, may move within it, and never leaves it. The term "cyclically stable" stems from the intuitive notion of cycles within a CSS. However, the paths may, of course, be much more complicated.

4. RESULTS

This section presents two main theorems the proof of which will be given in the following section. The first serves the optimality result in the games with cheap talk, and the second theorem shows the existence of a cyclically stable set in the games both with and without cheap talk. The following is the first theorem stating that in the games with cheap talk, any non-optimal outcome is not stable.

THEOREM 1: Suppose that A-2 holds. If f is in a cyclically stable set, then $\Pi(f)=\pi^*$ holds.

Observe that the statement of the theorem does not hold in a game without cheap talk. Consider the game of Figure 1. If all the people in the society take R, then no individual takes L to get zero instead of one. Therefore, ([R],[R]) is a socially stable strategy, *a fortiori*, it forms a cyclically stable set as a singleton. On the other hand, in the game with cheap talk, the cooperation toward the Pareto optimal outcome is possible because even if all the other take R, one can take the strategy which expects a "signal" and takes L if one gets the signal and remains R if one does not, which is followed by the opponent's change of the strategy to the one which actually sends a signal for cooperation.

The second theorem states that there exists at least one cyclically stable set which consists of Pareto optimal outcomes in both games with and without cheap talk.

THEOREM 2: *Suppose that either* A-1 *or* A-2 *holds. Then there exists at least one cyclically stable set the payoff of each element of which is* π^*.

5. PROOFS

The proofs of the main theorems are given in this section. Before it, we present some lemmata. First, we present the following lemma the proof of which is a direct calculation.

LEMMA 1: *Suppose that either* A-1 *or* A-2 *holds. If* $g \in F$ *is accessible from* $f \in F$, *then* $\Pi(g) \geq \Pi(f)$.

Proof: Suppose that $g \in F$ is accessible from $f \in F$. Then there exist functions $p:[0,1] \to F$ continuous, $h:[0,1) \to F$ continuous from the right, and $\alpha:[0,1) \to [0,\infty)$ continuous from the right such that $P(0)=f$, $p(1)=g$,

$$(d^+/dt)p(t) = \alpha(t)\{h(t)-p(t)\} \quad \text{for } t \in [0,1), \text{ and}$$

$$h(t) \in Br(p(t)) \quad \text{for } t \in [0,1).$$

We now calculate $(d^+/dt)\Pi(p(t))$ for $t \in [0,1)$.

$$(d^+/dt)\Pi(p(t)) = [\partial\Pi(p(t))/\partial z]_{z \in S} \cdot (d^+/dt)p(t)$$

$$= [\partial\Pi(p(t))/\partial z]_{z \in S} \cdot [\alpha(t)\{h(t)-p(t)\}]$$

$$= \alpha(t)\Sigma_{r \in S_1}\Sigma_{s \in S_2} p_2(t)(s)\{h_1(t)(r)-p_1(t)(r)\}\pi(r,s)$$

$$+ \alpha(t)\Sigma_{r \in S_2}\Sigma_{s \in S_1} p_1(t)(s)\{h_2(t)(r)-p_2(t)(r)\}\pi(s,r)$$

$$= \alpha(t)[\Pi_1(p(t);h_1(t))-\Pi_1(p(t))]$$

$$+ \alpha(t)[\Pi_2(p(t);h_2(t))-\Pi_2(p(t))]$$

The last expression is nonnegative since $\alpha(t) \geq 0$ and $h(t)$ is a best response to $p(t)$. Therefore, we have

$$\Pi(g)-\Pi(f) = \int_0^1 (d^+/dt)\Pi(p(t))dt \geq 0. \qquad \text{Q.E.D.}$$

The next two lemmata show that if a strategy profile f is in a CSS, then a pure strategy profile $[\hat{s}]$ is accessible from f whenever $\pi(\hat{s})$ is the maximum payoff among those which are best responses to f.

LEMMA 2: _Suppose that either_ A-1 _or_ A-2 _holds. Given_ f, _suppose that_ f _is in a cyclically stable set. Let_

$$\hat{s} \in \text{argmax}_{s \in S}\{\pi(s): [s] \in Br(f)\}.$$

Then there exists $\bar{\varepsilon} > 0$ _such that_ $p:[0,1] \to F$ _defined by_

$$p(t) = t\bar{\varepsilon}[\hat{s}] + (1-t\bar{\varepsilon})f \quad \text{\underline{for} } t \in [0,1]$$

is an accessible path from f *to* $\bar{\varepsilon}[\hat{s}]+(1-\bar{\varepsilon})f$.

Proof: First, given $\varepsilon\in(0,1)$, define $p_\varepsilon:[0,1]\to F$ by

$$p_\varepsilon(t)=t\varepsilon[\hat{s}]+(1-t\varepsilon)f.$$

Then $p_\varepsilon(0)=f$, $p_\varepsilon(1)=\varepsilon[\hat{s}]+(1-\varepsilon)f$, and

$$(d^+/dt)p_\varepsilon(t)= [\varepsilon/(1-t\varepsilon)]\{[\hat{s}]-p(t)\} \quad \text{for } t\in[0,1)$$

hold. Therefore, what we have to prove is that there exists $\bar{\varepsilon}>0$ such that for all $\varepsilon\in[0,\bar{\varepsilon})$, $[\hat{s}]\in Br(\varepsilon[\hat{s}]+(1-\varepsilon)f)$ holds. Consider $\varepsilon[\hat{s}]+(1-\varepsilon)f$, and compare

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[\hat{s}_i]) = (1-\varepsilon)\Pi(f;[\hat{s}_i]) + \varepsilon\pi(\hat{s})$$

with

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[s_i']) = (1-\varepsilon)\Pi(f;[s_i']) + \varepsilon\pi(\hat{s}\backslash s_i'). \quad (i=1,2)$$

We have $\Pi(f;[\hat{s}_i])\geq\Pi(f;[s_i'])$ for all $s_i'\in S_i$ and $\pi(\hat{s})\geq\pi(\hat{s}\backslash s_i')$ for all $s_i'\in S_i$ whenever $\Pi(f;[\hat{s}_i])=\Pi(f;[s_i'])$ holds. Therefore, for sufficiently small $\varepsilon>0$,

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[\hat{s}_i]) \geq \Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[s_i'])$$

holds for all $s_i'\in S_i$ and any $i=1,2$. Q.E.D.

LEMMA 3: *Suppose that either* A-1 *or* A-2 *holds. Suppose further that* f *is in a cyclically stable set. Given* f, *let*

$$\hat{s} \in \operatorname{argmax}_{s\in S}\{\pi(s): [s]\in Br(f)\}.$$

Then $p^*:[0,1]\to F$ *defined by*

$$p^*(t)=t[\hat{s}]+(1-t)f \quad \textit{for } t\in[0,1]$$

is an accessible path from f *to* $[\hat{s}]$.

