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NUMERICAL REPRESENTATIONS OF IMPERFECTLY ORDERED PREFERENCES (A UNIFIED GEOMETRIC EXPOSITION)*

by

Avraham Beja
Tel Aviv University
and
Massachusetts Institute of Technology

and
Itzhak Gilboa
Northwestern University

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* After having written a first draft of this paper we are privileged with access to Chapter 16 of the forthcoming volume II of Theory of Measurement by Krantz, Luce, Suppes and Tversky (henceforth KLST), which is an almost definitive treatment of the issues we deal with here. The present more limited and more focused effort offers a unified framework that we believe to be insightful (especially for the non-expert) and a number of new results dealing with some subtle aspects of the theory. We are grateful to Amos Tversky for his encouragement to continue with this work. We also wish to thank Peter C. Fishburn and David Schmeidler for helpful comments and references.

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Abstract

This paper uses "generalized numerical representations" to extend some of the ideas underlying classical utility theory, applying them to imperfectly ordered preferences in general and semiordered preferences in particular. It offers a unified geometric approach, where the representations help visualize the relationship between suborders, interval orders, semiorders, and weak orders: increasingly stringent conditions on the preference relation give rise to increasingly intuitive association between the stated preferences and the "utility numbers" assigned to alternatives. Technically, the main new results are axiomatizations for fixed threshold representations of the "just noticeable difference" when the set of alternatives is not necessarily countable. For the special case of "lottery spaces", additional axioms that relate to the special properties of the space characterize the semiordered preferences whose (fixed threshold) representation can be made to conform with the familiar "expected utility" structure.

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1.INTRODUCTION.

The representation of pairwise preferences by assigning numerical values to alternatives is central to the scientific study of choice behavior. The classical representation assigns "utility numbers" to all elements in a given set of alternatives so that the number (utility) u(x) assigned to an alternative labeled x is higher than the number u(y) assigned to any alternative y if and only if x is preferred to y. This paper is concerned with the modern extensions of utility theory, dealing with preferences that are not completely ordered and thus do not admit a classical representation.[1]

The significance of partially ordered preferences in almost all of the social sciences can hardly be overstated: they are prominent in such diverse fields as the study of stimulus-response behavior in psychology, collective choice in economics and political science, and multiple-criteria decision-making in management science. But results on the representation of imperfectly ordered preferences have typically been somewhat fragmented (and have sometimes involved only "partial" representations). In an attempt to enhance and unify the theory, we use in this paper a formal structure that we call "generalized numerical representation" of preferences, offering the following contributions.

* A unified framework that puts the individual results in proper perspective.
* A geometric interpretation that makes the results, and especially the relationships between them, more intuitive.
* A separation of various distinct elements which are inherent in the classical utility representation, showing how each of these is associated with different properties of

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1 Besides the forthcoming KLST, surveys with extensive references are given in, e.g., Roberts (1970) or Fishburn (1970b, 1985).

the preferences that are to be represented.

* A strengthening of some of the earlier results, especially for very rich sets of alternatives (sets of high cardinality).
* A characterization of the semiordered preferences over lotteries which can be represented by utility numbers that satisfy the mathematical expectations structure.

The classical representation of preferences by a utility function u(.) comprises a number of intuitive features, which have been treated jointly for so long that their distinct characteristics are often overlooked. Suppose that some alternative x is (strictly) preferred to another alternative y. A classical utility function satisfies all of the following properties.

1. $u(x)>u(y)$.
2. If z is not preferred to y, then $u(z)<u(x)$.
3. If x is not preferred to z, then $u(z)>u(y)$.
4. $u(z)>u(w)$ if and only if there is some alternative t such that t is preferred to w but not to z, or z is preferred to t while w is not.
5. If z is not preferred to w, then $u(z)-u(w)<u(x)-u(y)$.
6. There is a positive number ε such that if z is not preferred to w then $|u(z)-u(x)|+ |u(w)-u(y)|>\varepsilon$.

We shall show below that these features can be attained also for preferences that do not admit a classical utility function. We shall also identify which properties of the preference relation are essential for the attainability of each of these features.

