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NOISY SEARCH AND THE DIAMOND PARADOX

by

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ABSTRACT

It is a paradoxical feature of Diamond's (1971) search model that the market equilibrium does not approach the Betrand outcome in the limit as the search cost becomes arbitrarily small. We show that the Diamond equilibrium is not the limit of the equilibrium with noisy search as the amount of noise goes to zero. Specifically, we demonstrate that if the sequential search technology is replaced by a noisy search technology, the market equilibrium converges to the competitive one when the search cost goes to zero even for an arbitrarily small amount of noise.

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Introduction

The Bertrand model of price competition for a homogeneous product predicts the 'competitive' outcome in which firms' profits are zero. This is subject to the caveat that consumers are perfectly informed about each firms' price. A well-known result due to Diamond (1971) states that if consumers search sequentially and incur a positive cost, however small, to receive a price quotation, the monopoly outcome obtains no matter how large the number of firms is. It is a paradoxical feature of this result that the market equilibrium does not approach the Bertrand outcome in the limit as the search cost becomes arbitrarily small.

In response to the preceding there has arisen a large literature which demonstrates the existence of market equilibria characterized by price disperesion on the supply side and costly search on the demand side. While most models of this genre require heterogeneity of buyers, sellers or both, Burdett and Judd (1983) (B-J) have demonstrated that price dispersion persists even when all agents are identical if some 'noise' is introduced into the sequential search technology.[1] Specifically, price dispersion is the only equilibrium if there is a positive probability less than one that the payment of a search cost elicits more than one price. For example, the search cost may represent

[1]B-J present two different search models, namely noisy search and non-sequential search. The reference is to the former.

the buyer's time and expense involved in driving to the shopping mall. Once she has incurred this cost and arrived at the mall, however, the buyer may observe the same item on display at different shops. Alternatively, the search cost may be the cost of buying a newspaper which reports the prices of a random number of sellers. Moreover, B-J show that the perfectly competitive outcome obtains if this probability is one, i.e., the competitive outcome is sustainable even if buyers are imperfectly informed as long as each observes at least two prices.

The primary purpose of this paper is to show that the Diamond equilibrium is not the limit of the market equilibrium with noisy search as the amount of noise goes to zero. Specifically, we show that whenever there is a positive probability, however small, that a randomly selected price sample contains more than one price, the market equilibrium is arbitrarily near the Bertrand outcome as search costs become sufficiently small. We achieve this by providing a dynamic model of noisy search which explicitly takes into account that search is time consuming and firms may want to change prices during the course of consumer search. In each period a consumer may pay for one price sample, containing a random number of prices, and firms post new prices which are binding for that period only. We analyze both a finite horizon search problem, and the infinite horizon problem and show the following: Let $1-q$, $0 < q < 1$, be the probability that more than one price is elicited upon payment of the search cost. Then in the case of the finite horizon, as the number of periods becomes arbitrarily large, there exists for every q, arbitrarily close to 1, a region of sufficiently small search costs for which the average market price is arbitrarily near the competitive (Bertrand) price. In the case of the finite horizon, the equilibrium (non degenerate) price distribution is **not** stationary over time, but converges to a unique stationary distribution with the preceding property when the remaining number of periods is unboundedly large. The latter distribution represents a stationary equilibrium for the infinite horizon search problem.

It is interesting to compare our analysis to a closely related recent paper by Douglas Gale (1988). While in the present paper the distinction between the Bertrand and Diamond models relates to buyers' information costs, Gale interprets the distinction between the Diamond and Bertrand models as one of timing. Bertrand competition is characterized as a game of ex-ante pricing in which sellers move first and buyers, who know all the prices before choosing a firm are at an advantage. The Diamond model is interpreted as an ex-post pricing game; buyers choose a firm before a price is quoted. Gale proceeds to define a game in which sellers are uncertain whether they are playing the ex-ante or ex-post pricing game. Buyers know which game is being played, have no search costs but are impatient to buy. In a dynamic version of the model it is shown that when buyers are sufficiently patient, the equilibrium price converges to the competitive one.