Proof: Let f, $\hat{s}$, and p^* be as in the statement of the lemma. Note that $p^*(0)=f$, $p^*(1)=[\hat{s}]$, and that

$$(d^+/dt)p^*(t) = 1/(1-t)\{[\hat{s}]-p^*(t)\} \quad \text{for } t\in[0,1).$$

Therefore, what we have to show is that for any $t\in[0,1)$, $[\hat{s}]\in Br(p^*(t))$ holds. Denote $g^{\mu}=\mu[\hat{s}]+(1-\mu)f$. Then it is equivalent to showing that $[\hat{s}]\in Br(g^{\mu})$ for all $\mu\in[0,1)$.

Suppose the contrary, i.e., that there exists $\mu'\in[0,1)$ such that $[\hat{s}]\notin Br(g^{\mu'})$ holds. Let $\tilde{\mu}=\inf\{\mu': [\hat{s}]\notin Br(g^{\mu'})\}$. Observe that $g^{\tilde{\mu}}$ is accessible from f. If $[\hat{s}]\in \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$ holds, then by lemma 2 there exists $\bar{\varepsilon}>0$ such that $[\hat{s}]\in Br(g^{\tilde{\mu}+\varepsilon})$ holds for all $\varepsilon\in[0,\bar{\varepsilon})$, which is a contradiction. Thus, $[\hat{s}]\notin \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$ must hold. Then there exists $s'\neq\hat{s}$ with $s'\in \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$. From lemma 2, there exists $\bar{\varepsilon}>0$ such that $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$ is accessible from $g^{\tilde{\mu}}$ _a fortiori_ from f for all $\varepsilon\in[0,\bar{\varepsilon})$. On the other hand, it must be the case that either

$$\Pi(g^{\tilde{\mu}};[s_i']) > \Pi(g^{\tilde{\mu}};[\hat{s}_i]), \text{ or} \tag{1}$$

$$\pi(s')>\pi(\hat{s}) \text{ with } \Pi(g^{\tilde{\mu}};[s_i']) = \Pi(g^{\tilde{\mu}};[\hat{s}_i]) \tag{2}$$

holds and note that $\Pi(g^{\tilde{\mu}};[s_i']) \geq \Pi(g^{\tilde{\mu}};[\hat{s}_i])$ for $i=1,2$. Then there exists $\varepsilon\in(0,\bar{\varepsilon})$ such that

$$\begin{aligned}\Pi(\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}) &= (1-\varepsilon)^2\Pi(g^{\tilde{\mu}}) + \varepsilon(1-\varepsilon)\Pi(g^{\tilde{\mu}};[s_1'])\\ &\quad + \varepsilon(1-\varepsilon)\Pi(g^{\tilde{\mu}};[s_2']) + \varepsilon^2\pi(s')\\ &> \Pi(f)\end{aligned} \tag{3}$$

holds by virtue of (1) and (2). From lemma 1, (3) implies that f is not accessible from $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$. Thus, $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$ is accessible from f but not vice versa, which is a contradiction to the assumption that f is in a cyclically stable set. Q.E.D.

We are now in a position to present the proof of Theorem 1, which serves the optimality result.

The Proof of Theorem 1: Suppose first that f is in a cyclically stable set. From lemma 3, there exists $\tilde{s}\in S$ such that $[\tilde{s}]$ is accessible from f. We construct another strategy profile $[\hat{s}]$ (which may be identical to $[\tilde{s}]$) in the following manner:

$$\hat{s}_i(h) = \begin{cases} \tilde{s}_i(e) & \text{if } h=e, \\ \tilde{s}_i(h) & \text{if } h=c^1(\tilde{s}), \\ c_i^* & \text{otherwise}, \end{cases} \quad \text{for } i=1,2.$$

Observe that $\Pi([\tilde{s}];[\hat{s}_i])=\pi(\hat{s})=\pi(\tilde{s})$.

We claim that $[\hat{s}]\in\text{argmax}_{s\in S}\{\pi(s)\colon [s]\in Br([\tilde{s}])\}$. Suppose the contrary. Then there exists $s'\neq\hat{s}$ such that $[s']\in\text{argmax}_{s\in S}\{\pi(s)\colon [s]\in Br([\tilde{s}])\}$. From lemma 3, $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$ is accessible from $[\tilde{s}]$ for all $\varepsilon\in[0,1]$. On the other hand, it must be the case that either

$$\Pi([\tilde{s}];[s_i']) > \Pi([\tilde{s}];[\hat{s}_i]) = \pi(\tilde{s}) \quad \text{for some } i=1,2, \text{ or} \tag{4}$$

$$\pi(s')>\pi(\hat{s}) \text{ with } \Pi([\tilde{s}];[s_i']) = \Pi([\tilde{s}];[\hat{s}_i]) \quad \text{for } i=1,2 \tag{5}$$

holds and note that $\Pi(\tilde{g}^{\mu};[s_i']) \geq \Pi(\tilde{g}^{\mu};[\hat{s}_i])$ for $i=1,2$. Then there exists $\varepsilon\in(0,\bar{\varepsilon})$ such that

$$\begin{aligned}\Pi(\varepsilon[s']+(1-\varepsilon)[\tilde{s}]) = {} & (1-\varepsilon)^2\pi([\tilde{s}]) + \varepsilon(1-\varepsilon)\Pi([\tilde{s}];[s_1']) \\ & + \varepsilon(1-\varepsilon)\Pi([\tilde{s}];[s_2']) + \varepsilon^2\pi(s') \\ & > \pi(\tilde{s}) \end{aligned} \tag{6}$$

holds by virtue of (4) and (5). From lemma 1, (6) implies that $[\tilde{s}]$ is not accessible from $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$. Thus, $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$ is accessible from $[\tilde{s}]$ but not vice versa, which is a contradiction to the assumption that f is in a cyclically stable set. Thus, $[\hat{s}]\in\text{argmax}_{s\in S}\{\pi(s)\colon [s]\in Br([\tilde{s}])\}$ must hold. Hence by lemma 3, $[\hat{s}]$ is accessible from $[\tilde{s}]$.

Next, we construct a strategy profile $[s^*]$ satisfying

$s_i^*(e)\neq\hat{s}_i(e)$, and

$s_i^*(h)=c_i^*$ if $h\neq e$, for $i=1,2$.