The six conditions stated above do not imply one of the key features of classical utility, namely that $u(z)>u(w)$ *only if* z is (strictly) preferred to w. As will be shown below, in the generalized numerical representations $u(z)>u(w)$ can sometimes reflect an advantage of z over w which is implicit in the stated preferences even though it is not explicitly expressed. In such cases, the failure to give a clear preference statement can be attributed to limited powers of discrimination, e.g. the advantage of z over w may be said to be below the "just noticeable difference (**jnd**)". Our discussions below include both variable-jnd and (the more restrictive) fixed-jnd representations.

An especially extensive research effort has been devoted in the literature to representations of preferences that reflect attitudes towards risk. In this case the objects of choice are "lotteries", specified by alternative probability distributions over a common set of "outcomes". The classical analysis concentrates on the so-called "expected utility hypothesis", i.e. the specification of conditions on the preferences over lotteries (e.g. transitivity, continuity and independence) that guarantee that the utility value assigned to a lottery can be written as the expected value of utility numbers corresponding to the possible outcomes. Here the modern extensions tend to relax the "expected utility" structure, but typically they maintain the basic premise of classical utility, i.e that the utility number assigned to one lottery be higher than the number assigned to another lottery if and only if the first is preferred to the second. In this paper we also show how (and when) "non expected utility" can be extended in a different way that maintains the mathematical expectations structure, but relaxes the classical condition that any two lotteries for which no preference is stated be assigned exactly the same utility value.

The paper is organized as follows. Section 2 gives some preliminaries on framework and notation. Section 3 gives a first overview of our geometric interpretation and unified perspective of numerical representations in the limited context of countable sets. We show how increasingly stringent conditions on the preference relation give rise to numerical representations that exhibit increasingly intuitive properties of monotonicity between the stated preferences and the numerical values assigned to alternatives, as indicated by properties 1-5 above. This is based to a large extent on previously published results, but the unified framework makes these results more intuitive and they are also often stated in slightly stronger versions than before. We also note that property 6 above is in effect non-restrictive in the context of countable sets. Section 4 introduces the notion of "limited discrimination preference order". It is shown how such preference orders can be partially ranked by their "powers of discrimination" (and also how some seemingly intuitive ways to devise such a ranking can easily fail). Section 5 extends and modifies the analysis of section 3 to uncountable sets. Section 6 is devoted to preferences over the (uncountable) set of all lotteries with a common set of outcomes, showing when a limited discrimination preference order can be represented by a (generalized) utility function that maintains the expected value structure. The last section 7 offers a few concluding remarks on "uniqueness", more specifically on the range of permissible transformations of generalized numerical representations. The formal proofs of our results are presented separately in an appendix.

2. BINARY RELATIONS ASSOCIATED WITH PREFERENCES.

The notational framework for this study is as follows. Let A be a non-empty set of **alternatives.** Preferences among alternatives are considered as pairwise comparisons; comprehensive collections of such comparisons are formalized by binary relations. A **binary relation on A** is a subset B of the set $A^2 = A \times A$ of all ordered pairs of alternatives. xBy is synonymous with $(x,y) \in B$, i.e. the said relation applies in a comparison between the alternatives x and y. We also adopt the useful notation whereby a concatenation B_1B_2 of any two binary relations B_1 and B_2 (on a common set A) denotes a binary relation on A by

$$B_1B_2 = \{(x,y) \in A^2 : xB_1z \text{ and } zB_2y \text{ for some } z \in A\}$$

This concatenation can be extended recursively.

Our primitive binary relation, to be denoted by P, will be strict preference, which is asymmetric (i.e., xPy implies [not yPx]). Absence of preference in both directions is denoted by I, viz.

$$I = \{(x,y) \in A^2 : \text{not } xPy \text{ and not } yPx\}$$

The relation 'I' is commonly referred to as **indifference**. Note that here it includes pairs which in common parlance might be considered "incomparable": our choice of strict preference as the primitive relation shuns the distinction between pairs from which choice seems impossible, and pairs from which it is considered a matter of indifference. As defined, indifference is necessarily reflexive (xIx for all $x \in A$) and symmetric (xIy implies yIx), but not necessarily transitive (xIy and yIz need not imply xIz).