Statement of The Model and the Main Result

Time is discrete and indexed by t. We analyze both the finite horizon model in which there is a terminal period $T \geq 1$ following which no trade occurs and the infinite horizon model. There is a continuum of sellers. Sellers compete in prices and each is able to supply an unlimited quantity of a homogenous good at constant marginal cost, assumed without loss of generality to be zero.

In each period, a new cohort of buyers of measure $\mu > 0$ per firm enters the market. Each buyer has a demand for one unit for which she is willing to pay $p^* > 0$ at the most (i.e., p^* is the monopoly price). Buyers know only the distribution of prices but not the prices charged by particular sellers. In each period, a buyer can obtain a single random price sample from the price distribution by incurring a search cost $c > 0$. Following B-J (1983) we assume that search is noisy such that the number of prices contained in a sample is a discrete random variable: With probability q, $0 < q < 1$, a sample contains exactly one price while with probability 1-q a sample contains two prices. Buyers are assumed to minimize the expected cost of purchase, including the sellers' price and search costs. It is assumed that at any date, buyers in the market correctly anticipate the future sequence of price distributions. At any date, define a reservation price $\tilde{X}_t$ with the property: buy if and only if the lowest price observed at t does not exceed $\tilde{X}_t$. If only prices exceeding $\tilde{X}_t$ are observed at $t < T$, the buyer remains in the market for an additional period.

In each period, sellers simultaneously post prices which are binding only for that period. Since sellers are free to post new prices at each period, any past prices a buyer may have observed are obsolete. This implies that one can buy only from a seller whose price has been sampled in that period.

Let $\mu^t \geq \mu$ be the measure of buyers per firm in the market at the beginning of period t who have not yet made a purchase. μ^t may exceed μ if there are buyers who have previously entered the market but have not yet purchased a unit. We denote by $F^t(\cdot)$ the distribution of prices at t, possibly degenerate. For simplicity, discounting is ignored.

Let $\Pi^t(p)$ denote the expected profit of a firm whose price is p when the price distribution is $F^t(p)$. Obviously, $\Pi^t(p) = 0$ if $p > \tilde{X}^t$. For $p \leq \tilde{X}^t$, a sale is made with probability 1 if its customer has observed only 1 price. If its customer observes 2 prices, a sale is made with probability $1 - F^t(p)$. Thus:

$$(1) \qquad \Pi^t(p) = p\mu_t(q + 2(1-q)(1 - F^t(p))).$$

Let $\bar{p}^t$ and $\underline{p}^t$ respectively denote the highest (sup) and lowest (inf) price in the market at period t and let $A^t = \int_{\underline{p}^t}^{\bar{p}^t} p \, dF^t(p)$ be the average price in the market at period t.

In equilibrium it is required that sellers choose prices to maximize profits, given the pricing behavior of all other sellers and the search behavior of consumers, and consumers search to minimize purchase costs given the current price distribution and the sequence of expected future price distributions. To close the model it is required that in every period the individually optimal behavior of sellers reproduces the price distribution in response to which it arose. Formally:

Definition 1: A finite horizon noisy search equilibrium is defined as the tuple of sequences:

$$\left\{F^t(\cdot),\quad \tilde{X}^t,\quad \mu^t,\quad \tilde{\Pi}^t\right\}_{t=1}^{T}$$

such that:

(i) At every period t, given μ^t and $\tilde{X}^t$, a firm's profit is $\tilde{\Pi}^t$ if its price is contained in the support of $F^t(\cdot)$ and does not exceed $\tilde{\Pi}^t$ if its price lies outside this support.

(ii) At every period t, $\tilde{X}^t$ represents the optimal search strategy of consumers in the market, given the sequence of future price distributions, $F^{t+1}(\cdot),\ldots,F^T(\cdot)$ □

Definition 2: A finite horizon noisy search equilibrium is said to be stationary if for each $t_1, t_2 \leq T$, $F^{t_1}(\cdot) = F^{t_2}(\cdot)$. If this is not the case, the equilibrium is said to be nonstationary. □

In the case of an infinite horizon, the definitions of equilibrium and stationarity are analogous to definitions 1 and 2, applying to the infinite sequences $F^t(\cdot)$, etc.