Suppose now that $\pi^*>\Pi(f)$ holds. Then $\pi(\hat{s}\backslash s_i^*)=\pi^*>\pi(\hat{s})=\Pi(f)$ holds for $i=1,2$. If we take any s^{**} with $[s^{**}]\in\operatorname{argmax}_{s\in S}\{\pi(s): [s]\in Br([\hat{s}])\}$, we have $\pi(\hat{s}\backslash s_i^{**})>\pi(\hat{s})$ for $i=1,2$. Then for sufficiently small $\varepsilon>0$, we have

$$\Pi(\varepsilon[s^{**}]+(1-\varepsilon)[\hat{s}]) = (1-\varepsilon)^2\pi(\hat{s}) + \varepsilon(1-\varepsilon)\pi(\hat{s}\backslash s_1^{**})$$
$$+ \varepsilon(1-\varepsilon)\pi(\hat{s}\backslash s_2^{**}) + \varepsilon^2\pi(s^{**}) > \pi(\hat{s}) = \Pi(f).$$

By lemma 1, f is not in a cyclically stable set, which is a contradiction.

Q.E.D.

Note that the logic used in the proof cannot be applied to a game without cheap talk, and one may easily verify that in the example of Figure 1, (1,1) as well as (2,2) is a payoff pair in a cyclically stable set.

Next, we prove the existence of cyclically stable set. To this aim, we define the following. Given $f_i\in F_i$, let $S_i(f_i)\subset S_i$ $(i=1,2)$ be the support of f_i, i.e.,

$$S_i(f_i)=\{s_i\in S_i: f_i(s_i)>0\}.$$

We write $S(f)=S_1(f_1)\times S_2(f_2)$.

The Proof of Theorem 2: Suppose the contrary, i.e., that there exists no $f\in F$ in any cyclically stable set which satisfies $\Pi(f)=\pi^*$. Given f with $\Pi(f)=\pi^*$, we first claim that g is accessible from f whenever $S(g)\subset S(f)$. We show that $p:[0,1]\to F$ with

$$p(t)=(1-t)f+tg \quad t\in[0,1]$$

is an accessible path from f to g. Since $p(0)=f$, $p(1)=g$, and

$$(d^+/dt)p(t)=[1/(1-t)]\{g-p(t)\},$$

what we have to show is that g is in $Br(p(t))$ for all $t\in[0,1)$. Observe that $\Pi(f)=\pi^*$ implies that $\pi(s)=\pi^*$ for all $s\in S(g)\subset S(f)$ since π^* is a maximum. Then we have

$$\Pi(p(t);g_i)=\Pi((1-t)f+tg;g_i)$$

$$=\Sigma_{r_i\in S_i}\Sigma_{s_j\in S_j}g_i(r_i)\{(1-t)f_j(s_j)+tg_j(s_j)\}\pi(r_i,s_j)=\pi^*,\ i=1,2,$$

which is always a maximum.

Since f is not in a cyclically stable set, there exists $g^1\in F$ which is accessible from f but not vice versa. Suppose not. Then we can claim that $F^*=\{g\in F|g$ is accessible from $f\}$ is a cyclically stable set. Indeed, take any f_1 and f_2 in F^*. Then f is accessible from f_1, and f_2 is accessible from f. By transitivity of accessibility, f_2 is accessible from f_1. On the other hand, take any $g\in F^*$ and $g'\notin F^*$. If g' is accessible from g, then again by transitivity g' is accessible from f, a contradiction to the definition of F^*.

Let g^1 be a strategy profile which is accessible from f but not vice versa. We know that $S(g^1)\not\supset S(f)$, in particular, $S(g^1)\neq S(f)$. From lemma 1, $\Pi(g^1)=\pi^*$. Therefore, g^1 is not in a cyclically stable set, either. Then there exists $g^2\in F$ which is accessible from g^1 but not vice versa. Thus, $S(g^2)\neq S(g^1)$ holds, and by transitivity $S(g^2)\neq S(f)$ also holds (otherwise, f must be accessible from g^1). Continuing the process. There must be an infinite sequence $\{g^m\}_{m=1}^{\infty}$ such that $S(g^m)\neq S(g^{m'})$ for all $m\neq m'$. This is impossible since the number of elements $S(\cdot)$'s is finite. Thus, there exists $f\in F$ which is in a CSS. Hence, there exists a cyclically stable set the payoff of the element in which is π^*. Q.E.D.

FOOTNOTES

1. Kaneko (1987) also proposes a solution concept called conventionally stable sets to cope with the question of social stability.
2. One may ask how they see the relative population of the second strategy to the first one since the actual outcomes are the same for both strategies unless one announces L in the first stage. One possible scenario is that negligible portion of individuals announce L in the first stage, which reveals the relative size of the population who take the second strategy and that this information gradually spreads in the population.
3. In that case, some studies have shown that the optimal outcome is necessarily attained under some qualifications; among those studies are Aumann and Sorin (1989) and Matsui (1989).

REFERENCES

Aumann, R. and S. Sorin (1989), "Cooperation and Bounded Recall," Games and Economic Behavior, vol.1, pp.5-39.

Bernheim, B.D. (1984), "Rationalizable Strategic Behavior," Econometrica, vol.52, pp.1007-28.

Gilboa, I. and A. Matsui (1989), "Social Stability and Equilibrium," mimeo.

Kaneko, M. (1987), "The Conventionally Stable Sets in Noncooperative Games with Limited Observations I: Definitions and Introductory Arguments," Mathematical Social Sciences, vol.13, pp.93-128.

Kreps, D. and R. Wilson (1982), "Sequential Equilibria," Econometrica,

Matsui, A. (1989), "Information Leakage Forces Cooperation," Games and Economic Behavior, vol.1, pp.94-115.

Nash, J. (1951), "Non-cooperative Games," Annals of Mathematics, vol.54, pp.286-95.

Pearce, D.G. (1984), "Rationalizable Strategic Behavior and the Problem of Perfection," Econometrica, vol.52, pp.1029-50.

	L		R	
L	2	2	0	0
R	0	0	1	1

Figure 1

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CHEAP TALK AND COOPERATION IN THE SOCIETY*

by

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Abstract

This paper considers a society which consists of many individuals. They are divided into two types, and two individuals of different types are randomly matched to play a pure coordination game with cheap talk; that is, in the first stage, each individual announces one of his/her own actions simultaneously, and in the second stage, knowing the announcement of the opponents, they actually play a one-shot pure coordination game. We apply a noncooperative solution concept called cyclically stable sets to this society. The basic concept is accessibility which is defined, roughly speaking, as follows: a strategy profile g is accessible from another strategy profile f if there is a path from f to g where the direction of the path at each point on it is a best response to that point. A <u>cyclically stable set</u> is a set of strategy profiles which is closed under accessibility and for which any two members are accessible from each other. It is shown that cyclically stable sets always exist and that any cyclically stable set contains only Pareto optimal outcomes. In attaining a Pareto optimal outcome, cheap talk plays an important role.