The pairwise preference statements are often taken to reflect to one degree or another some underlying well ordered "standings" of all alternatives. This is clearly the case when P is a **weak order**, i.e. PP implies P and II implies I (equivalently $(P \cup I)(P \cup I)$ implies $P \cup I$). But even when P is not itself a weak order, there are some inferences that can be drawn from the stated preferences about the presumed underlying standings of the alternatives. To formalize this, let

xQy if xPz and not yPz, or zPy and not zPx, for some $z \in A$.
i.e. $Q = P(P \cup I) \cup (P \cup I)P = PI \cup IP \cup PP$.

Borrowing a term from consumer demand theory, we interpret Q as "**revealed preference**": xQy may apply even when xPy does not, in which case an advantage of x over y is indirectly revealed by comparisons with some other alternative.[1] Since I is reflexive, P implies Q, i.e. Q augments the **stated preferences** P by pairs of alternatives from I whenever an **implicit preference** is detectable. As defined, Q is irreflexive but not necessarily asymmetric. The symmetric part of Q represents pairs of alternatives for which there is contradicting evidence as to their relative standing: one alternative appears to be better in some contexts (comparisons with some z) while the other alternative seems better in other contexts (in general, the

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[1]The concept of implicitly revealed preferences appears in the literature in various forms and under a variety of different names. Luce (1956) suggested $PI \cup IP$, calling it "the relation induced by P", and Fishburn (1985) termed the asymmetric part of this relation "the sequel of P". KLST call the dual of the complement of Q "the two-sided quasiorder induced by P".

implicit preferences may even contradict the stated preferences, as xPy does not preclude yQx). Such contradictions do not appear in the asymmetric part of Q and in the symmetric complement of Q, defined by

xP^*y if xQy and not yQx

xEy if [not xQy] and [not yQx].

The relation Q need not be transitive even when P is (for example, consider $A=\{x,y,a,b,c\}$ with $P=\{(a,b),(b,c),(a,c)\}$, where xQb and bQy but not xQy). On the other hand, the complement of Q is always transitive, even if P is not (this is apparent when one observes that [not yQx] applies if and only if $yPz \Rightarrow xPz$ and $zPx \Rightarrow zPy$). It follows that P* is always transitive: suppose xP*y and yP*z, then [not zQy] and [not yQx] imply [not zQx], and either $xPw(P \cup I)y$, implying [not zPw] and subsequently xQz, or $x(P \cup I)wPy$, implying wPz and again xQz. Thus P* gives a (possibly partial) ordering of the underlying standings of all alternatives on the basis of uncontradicted evidence, either direct (in P) or implied (in Q).

When xEy the two alternatives are said to be "equivalent", because there is no evidence, not even implied or contradictory, of any advantage of one over the other. Indeed, E is an **equivalence relation**, being reflexive, symmetric and transitive.

Remark: It is possible to extend the inferences about the relative standings of the alternatives to the transitive closure of Q (rather than limit them to the one-intermediate-step comparisons used for Q). This can only generate contradicting inferences, and only for pairs in E (not for pairs in P*).[1] We shall not follow this line, but only note that the subsequent results on the numerical representation of Q also apply to the extended interpretation of revealed preference.

When there are no contradictory inferences about the relative standings of the alternatives, i.e. when Q=P*, the preference relation P is a "semiorder", a concept

[1] I.e., if the transitive closure of Q is denoted by $\underline{Q}$, and its asymmetric part and its symmetric complement are denoted by $\underline{P}^*$ and $\underline{E}$, respectively, then $\underline{P}^*=P^*$ and $\underline{E} \subseteq E$.

that is discussed at some length in the next section. When P is a weak order then it is readily verified that also Q=P and E=I, so that all potential inferences are not only non-contradictory but also explicitly stated.

3. NUMERICAL REPRESENTATIONS OF PREFERENCES OVER COUNTABLE SETS.

The essence of numerical representation for preference relations is the assignment of numbers ("utilities") to the decision-relevant alternatives so that the assigned utilities represent the given preference relation as intuitively as possible on the basis of the familiar ordering of the real numbers. In this section we show how increasingly stringent conditions on the preference relation give rise to increasingly intuitive representations. To put the analysis in clearest perspective, we separate issues that are associated with the level of consistency in preference from the more technical difficulties that can arise if the set of alternatives has very high cardinality. Accordingly, we start by limiting the discussion in this section to preferences over countable sets, i.e., sets which are either finite or denumerably infinite (like the integers). The presentation is based on previously published results, some of which are given here in slightly stronger versions. The proofs for the strengthened versions are indicated in the appendix.