We now state our main result:

Theorem 1:

(a) **Finite horizon noisy search**:

For any q, however close to 1, there exists $c^*(q) > 0$ such that for $c < c^*(q)$, the unique equilibrium average market price at any period $t < T$ is monotonically decreasing in:

(i) the magnitude of the search cost

and

(ii) the number of periods which remain until the terminal period T.

In the limit as c goes to zero and the remaining number of periods goes to infinity, the average market price approaches the competitive price.

(b) **Infinite horizon noisy search**:

Consider the finite horizon noisy search equilibrium discussed in (a) for some $c < c^*(q)$. As the remaining number of periods goes to infinity, the limit of the average price is the equilibrium average price of the stationary equilibrium for the infinite horizon model. Thus as $c \to 0$ the competitive outcome is the limit of the average price of the infinite horizon equilibrium. □

The preceding theorem is proved by means of a number of claims which are stated and proved in the following section.

Analysis and Discussion

Consider any period at which prices are non degenerately distributed. Using standard arguments, (e.g. B-J), it is easy to establish the following:

Claim 1: (i) At any period t such that $F^t(\cdot)$ is non degenerate, $F^t(\cdot)$ is continuous (i.e. contains no mass points) with connected support. (ii) At any period t, $\bar{p}^t = \tilde{X}^t$. That is, if $F^t(\cdot)$ is not degenerate, $\tilde{X}^t$ is the supremum of $F^t(\cdot)$'s support. □

Claim 2: At any period t, $F^t(\cdot)$ is non degenerate.

Proof: Suppose not. Then each firm i charges $\tilde{X}^t$. A prospective buyer who has observed no other price buys from firm i with probability 1. If the buyer has observed the price of another seller (which by assumption is also $\tilde{X}^t$), she chooses one of the two sellers at random. Thus firm i's expected profit is

$$\mu^t \tilde{X}^t [q + 2(1-q)] \cdot \frac{1}{2}] = \mu^t \tilde{X}^t.$$

By deviating to $\tilde{X}^t - \epsilon$, firm i makes a sale with probability 1. Thus its expected profit is:

$\mu_t(\tilde{X}^t - \epsilon)[q + 2(1-q)] = \mu_t(\tilde{X}^t - \epsilon)(2-q)$ which exceeds $\mu_t \tilde{X}^t$ for sufficiently small ϵ, □

To summarize, claims 1 and 2 establish that at any period prices are continuously distributed and the highest price charged is the reservation price, $\tilde{X}^t$.

In what follows we find it convenient to index periods by their proximity to the terminal period T. Thus, $T - \tau$ is the τth from last period.

We first construct the equilibrium price distribution at $T-\tau$, $\tau = 0,1,\ldots,T$. By claim 2, $\bar{p}^{T-\tau} = \tilde{X}^{T-\tau}$. By the equal profit condition, $\Pi(\tilde{X}^{T-\tau}) = \Pi(p)$ for any p in the support of $F^{T-\tau}(\cdot)$. Substituting from (1), this yields:

$$(2) \qquad \tilde{X}^{T-\tau} q = p\{q + 2(1-q)(1 - F^{T-\tau}(p)).$$

Substituting $\Pi(\underline{p}^{T-\tau}) = \mu^{T-\tau} \underline{p}^{T-\tau}(q+2(1-q))$ into (2) and simplifying yields:

$$(3) \qquad \underline{p}^{T-\tau} = \tilde{X}^{T-\tau} q(q + 2(1-q))^{-1}.$$

Solving (2) for $F^{T-\tau}(p)$ yields:

$$(4) \quad F^{T-\tau}(p) = \begin{cases} 1 & \text{if} \quad p > \tilde{X}^{T-\tau} \\ 1 - \dfrac{q(\tilde{X}^{T-\tau} - p)}{2p(1-q)} & \text{if} \quad \underline{p}^{T-\tau} \le p \le \tilde{X}^{T-\tau}. \\ 0 & \text{otherwise} \end{cases}$$