INTRODUCTION

In the field of noncooperative game theory, an equilibrium is defined as a strategy profile from which no one has an incentive to make a unilateral deviation. Many solution concepts require that players make rational choices, i.e., determine their strategies so as to maximize their expected payoffs calculated based on their beliefs and that those beliefs must satisfy a certain consistency requirement. The criteria of rational choices and consistency of beliefs vary from one concept to another. For example, Nash equilibrium (Nash(1951)) requires the rationality (expected payoff maximization) of strategies and the consistency of beliefs with the equilibrium strategies (i.e., calculated according to Bayes' rule) at any information set on equilibrium paths, that is to say, those which are reached with positive probability under equilibrium strategy profile. On the other hand, sequential equilibrium (Kreps and Wilson(1982)) require, in addition to those conditions required in Nash equilibrium, a certain consistency of beliefs with the equilibrium strategies at any information set both on and off equilibrium paths. Regarding normal form games, players' beliefs must be consistent with each other as well as with the equilibrium strategy profile when we use Nash equilibrium or its refined concepts; on the other hand, rationalizability (Bernheim(1984) and Pearce(1984)) requires rational choices but only internal consistency of beliefs with rational choices, allowing two players' beliefs to be inconsistent.

In spite of such a widespread spectrum, what is common to these concepts is that players' systems of beliefs and strategies do not change throughout the entire game. The game is played exactly once (if it is a repeated game, the repetition occurs once), and if they change their beliefs or strategies, the changes are incorporated in larger systems of beliefs and strategies. In this sense, each player is treated as if he/she had a complete contingent plan of beliefs and strategies. The stability of strategies discussed in this context is called <u>strategic stability</u>.

In many daily life situations, on the other hand, people do not know and/or do not care about the entire structure of a game. Nevertheless, they behave so as to maximize their payoffs. In order to behave optimally, they do not necessarily have to know the entire structure of the game. What they have to know are their own payoffs and the opponents' strategies, or more extremely, they only have to know their expected payoffs from each of the actions available to them. One of various plausible stories of how they learn to behave optimally is that the game is repeated many times, and people use trials and errors in determining their actions. In this process, since the behavior pattern necessarily changes as time goes on, the belief system changes as well. <u>Social stability</u> refers to the stability of the stationary point in this repeated situation. Gilboa and Matsui (1989) suggest a new solution concept called cyclically stable set on the basis of this way of viewing the world.[1/] Cyclically stable sets are applied to the general class of normal form games with finite number of types each of which consists of many individuals who are matched randomly to play a single game. In the course of long time repetition, a behavior pattern of the society changes gradually to a certain class of strategy profiles. A cyclically stable set is a set of strategy profiles of the society such that once an once an actual behavior pattern falls in the set, it never leaves the set, and any strategy profile in the set may always be realized. The purpose of this paper is to apply this solution concept to a pure coordination game with cheap talk. The main result is that cheap talk forces players to cooperate to attain Pareto optimal outcomes.

We say that a game is of <u>pure coordination</u> if the participants of the game always have the same payoff. In a game of pure coordination, it has been thought intuitive that cooperation is likely to be an outcome if there is a cheap talk before the game begins. Consider the following two-person two stage game. In the first stage, players announce either L or R. In the second stage they choose either L or R after observing the first stage announcements. Payoffs are determined by the actions they take in the second stage, which are shown in the payoff matrix of Figure 1. This two stage game is called a game with cheap talk in the sense that the actions taken in the first stage do not directly affect the payoffs nor the actions available to players in the second stage of the game. The game which consists only of the second stage of the original game is called the game without cheap talk.

Figure 1

Even in this class of games, which involves no conflict of interests between players, no rigorous noncooperative theory predicts that cheap talk has ([L],[L]) chosen as a unique equilibrium outcome of the second stage.

Indeed, from the viewpoint of strategic stability, ([R],[R]) is an equally good equilibrium outcome of the second stage even in the game with cheap talk since players strictly prefer R to L if they believe that the opponents take R no matter what announcements are made in the first stage of the game.

In the context of social stability, however, when this two stage game itself is repeated in the society, it might be the case that the norm which puts a certain meaning on cheap talk develops during the process of change in behavior patterns, which leads to the Pareto optimal outcome. In the example of Figure 1, cooperation emerges as in the following story.

Suppose that the initial behavior pattern is such that all the players announce R in the first stage and take R in the second stage no matter what is announced in the first stage. Note that this strategy is a best response to itself. Through a long time of trials and errors, there develops a new strategy in which one announces R, takes R if the opponent's announcement is R, and takes L if the opponent's announcement is L. This strategy may well develop since it is also a best response to the behavior pattern. If, for instance, more than a half of the population switched to this strategy, which is likely to occur at some point in time, the best response is to announce L and take L since the one who takes this strategy gains two if he/she meets another who takes the second strategy and gains zero if he/she meets another who takes the original strategy, and then his/her expected payoff is more than one.[2/] And once this strategy prevails in the society, to take R in the second stage can never be a best response except in unreached nodes.

This verbal description essentially corresponds to a formal argument in the subsequent sections. Its logic involves three important points concerning social stability as distinguished from strategic stability (the first two points) and from evolutionary stability (the third point). First, a society we are interested in consists of many individuals, who are matched randomly to play the game with cheap talk and are never matched again in the future so that they are not involved in strategic interaction beyond a single matching situation. If the game is repeated infinitely many times between the same individuals, then it is unavoidable for the analysis to suffer from strategic interaction between different two stage games, consequently, the game should be considered as an infinitely repeated game.[3/] Our setting avoid these complicated situations. An important remark is that we do not necessarily need infinite number of individuals to cope with the problem; rather, it is enough to consider the players who ignore the small probability of rematching.

Second, we deal with the situation in which people try to figure out their opponents' strategies on the basis of what happens in the current world. If individuals examine the two stage game and try to find an optimal strategy independent of what happens in the society, the analysis is not different from the one concerning strategic stability, and there seems to be no room for cheap talk to play a role in attaining the Pareto optimal result.

The third point is that after sufficient trials and errors, only best response strategies may increase their relative frequency in the society we deal with—as distinguished from a competition among genes, in which a better-response gene, namely, better than the major existing gene, may increase its relative frequency without being a best response.

The rest of this paper is organized as follows. Section 2 presents some notations and definitions. Section 3 defines and discusses the notion of social stability. Section 4 gives the main theorems which basically state that in the pure coordination games with cheap talk only Pareto efficient outcomes are in cyclically stable sets.