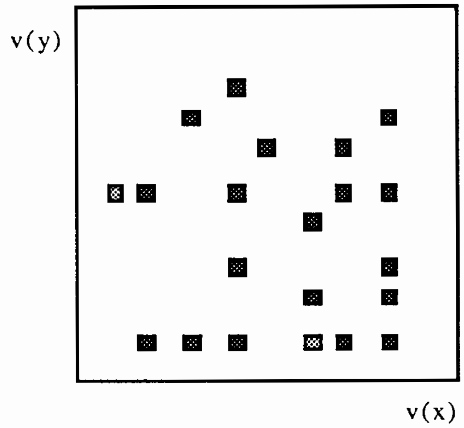
The following basic notion is the cornerstone of our analysis.

Definition G1 A **generalized numerical representation (GNR)** of a binary relation $B \subseteq A^2$ is a pair (v,S), where $v: A \rightarrow \Re$ is a mapping from A to the real line $\Re$ and S is a subset of $\Re^2$, such that $(x,y) \in B$ if and only if $(v(x),v(y)) \in S$.

Example. Figure 1 below gives a binary relation comprising pairwise comparisons based on the encounters in a chess tournament (the "candidates" - for world championship - tournament held in Bled, Yugoslavia, 1954). The left side gives the conventional representation used in the chess world (1 means a win for the row player and 0.5 means a draw), and the right side a GNR where each player is simply assigned his serial number on the list.

	1	2	3	4	5	6	7	8
1. Benko	-	0	0	0	1	0	0	0
2. Fischer	1	-	.5	.5	1	0	.5	0
3. Gligoric	1	.5	-	0	.5	.5	1	0
4. Keres	1	.5	1	-	1	0	.5	1
5. Olafson	0	0	.5	0	-	1	0	0
6. Petrosian	1	1	.5	1	0	-	0	.5
7. Smyslov	1	.5	0	.5	1	1	-	0
8. Tal	1	1	1	0	1	.5	1	-

(a) Conventional statement of pairwise comparisons in a chess tournament



(b) A GNR of the chess tournament data

Figure 1

Observation 3.1 Every binary relation on a countable set has a GNR.

Classical utility theory deals with numerical representations where S takes the special form H, defined by

$$H= \{(\alpha,\beta)\in \Re^2 : \alpha>\beta\}$$

and thus $xPy \Leftrightarrow (v(x),v(y))\in H \Leftrightarrow v(x)>v(y)$. When it is possible, such a representation is very attractive, because it directly associates "higher" (utility) with "better" (alternatives). When this is not possible, it would still be desirable to maintain at least some of the intuitive monotonicity between the underlying preferences and the numbers that represent them. Without it, a numerical representation would be merely a labeling of the alternatives, devoid of any connotation of "utility".

Looking for a reasonable monotonicity condition as a starting point, we note that in any GNR (v,S) the asymmetry of P implies that $(v(x),v(x))\notin S$ for all $x\in A$, hence $xPy \Rightarrow v(x)\neq v(y)$. Therefore, the condition $xPy \Rightarrow v(x)>v(y)$ suggests itself as a minimal monotonicity requirement. This condition means that the relevant part of S, i.e. $S\cap(\text{range } v)^2$, should be included in H, and (since the irrelevant part of S can be chosen at will) this suggests the condition $S \subseteq H$. The preferences that admit representations satisfying this condition constitute a familiar class.

Definition P1 An asymmetric binary relation P is a **suborder** if $x_i P x_{i+1}$ for $i=1,\dots,n$ implies [not $x_{n+1} P x_1$]. Any binary relation whose asymmetric part is a suborder is termed **acyclical**.

The results on the representation of suborders will be stated in two versions. The distinction between them highlights an important difference between our generalized utility representations by pairs (v,S) and the weaker representations by a numerical function v alone.

Theorem 3.2 (Adams, 1965).

Let B be a binary relation on a countable set A. Then there is a functon $v: A \to \Re$ such that $v(x)>v(y)$ whenever xBy if and only if B is a suborder.

Theorem 3.2(a)

A relation B on a countable set has a GNR (v,S) such that $S \subseteq H$ if and only if B is a suborder.