Let $E^{T-\tau}$ be the expected price paid by a consumer who buys at $T-\tau$. With probability q only 1 price is observed, in which case the expected price is simply $A^{T-\tau}$, the mean of $F^{T-\tau}(p)$. With probability $(1-q)$, two prices are observed in which case the expected price is the expected minimum of two prices. Noting that $1 - (1 - F^{T-\tau}(p))^2$. is the probability that the minimum of the two randomly drawn prices does not exceed p, we obtain:

$$E^{T-\tau} = q\int_{\underline{p}^{T-\tau}}^{\tilde{X}^{T-\tau}} p\,dF^{T-\tau}(p) + (1-q)\int_{\underline{p}^{T-\tau}}^{\tilde{X}^{T-\tau}} p\,d(1-(1 - F^{T-\tau}(p))^2).$$

Integrating by parts:

$$(5) \quad E^{T-\tau} = \tilde{X}^{T-\tau} - \{q \int_{\underline{p}^{T-\tau}}^{\tilde{X}^{T-\tau}} F^{T-\tau}(p)dp + (1-q) \int_{\underline{p}^{T-\tau}}^{\tilde{X}^{T-\tau}} 1 - (1-F^{T-\tau}(p))^2 dp\}$$

A buyer who has sampled at any period $T-\tau$ will accept the smallest price sampled only if this does not exceed $E^{T-\tau-1}$; the latter is the expected cost of buying in the following period, including the extra search cost and the expected price. Therefore

$$(6) \qquad \tilde{X}^{T-\tau} = \min\{p^*,\ E^{T-\tau-1}\}, \qquad \tau = 0,1,\dots,T.$$

Claim 3:

$$E^{T-\tau} = \tilde{X}^{T-\tau}\varphi(q), \quad A^{T-\tau} = \tilde{X}^{T-\tau}\beta(q)$$

where:

$$\varphi(q) = \frac{-q^2}{2(1-q)} + \frac{q}{2-q}\left(1 + \frac{q^2}{2(1-q)} - \frac{q^2}{4(1-q)^2} + \frac{q^3}{4(1-q)^2}\right) +$$

$$+ \frac{(2-q)q}{4(1-q)} + \ln\left(\frac{2-q}{q}\right)\left[\frac{q^2}{2(1-q)} - \frac{2q^2}{4(1-q)^2} + \frac{2q^3}{4(1-q)^2}\right] < 1$$

and

$$\beta(q) = \frac{q\ \ln\left(\frac{2-q}{q}\right)}{2(1-q)} < 1.$$

Proof: See Appendix 1.

Claim 4: Define $c^* = p^*(1 - \varphi(q))$.

If $c \geq c^*$, then $\tilde{X}^{T-\tau} = p^*$, $\tau = 0,1,\ldots,T$.

If $c < c^*$, then

$$(7) \qquad \tilde{X}^{T-\tau} = (\varphi(q))^{\tau} p^* + c\left[1 + (\varphi(q) + (\varphi(q))^2 + \ldots + (\varphi(q))^{\tau-1}\right]$$

and

$$\tilde{X}^{T-\tau} > \tilde{X}^{T-\tau-1}, \qquad \tau = 0,1,\ldots,T.$$

Proof: $\tilde{X}^T = p^*$ by assumption and $\tilde{X}^{T-1} = \min\{p^*, \varphi(q)\, p^* + c\}$ by claim 3. Thus

$$\tilde{X}^{T-1} < p^* \text{ iff } \varphi(q)p^* + c < p^* \iff c < p^*(1 - \varphi(q)).$$

This proves that $\tilde{X}^{T-1} = p^*$ if $c \geq c^*$. A recursive application of this argument proves the first part of the claim.