2. PURE COORDINATION GAMES WITH AND WITHOUT CHEAP TALK

In a society, which is called a game, there are two types of individuals: type 1 and type 2. Some individuals, who are assumed to be anonymous, are matched randomly to take some actions. In each matching situation, one individual from each type is selected, and they are matched to play a finitely repeated game.

Let C_1 and C_2 be finite action spaces for type 1 individual and type 2 individual respectively. We assume that $C_1 \cap C_2 = \phi$ and $|C_i| \geq 2$ for i=1,2. A mapping $u: C = C_1 \times C_2 \rightarrow \Re$ is a utility function to both type individuals. We consider once or twice repetition of this game. Let K denote the number of repetitions: K=1 or 2. A pure strategy of type i individual (i=1,2) is a mapping $s: H^K \rightarrow C_i$ where $H^1 = \{e\}$ and $H^2 = \{e\} \cup C$ with e denoting the empty history. We denote by S_i the set of all pure strategies of a type i individual (i=1,2). A strategy profile for type i individuals (i=1,2) is a probability distribution over S_i. Let F_i denote the set of all strategy profile for type i, i.e.,

$$F_i = \{f_i : S_i \rightarrow \Re :\ \Sigma_{s \in S_i} f_i(s) = 1 \text{ and } f_i(s) \geq 0 \text{ for all } s \in S_i\}.$$

We write $S = S_1 \times S_2$ and $F = F_1 \times F_2$. F is called the set of strategy profiles (of the society). Given f and g in F, define $(1-\lambda)f + \lambda g$ as $h \in F$ such that

$h_i(s)=(1-\lambda)f_i(s)+\lambda g_i(s)$ for all $s\in S_i$ and $i=1,2$. Given f in F, there are two possible scenarios concerning the choice of strategies by the individuals. One is that $f_i(s)$-portion of the entire population of type i take pure strategy s for each s; and the other is that every type i individual takes the mixed strategy f_i. This distinction does not affect our analysis in the sequel. However, we find the former more appealing than the latter and prefer to keep it in mind. In considering the dynamic adjustment process, the current strategy profile will be often referred to as a behavior pattern. F is considered as a subset of a $(|S|-2)$-dimensional space on which Euclidean norm, $\|\cdot\|$, and linear operations are defined.

The payoff function in a single matching situation is a mapping $\pi:S\to\Re$ calculated based on u. We shall consider in the sequel two cases: games with and without cheap talk. We make the following two alternative assumptions. Let $c^k(s)$ $(k=1,2)$ be a pair of actions at the k-th stage induced by $s\in S$. Then

A-1(without cheap talk): K=1, and

$$\pi(s)=u(c^1(s)).$$

A-2(with cheap talk): K=2, and

$$\pi(s)=u(c^2(s)).$$

We say a game is <u>without (with) cheap talk</u> if assumption A-1 (resp. A-2) holds. In the sequel, we assume that either A-1 or A-2 (but not both) holds.

Let c^* and $\underline{c}$ satisfy $c^*\in\operatorname{argmax}_{c\in C}u(c)$ and $\underline{c}\in\operatorname{argmin}_{c\in C}u(c)$. Let $\pi^*=u(c^*)$

and $\underline{\pi}=u(\underline{c})$. Note that $\pi^*=\max_{s\in S}\pi(s)$ and $\underline{\pi}=\min_{s\in S}\pi(s)$. Since we assume $C_1\cap C_2=\phi$, we may write $\pi(s_1,s_2)=\pi(s_2,s_1)$, in which case we define the payoff function as $\pi:S_1\times S_2\cup S_2\times S_1\to\Re$. We write $\pi(s\backslash s_i')=\pi(s_i',s_j)$ (i=1,2). Given a strategy profile $f\in F$, the expected payoff for an individual of type i (i=1,2) if he/she takes a strategy $g_i\in F_i$ is:

$$\Pi_i(f;g_i) = \Sigma_{r_i\in S_i}\Sigma_{s_j\in S_j}g_i(r_i)f_j(s_j)\pi(r_i,s_j)$$

where j denotes not i. This is also the expected payoff of type j individual if type i takes g_i and j takes f_j. We denote $\Pi_i(f)=\Pi_i(f;f_i)$. Note that $\Pi_1(f)=\Pi_2(f)$. Therefore, we occasionally write $\Pi(\cdot)=\Pi_i(\cdot)$ (i=1,2) in the sequel. This should cause no confusion. Let $Br_i(f)$ be the set of strategy profiles for individuals of type i (i=1,2) that are best responses to f, i.e.,

$$Br_i(f)=\text{argmax}_{g_i\in F_i}\Pi_i(f;g_i).$$

We may write $Br(f)=Br_1(f)\times Br_2(f)$. Let a function $[\cdot]:S_1\cup S_2\cup S\to F_1\cup F_2\cup F$ satisfy $[s](s)=1$ for all $s\in S_1\cup S_2\cup S$. Notice that if s and s_i' are a pure strategy pair and a pure strategy respectively, then $\Pi([s])=\pi(s)$ and $\Pi([s];[s_i'])=\pi(s\backslash s_i')$. We use $\Pi(\cdot)$ and $\pi(\cdot)$ interchangeably.

3. SOCIAL STABILITY AND CYCLICALLY STABLE SET

This section defines and discusses the concept of cyclical stability. To capture the idea of social stability, we consider the following two points: (1) there are no strategic considerations between any two matchings (there may be strategic interaction within each matching situation); and (2) unlike a deviation made by a single player, a change in behavior pattern is likely to be continuous. The former reflects the fact that individuals are anonymous and are matched randomly. The latter expresses the idea that within a small time interval, only a small portion of individuals change their strategies. In order to express these points, the notion of accessibility is given, which is an unperturbed version of accessibility as defined in Gilboa and Matsui(1989).

DEFINITION: A strategy profile g is <u>accessible</u> from another strategy profile f if there exist a continuous $p:[0,1]\to F$, $h:[0,1)\to F$ which is continuous from the right, and $\alpha:[0,1)\to[0,\infty)$ which is continuous from the right such that $p(0)=f$, $p(1)=g$,

$$(d^{+}/dt)p(t)=\alpha(t)\{h(t)-p(t)\} \quad \text{for } t\in[0,1)$$

and

$$h(t)\in Br(p(t)) \quad \text{for } t\in[0,1).$$

The definition says that, in case of $\alpha>0$, a behavior pattern moves in the direction of a best response to the current behavior pattern, and it may stay at the same place only when the behavior pattern is a best response to itself. By including the case of $\alpha=0$, we assure that a strategy profile is always accessible from itself. We call the function p an accessible path from f to g. It is easy to verify that accessibility satisfies transitivity.