The function v in theorem 3.2 is only a *partial* representation of the binary relation B, because it is not guaranteed that one can fully recover B from the numerical representation. For example, suppose that a suborder P on the set $A=\{a,b,c\}$ is represented by a function v where $v(a)=1$ and $v(b)=v(c)=0$. If this function is only known to be a partial representation as in 3.2 then it is consistent both with $P=\{(a,b),(a,c)\}$ and with $P=\{(a,b)\}$. But if v is known to be part of a GNR as in 3.2(a) then it is consistent only with $P=\{(a,b),(a,c)\}$, not with $P=\{(a,b)\}$. To qualify for a GNR, a function v must satisfy the preliminary condition that $v(x)\neq v(y)$ whenever $(x,y)\notin E$. As we shall see in section 5 below, the distinction is crucial when A is not a countable set.

Note that the acyclicity and asymmetry of a suborder are obviously necessary for $S \subseteq H$. The more substantive contribution of theorems 3.2-3.2(a) is the observation that this is also sufficient. A similar relationship between necessity and sufficiency seems to apply also in the other characterizations that follow.

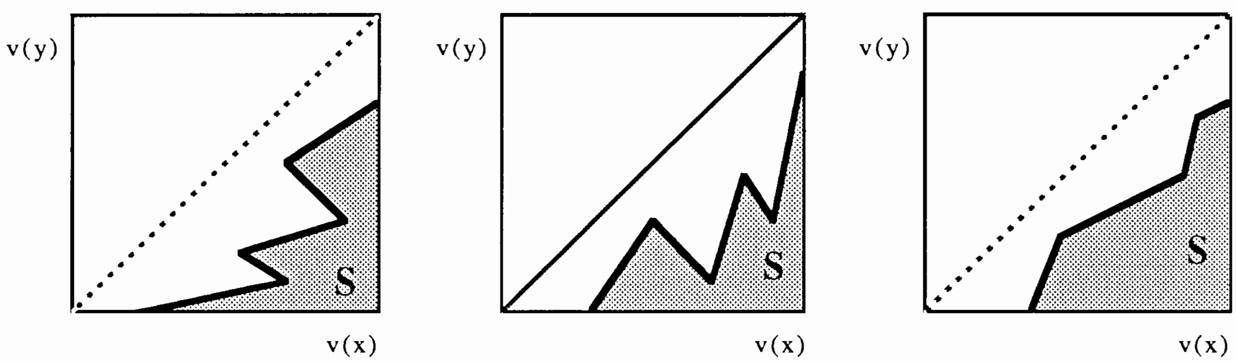
The next definition introduces additional aspects of monotonicity.

Definition G2.

(i) A GNR (v,S) satisfies **upward monotonicity** if $(\alpha,\beta)\in S$ implies $(\alpha',\beta)\in S$ for all $\alpha'>\alpha$.

(ii) A GNR (v,S) satisfies **downward monotonicity** if $(\alpha,\beta)\in S$ implies $(\alpha,\beta')\in S$ for all $\beta'<\beta$.

(iii) A GNR (v,S) satisfies **bilateral monotonicity,** or in short is **monotonic,** if it satisfies both upward monotonicity and downward monotonicity.



Upward monotonicity Downward monotonicity Bilateral monotonicity

Figure 2

By the asymmetry of P, either part of definition G2 implies the minimal monotonicity condition $S \subseteq H$. To have representations that satisfy these stronger conditions, it is not sufficient that preferences be acyclical. The next two theorems specify precisely when such representations are possible. Each of these is preceded by a definition that summarizes the corresponding conditions.

Definition P2. An asymmetric binary relation P is an **interval order** if PIP implies P (i.e. xPy, yIz and zPw imply xPw).

Remark. Fishburn, who introduced the term 'interval order', motivated his suggestion by the following example, which dates back to Norbert Wiener. Suppose that the set A represents events in time, where each element $x \in A$ is defined by a real time interval $[b(x),e(x)] \subseteq \Re$, and suppose that the relation P stands for 'precedence in time' with xPy defined to apply between x and y whenever $e(x)<b(y)$. Then it is readily verified that P satisfies the "interval order condition". Furthermore, given any set A with a binary relation P on A, the above condition is necessary for the possibility of assigning real intervals to alternatives in A so that xPy if and only if the interval assigned to x precedes the one assigned to y (in the sense defined above).