If $\tilde{X}^{T-\tau} > \tilde{X}^{T-\tau-1}$, $\tau = 0,1,\ldots,T$, then from the preceding, (7) obtains for T-1. (7) is then shown to obtain for any τ by induction. Using (7), $\tilde{X}^{T-\tau} - \tilde{X}^{T-\tau-1} = (\varphi(q))^{\tau}\left[p^*(1 - \varphi(q)) - c\right]$

which is positive iff $c < c^*$. This completes the proof of the second part of the claim. □

A corollary of claim 4 is that the equilibrium is stationary iff $c \geq c^*$; in this case $\tilde{X}^t = p^*$, $t = 1,2,\ldots,T$, and using (4) the construction of $F^t(\cdot)$ is identical at each period. If $c < c^*$, $\tilde{X}^t \neq \tilde{X}^{t+1}$ and so $F^t(\cdot) \neq F^{t+1}(\cdot)$. Using claim 3 and equation (7) yields:

$$A^{T-\tau} = \beta(q)\left\{(\varphi(q))^{\tau}p^* + c[1 + \varphi(q) + \ldots + (\varphi(q))^{\tau-1}]\right\}. \tag{8}$$

Taking limits (note that the order in which the double limit is taken is immaterial):

$$\lim_{\substack{T-\tau\to\infty \\ c\to 0}} A^{T-\tau} = 0$$

This completes the proof of part (a) of theorem 1.

The following claim applies to the infinite horizon model.

Claim 5: The price distribution:

$$\hat{F} = \begin{cases} 1 & \text{if} \quad p > c[1-\varphi(q)]^{-1} \\ 1 - \dfrac{q(c(1-\varphi(q))^{-1} - p)}{2p(1-q)} & \text{if} \quad \hat{\underline{p}} < p < c(1-\varphi(p))^{-1} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $\hat{\underline{p}} = c[1 - \varphi(q)]^{-1}\dfrac{q}{q + 2(1-q)}$ is a stationary infinite horizon noisy search equilibrium.

Proof: Observe that $\hat{F}(\cdot)$ is of the type defined by (4) with $\hat{\bar{p}} = \frac{c}{1-\varphi(q)}$ being the highest price in the support and $\hat{\underline{p}}$ the lowest price in the support. Thus $\hat{F}(\cdot)$ is an equilibrium if $\hat{\bar{p}}$ is the (constant) reservation price of buyers. Define $\hat{E}$ as the expected price paid by a buyer who draws a single random price sample from $\hat{F}(\cdot)$ and pays the lowest price observed, i.e., $\hat{E}$ is defined analogously to $E^{T-\tau}$ in (5). It is clear that claim 5 is applicable to $\hat{F}$ to the effect that $\hat{E} = \varphi(q)\frac{c}{1-\varphi(q)}$. It follows that the buyers' reservation price is $\frac{c}{1-\varphi(q)}$; the expected purchasing cost associated with one more search is $\hat{E} + c = \frac{\varphi(q)c}{1-\varphi(q)} + c = \frac{c}{1-\varphi(q)}$. Therefore $\hat{F}(\cdot)$ is a stationary equilibrium. □

It is clear that claim 4 applies to $\hat{F}$ to the effect that $\hat{A}$, the mean of $\hat{F}$, is $\beta(q)\frac{c}{1-\varphi(q)}$. But the last expression is $\lim_{T-\tau\to\infty} A^{T-\tau}$, as seen by taking the limit in (8). This proves part (b) of Theorem 1.

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APPENDIX

Proof of Claim 3

I. **Proof that** $\mathbf{E}^{T-\tau} = \tilde{X}^{T-\tau}\varphi(q)$: (For convenience, superscripts and limits of integrals are deleted in what follows when no ambiguity results.)