The interpretation of this definition is that only small and equal potions of individuals in each type realize the current behavior pattern and change their behavior pattern to another which is a best response to it. Using this notion of accessibility, we are now in a position to present the definition of cyclical stability.

DEFINITION: A nonempty subset F^* of F is an (unperturbed) cyclically stable set (CSS) if

no $g \notin F^*$ is accessible from any $f \in F^*$, and

every g in F^* is accessible from all f in F^*.

A cyclically stable set (CSS) is stable in the sense that once the actual behavior pattern falls into it, another strategy profile may be realized if and only if it is within the CSS. The interpretation of this concept is as follows: for a long time, individuals have sought better strategies. After they experience enough, a behavior pattern falls into a CSS, may move within it, and never leaves it. The term "cyclically stable" stems from the intuitive notion of cycles within a CSS. However, the paths may, of course, be much more complicated.

4. RESULTS

This section presents two main theorems the proof of which will be given in the following section. The first serves the optimality result in the games with cheap talk, and the second theorem shows the existence of a cyclically stable set in the games both with and without cheap talk. The following is the first theorem stating that in the games with cheap talk, any non-optimal outcome is not stable.

THEOREM 1: Suppose that A-2 holds. If f is in a cyclically stable set, then $\Pi(f)=\pi^*$ holds.

Observe that the statement of the theorem does not hold in a game without cheap talk. Consider the game of Figure 1. If all the people in the society take R, then no individual takes L to get zero instead of one. Therefore, ([R],[R]) is a socially stable strategy, a fortiori, it forms a cyclically stable set as a singleton. On the other hand, in the game with cheap talk, the cooperation toward the Pareto optimal outcome is possible because even if all the other take R, one can take the strategy which expects a "signal" and takes L if one gets the signal and remains R if one does not, which is followed by the opponent's change of the strategy to the one which actually sends a signal for cooperation.

The second theorem states that there exists at least one cyclically stable set which consists of Pareto optimal outcomes in both games with and without cheap talk.

THEOREM 2: Suppose that either A-1 or A-2 holds. Then there exists at least one cyclically stable set the payoff of each element of which is π^*.

5. PROOFS

The proofs of the main theorems are given in this section. Before it, we present some lemmata. First, we present the following lemma the proof of which is a direct calculation.

LEMMA 1: Suppose that either A-1 or A-2 holds. If $g \in F$ is accessible from $f \in F$, then $\Pi(g) \geq \Pi(f)$.

Proof: Suppose that $g \in F$ is accessible from $f \in F$. Then there exist functions $p:[0,1]\to F$ continuous, $h:[0,1)\to F$ continuous from the right, and $\alpha:[0,1)\to[0,\infty)$ continuous from the right such that $P(0)=f$, $p(1)=g$,

$$(d^+/dt)p(t)= \alpha(t)\{h(t)-p(t)\} \quad \text{for } t\in[0,1), \text{ and}$$

$$h(t)\in Br(p(t)) \quad \text{for } t\in[0,1).$$

We now calculate $(d^+/dt)\Pi(p(t))$ for $t\in[0,1)$.

$$(d^+/dt)\Pi(p(t))= [\partial\Pi(p(t))/\partial z]_{z\in S}\cdot(d^+/dt)p(t)$$

$$= [\partial\Pi(p(t))/\partial z]_{z\in S}\cdot[\alpha(t)\{h(t)-p(t)\}]$$

$$= \alpha(t)\Sigma_{r\in S_1}\Sigma_{s\in S_2}p_2(t)(s)\{h_1(t)(r)-p_1(t)(r)\}\pi(r,s)$$

$$+ \alpha(t)\Sigma_{r\in S_2}\Sigma_{s\in S_1}p_1(t)(s)\{h_2(t)(r)-p_2(t)(r)\}\pi(s,r)$$

$$= \alpha(t)[\Pi_1(p(t);h_1(t))-\Pi_1(p(t))]$$

$$+ \alpha(t)[\Pi_2(p(t);h_2(t))-\Pi_2(p(t))]$$

The last expression is nonnegative since $\alpha(t)\geq 0$ and $h(t)$ is a best response to $p(t)$. Therefore, we have

$$\Pi(g)-\Pi(f) = \int_0^1 (d^+/dt)\Pi(p(t))dt \geq 0. \qquad \text{Q.E.D.}$$

The next two lemmata show that if a strategy profile f is in a CSS, then a pure strategy profile $[\hat{s}]$ is accessible from f whenever $\pi(\hat{s})$ is the maximum payoff among those which are best responses to f.

LEMMA 2: _Suppose that either_ A-1 _or_ A-2 _holds. Given_ f, _suppose that_ f _is in a cyclically stable set. Let_

$$\hat{s} \in \operatorname{argmax}_{s\in S}\{\pi(s): [s]\in Br(f)\}.$$

Then there exists $\bar{\varepsilon}>0$ _such that_ $p:[0,1]\to F$ _defined by_

$$p(t) = t\bar{\varepsilon}[\hat{s}] + (1-t\bar{\varepsilon})f \quad \textit{for } t\in[0,1]$$

is an accessible path from f to $\bar{\varepsilon}[\hat{s}]+(1-\bar{\varepsilon})f$.

Proof: First, given $\varepsilon\in(0,1)$, define $p_\varepsilon:[0,1]\to F$ by

$$p_\varepsilon(t)=t\varepsilon[\hat{s}]+(1-t\varepsilon)f.$$

Then $p_\varepsilon(0)=f$, $p_\varepsilon(1)=\varepsilon[\hat{s}]+(1-\varepsilon)f$, and

$$(d^+/dt)p_\varepsilon(t)= [\varepsilon/(1-t\varepsilon)]\{[\hat{s}]-p(t)\} \quad \text{for } t\in[0,1)$$

hold. Therefore, what we have to prove is that there exists $\bar{\varepsilon}>0$ such that for all $\varepsilon\in[0,\bar{\varepsilon})$, $[\hat{s}]\in Br(\varepsilon[\hat{s}]+(1-\varepsilon)f)$ holds. Consider $\varepsilon[\hat{s}]+(1-\varepsilon)f$, and compare

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[\hat{s}_i]) = (1-\varepsilon)\Pi(f;[\hat{s}_i]) + \varepsilon\pi(\hat{s})$$

with

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[s_i']) = (1-\varepsilon)\Pi(f;[s_i']) + \varepsilon\pi(\hat{s}\backslash s_i'). \quad (i=1,2)$$

We have $\Pi(f;[\hat{s}_i])\geq\Pi(f;[s_i'])$ for all $s_i'\in S_i$ and $\pi(\hat{s})\geq\pi(\hat{s}\backslash s_i')$ for all $s_i'\in S_i$ whenever $\Pi(f;[\hat{s}_i])=\Pi(f;[s_i'])$ holds. Therefore, for sufficiently small $\varepsilon>0$,

$$\Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[\hat{s}_i]) \geq \Pi(\varepsilon[\hat{s}]+(1-\varepsilon)f;[s_i'])$$

holds for all $s_i'\in S_i$ and any $i=1,2$. Q.E.D.