Opening parenthesis in (5) gives

$$\text{(A.1)} \qquad E = \tilde{X} - q\int F dp - (1-q)\int dp + (1-q)\left[\int dp - 2\int F dp + \int F^2 dp\right]$$

Substituting $\int dp = \tilde{X} - \underline{p}$ and manipulating:

$$\text{(A.2)} \qquad E = P^* + (q-2)\int F dp + (1-q)\int F^2 dp.$$

Substituting from equation (5) and carrying out the integration gives:

$$\int F dp = \tilde{X} - \underline{p} - \gamma\left\{\tilde{X}\ \ln\left(\frac{\tilde{X}}{\underline{p}}\right) - \tilde{X} + \underline{p}\right\}$$

where $\gamma = \frac{q}{2(1-q)}$,

and

$$\int F^2 dp = \int\left[1 - \frac{q(\tilde{X} - p)}{2p(1-q)}\right]^2 dp = \int dp - 2\gamma\int \frac{(\tilde{X}-p)dp}{p} + \gamma^2\int \frac{(\tilde{X} - p)^2}{p^2}dp.$$

Calculating the second integral in the last expression:

$$\gamma^2 \int \frac{(\tilde{X} - p)^2}{p^2}\, dp = \gamma^2 \left\{ \tilde{X}^2 \int p^{-2} dp - 2\tilde{X}^2 \int p^{-1} dp + \int dp \right\} =$$

$$= \gamma^2 \left\{ -(\tilde{X})^2\, p^{-1} - 2\tilde{X} \ln p + p \,\Big|_{\underline{p}}^{\tilde{X}} \right\} = \gamma^2 \left\{ -\tilde{X} + \frac{\tilde{X}^2}{\underline{p}} - 2\tilde{X} \ln(\frac{\tilde{X}}{\underline{p}}) + \tilde{X} - \underline{p} \right\}.$$

Thus:

$$\int F^2 dp = \tilde{X} - \underline{p} - 2\gamma(\tilde{X} \ln(\frac{\tilde{X}}{\underline{p}}) - \tilde{X} + \underline{p}) + \gamma^2 \left\{ -\tilde{X} + \frac{\tilde{X}^2}{\underline{p}} - 2\tilde{X} \ln(\frac{\tilde{X}}{\underline{p}}) + \tilde{X} - \underline{p} \right\}.$$

Substituting back into (A.2) and simplifying gives:

$$E = \tilde{X} + (q-2)(\tilde{X} - \underline{p} - \gamma\tilde{X} \ln (\frac{\tilde{X}}{\underline{p}}) + \gamma\tilde{X} - \gamma\underline{p}) + (1-q)((1 + 2\gamma)\tilde{X} -$$

$$- (\gamma^2 + 2\gamma + 1)\underline{p} - 2(\gamma^2 + \gamma) \ln (\frac{\tilde{X}}{\underline{p}}) + \gamma^2 \frac{\tilde{X}^2}{\underline{p}}).$$

Substituting $\underline{p} = \tilde{X} \frac{q}{2-q}$ and simplifying gives:

$$E = \tilde{X} \Big[-\gamma q + \frac{q}{2-q}(1+\gamma q - \gamma^2 + q\gamma^2) + \frac{2-q}{q} \gamma^2 (1-q)$$

$$+ (q\gamma - 2\gamma^2 + 2q\gamma^2) \ln (\frac{2-q}{q}) \Big].$$

Substituting $\gamma = \frac{q}{2(1-q)}$ gives $\tilde{X}\varphi(q)$ as defined in the claim. It is easily verified that $\varphi(q) < 1$ for $0 < q < 1$.

II. **Proof that $A^{T-\tau} = \tilde{X}^{T-\tau}\beta(q)$.**

$A = \tilde{X} - \int F\, dp$ which from the preceding nalysis gives, after simplification:

$$A = \underset{\sim}{p} + \gamma\left\{\tilde{X} \ln \left(\frac{\tilde{X}}{\underset{\sim}{p}}\right) - \tilde{X} + \underset{\sim}{p}\right\}.$$

Substituting $\underset{\sim}{p} = \tilde{X}\frac{q}{2-q}$ and γ gives:

$$A = \tilde{X}\beta(q)$$

It is easily verified that $\beta(q) < 1$ for $0 < q < 1$. □