LEMMA 3: Suppose that either A-1 or A-2 holds. Suppose further that f is in a cyclically stable set. Given f, let

$$\hat{s} \in \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(f)\}.$$

Then $p^*:[0,1]\to F$ defined by

$$p^*(t)=t[\hat{s}]+(1-t)f \quad \text{for } t\in[0,1]$$

is an accessible path from f to $[\hat{s}]$.

Proof: Let f, $\hat{s}$, and p^* be as in the statement of the lemma. Note that $p^*(0)=f$, $p^*(1)=[\hat{s}]$, and that

$$(d^+/dt)p^*(t) = 1/(1-t)\{[\hat{s}]-p^*(t)\} \quad \text{for } t\in[0,1).$$

Therefore, what we have to show is that for any $t\in[0,1)$, $[\hat{s}]\in Br(p^*(t))$ holds. Denote $g^{\mu}=\mu[\hat{s}]+(1-\mu)f$. Then it is equivalent to showing that $[\hat{s}]\in Br(g^{\mu})$ for all $\mu\in[0,1)$.

Suppose the contrary, i.e., that there exists $\mu'\in[0,1)$ such that $[\hat{s}]\notin Br(g^{\mu'})$ holds. Let $\tilde{\mu}=\inf\{\mu': [\hat{s}]\notin Br(g^{\mu'})\}$. Observe that $g^{\tilde{\mu}}$ is accessible from f. If $[\hat{s}]\in \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$ holds, then by lemma 2 there exists $\bar{\varepsilon}>0$ such that $[\hat{s}]\in Br(g^{\tilde{\mu}+\varepsilon})$ holds for all $\varepsilon\in[0,\bar{\varepsilon})$, which is a contradiction. Thus, $[\hat{s}]\notin \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$ must hold. Then there exists $s'\neq\hat{s}$ with $s'\in \text{argmax}_{s\in S}\{\pi(s): [s]\in Br(g^{\tilde{\mu}})\}$. From lemma 2, there exists $\bar{\varepsilon}>0$ such that $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$ is accessible from $g^{\tilde{\mu}}$ _a fortiori_ from f for all $\varepsilon\in[0,\bar{\varepsilon})$. On the other hand, it must be the case that either

$$\Pi(g^{\tilde{\mu}};[s_i']) > \Pi(g^{\tilde{\mu}};[\hat{s}_i]), \text{ or} \tag{1}$$

$$\pi(s')>\pi(\hat{s}) \text{ with } \Pi(g^{\tilde{\mu}};[s_i']) = \Pi(g^{\tilde{\mu}};[\hat{s}_i]) \tag{2}$$

holds and note that $\Pi(g^{\tilde{\mu}};[s_i']) \geq \Pi(g^{\tilde{\mu}};[\hat{s}_i])$ for $i=1,2$. Then there exists $\varepsilon\in(0,\bar{\varepsilon})$ such that

$$\begin{aligned}\Pi(\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}) &= (1-\varepsilon)^2\Pi(g^{\tilde{\mu}}) + \varepsilon(1-\varepsilon)\Pi(g^{\tilde{\mu}};[s_1']) \\ &\quad + \varepsilon(1-\varepsilon)\Pi(g^{\tilde{\mu}};[s_2']) + \varepsilon^2\pi(s') \\ &> \Pi(f) \end{aligned} \tag{3}$$

holds by virtue of (1) and (2). From lemma 1, (3) implies that f is not accessible from $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$. Thus, $\varepsilon[s']+(1-\varepsilon)g^{\tilde{\mu}}$ is accessible from f but not vice versa, which is a contradiction to the assumption that f is in a cyclically stable set. Q.E.D.

We are now in a position to present the proof of Theorem 1, which serves the optimality result.

The Proof of Theorem 1: Suppose first that f is in a cyclically stable set. From lemma 3, there exists $\tilde{s}\in S$ such that $[\tilde{s}]$ is accessible from f. We construct another strategy profile $[\hat{s}]$ (which may be identical to $[\tilde{s}]$) in the following manner:

$$\hat{s}_i(h) = \begin{cases} \tilde{s}_i(e) & \text{if } h=e, \\ \tilde{s}_i(h) & \text{if } h=c^1(\tilde{s}), \\ c^*_i & \text{otherwise,} \quad \text{for } i=1,2. \end{cases}$$

Observe that $\Pi([\tilde{s}];[\hat{s}_i])=\pi(\hat{s})=\pi(\tilde{s})$.

We claim that $[\hat{s}]\in\text{argmax}_{s\in S}\{\pi(s): [s]\in Br([\tilde{s}])\}$. Suppose the contrary. Then there exists $s'\neq\hat{s}$ such that $[s']\in\text{argmax}_{s\in S}\{\pi(s): [s]\in Br([\tilde{s}])\}$. From lemma 3, $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$ is accessible from $[\tilde{s}]$ for all $\varepsilon\in[0,1]$. On the other hand, it must be the case that either

$$\Pi([\tilde{s}];[s'_i]) > \Pi([\tilde{s}];[\hat{s}_i]) = \pi(\tilde{s}) \quad \text{for some } i=1,2, \text{ or} \tag{4}$$

$$\pi(s')>\pi(\hat{s}) \text{ with } \Pi([\tilde{s}];[s'_i]) = \Pi([\tilde{s}];[\hat{s}_i]) \quad \text{for } i=1,2 \tag{5}$$

holds and note that $\Pi(\tilde{g}^{\mu};[s'_i]) \geq \Pi(\tilde{g}^{\mu};[\hat{s}_i])$ for i=1,2. Then there exists $\varepsilon\in(0,\bar{\varepsilon})$ such that

$$\begin{aligned}\Pi(\varepsilon[s']+(1-\varepsilon)[\tilde{s}]) = {} & (1-\varepsilon)^2\pi([\tilde{s}]) + \varepsilon(1-\varepsilon)\Pi([\tilde{s}];[s'_1]) \\ & + \varepsilon(1-\varepsilon)\Pi([\tilde{s}];[s'_2]) + \varepsilon^2\pi(s') \\ > {} & \pi(\tilde{s}) \end{aligned} \tag{6}$$

holds by virtue of (4) and (5). From lemma 1, (6) implies that $[\tilde{s}]$ is not accessible from $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$. Thus, $\varepsilon[s']+(1-\varepsilon)[\tilde{s}]$ is accessible from $[\tilde{s}]$ but not vice versa, which is a contradiction to the assumption that f is in a cyclically stable set. Thus, $[\hat{s}]\in\text{argmax}_{s\in S}\{\pi(s): [s]\in Br([\tilde{s}])\}$ must hold. Hence by lemma 3, $[\hat{s}]$ is accessible from $[\tilde{s}]$.

Next, we construct a strategy profile $[s^*]$ satisfying

$s_i^*(e) \neq \hat{s}_i(e)$, and

$s_i^*(h) = c_i^*$ if $h \neq e$, for $i=1,2$.

Suppose now that $\pi^* > \Pi(f)$ holds. Then $\pi(\hat{s}\backslash s_i^*) = \pi^* > \pi(\hat{s}) = \Pi(f)$ holds for $i=1,2$. If we take any s^{**} with $[s^{**}] \in \text{argmax}_{s \in S}\{\pi(s): [s] \in Br([\hat{s}])\}$, we have $\pi(\hat{s}\backslash s_i^{**}) > \pi(\hat{s})$ for $i=1,2$. Then for sufficiently small $\varepsilon > 0$, we have

$$\Pi(\varepsilon[s^{**}] + (1-\varepsilon)[\hat{s}]) = (1-\varepsilon)^2\pi(\hat{s}) + \varepsilon(1-\varepsilon)\pi(\hat{s}\backslash s_1^{**})$$
$$+ \varepsilon(1-\varepsilon)\pi(\hat{s}\backslash s_2^{**}) + \varepsilon^2\pi(s^{**}) > \pi(\hat{s}) = \Pi(f).$$

By lemma 1, f is not in a cyclically stable set, which is a contradiction.

Q.E.D.

Note that the logic used in the proof cannot be applied to a game without cheap talk, and one may easily verify that in the example of Figure 1, (1,1) as well as (2,2) is a payoff pair in a cyclically stable set.

Next, we prove the existence of cyclically stable set. To this aim, we define the following. Given $f_i \in F_i$, let $S_i(f_i) \subset S_i$ $(i=1,2)$ be the support of f_i, i.e.,

$$S_i(f_i) = \{s_i \in S_i : f_i(s_i) > 0\}.$$

We write $S(f) = S_1(f_1) \times S_2(f_2)$.

The Proof of Theorem 2: Suppose the contrary, i.e., that there exists no $f \in F$ in any cyclically stable set which satisfies $\Pi(f) = \pi^*$. Given f with $\Pi(f) = \pi^*$, we first claim that g is accessible from f whenever $S(g) \subset S(f)$. We show that $p:[0,1] \to F$ with

$$p(t) = (1-t)f + tg \quad t \in [0,1]$$

is an accessible path from f to g. Since $p(0)=f$, $p(1)=g$, and

$$(d^{+}/dt)p(t)=[1/(1-t)]\{g-p(t)\},$$

what we have to show is that g is in $Br(p(t))$ for all $t\in[0,1)$. Observe that $\Pi(f)=\pi^*$ implies that $\pi(s)=\pi^*$ for all $s\in S(g)\subset S(f)$ since π^* is a maximum. Then we have

$$\Pi(p(t);g_i)=\Pi((1-t)f+tg;g_i)$$
$$=\Sigma_{r_i\in S_i}\Sigma_{s_j\in S_j} g_i(r_i)\{(1-t)f_j(s_j)+tg_j(s_j)\}\pi(r_i,s_j)=\pi^*,\ i=1,2,$$

which is always a maximum.

Since f is not in a cyclically stable set, there exists $g^1\in F$ which is accessible from f but not vice versa. Suppose not. Then we can claim that $F^*=\{g\in F|g$ is accessible from $f\}$ is a cyclically stable set. Indeed, take any f_1 and f_2 in F^*. Then f is accessible from f_1, and f_2 is accessible from f. By transitivity of accessibility, f_2 is accessible from f_1. On the other hand, take any $g\in F^*$ and $g'\notin F^*$. If g' is accessible from g, then again by transitivity g' is accessible from f, a contradiction to the definition of F^*.

Let g^1 be a strategy profile which is accessible from f but not vice versa. We know that $S(g^1)\not\supset S(f)$, in particular, $S(g^1)\neq S(f)$. From lemma 1, $\Pi(g^1)=\pi^*$. Therefore, g^1 is not in a cyclically stable set, either. Then there exists $g^2\in F$ which is accessible from g^1 but not vice versa. Thus, $S(g^2)\neq S(g^1)$ holds, and by transitivity $S(g^2)\neq S(f)$ also holds (otherwise, f must be accessible from g^1). Continuing the process. There must be an infinite sequence $\{g^m\}_{m=1}^{\infty}$ such that $S(g^m)\neq S(g^{m'})$ for all $m\neq m'$. This is impossible since the number of elements $S(\cdot)$'s is finite. Thus, there exists $f\in F$ which is in a CSS. Hence, there exists a cyclically stable set the payoff of the element in which is π^*. Q.E.D.

FOOTNOTES

1. Kaneko (1987) also proposes a solution concept called conventionally stable sets to cope with the question of social stability.
2. One may ask how they see the relative population of the second strategy to the first one since the actual outcomes are the same for both strategies unless one announces L in the first stage. One possible scenario is that negligible portion of individuals announce L in the first stage, which reveals the relative size of the population who take the second strategy and that this information gradually spreads in the population.
3. In that case, some studies have shown that the optimal outcome is necessarily attained under some qualifications; among those studies are Aumann and Sorin (1989) and Matsui (1989).

REFERENCES

Aumann, R. and S. Sorin (1989), "Cooperation and Bounded Recall," Games and Economic Behavior, vol.1, pp.5-39.

Bernheim, B.D. (1984), "Rationalizable Strategic Behavior," Econometrica, vol.52, pp.1007-28.

Gilboa, I. and A. Matsui (1989), "Social Stability and Equilibrium," mimeo.

Kaneko, M. (1987), "The Conventionally Stable Sets in Noncooperative Games with Limited Observations I: Definitions and Introductory Arguments," Mathematical Social Sciences, vol.13, pp.93-128.

Kreps, D. and R. Wilson (1982), "Sequential Equilibria," Econometrica,

Matsui, A. (1989), "Information Leakage Forces Cooperation," Games and Economic Behavior, vol.1, pp.94-115.

Nash, J. (1951), "Non-cooperative Games," Annals of Mathematics, vol.54, pp.286-95.

Pearce, D.G. (1984), "Rationalizable Strategic Behavior and the Problem of Perfection," Econometrica, vol.52, pp.1029-50.

	L		R	
L	2	2	0	0
R	0	0	1	1

Figure 1