Theorem 3.3 (version of Fishburn, 1970b).

For an asymmetric relation P on a countable set, the following three statements are equivalent.

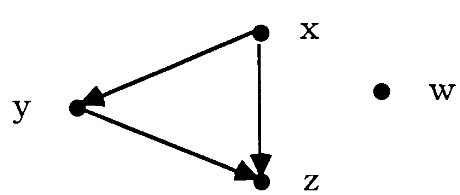
(i) P is an interval order.

(ii) P has a GNR that satisfies upward monotonicity.

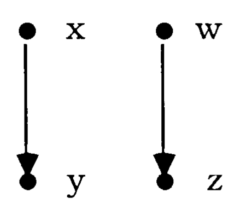
(iii) P has a GNR that satisfies downward monotonicity.

Definition P3. An asymmetric binary relation P is a **semiorder** if PQ implies P, equivalently [PIP or PPI] implies P.

Figure 3 illustrates the difference between the semiorder condition and the interval order condition (preference is indicated by an arrow).



(a) Violation of [PPI implies P]



(b) Violation of [PIP implies P]

Figure 3

Remark. The term 'semiorder', introduced by Luce, has its most common use in psychological measurement theory: e.g., if a person attempts to identify which of two stimuli is the stronger but can discriminate between the two only if the difference between them is sufficiently large (exceeds the 'just noticeable difference') then the stated comparisons form a semiorder. Luce motivated his introduction of semiorders by observing that the semiorder definition is a necessary condition for the possibility of assigning numbers to alternatives in such a way that the interval spanned by two alternatives for which preference applies is never a subinterval of the interval corresponding to two alternatives for which indifference applies. This is readily seen to be equivalent to bilateral monotonicity, a property which is the focus of the next theorem.

Theorem 3.4 (a stronger version of Luce, 1956).
An asymmetric relation on a countable set has a monotonic GNR if and only if it is a semiorder.

Every inference that can be drawn from a preference relation P can of course also be recovered from its GNR, and in this sense a GNR for P also gives a full representation of the revealed preferences Q. It would clearly be desirable to make these inferences more "direct" by applying a certain degree of monotonicity also to the representation of Q. In general, however, it would not be possible to let $v(x)>v(y)$ whenever xQy, because xQy does not preclude yQx. Therefore, the most that one can hope to achieve in general is an intuitive representation of P* and E. The next two extensions to our earlier theorems 3.2(a) and 3.3 state the maximal monotonicity that can be guaranteed for suborders and for interval orders.

Theorem 3.2(a)*. A suborder P on a countable set has a GNR (v,S) such that $v(x)>v(y)$ whenever xPy or xP*y and $v(x)=v(y)$ whenever xEy.

Theorem 3.3*. An interval order P on a countable set has a GNR (v,S) satisfying upward monotonicity (alternatively downward monotonicity) such that $v(x)>v(y)$ whenever xP^*y and $v(x)=v(y)$ whenever xEy.

When the stated preferences P are semiordered the revealed preferences exhibit no contradictions (i.e. the symmetric part of Q is empty and $Q=P^*$).[1] In this case Q is a weak order, being an asymmetric relation whose complement is transitive (see, e.g., Fishburn, 1985).[2] A GNR for P can then be made to reflect Q as intuitively as classical utility representations do for weak orders. This merits a special definition.

Definition G3 A **generalized utility representation** (**GUR**) of an asymmetric binary relation P is a monotonic GNR (u,S) of P which satisfies $xQy \Leftrightarrow u(x)>u(y)$. (The notational substitution of u for v is used to emphasize the fact that a GNR under consideration is indeed a GUR).

Theorem 3.5 (a stronger version of Luce, 1956). An asymmetric binary relation on a countable set has a GUR if and only if it is a semiorder.

Remark: While P has a GUR if and only if it has a monotonic GNR, P may also have monotonic GNRs which are not GURs. In a monotonic GNR for a semiorder P $v(x)>v(y)$ is mandatory whenever xQy (because if $v(x)\leq v(y)$ then xPz implies yPz and zPy implies zPx, hence [not xQy]), but there may be pair(s) x,y such that $v(x)>v(y)$ while xEy (rather than xQy).

For monotonic GNRs and GURs the boundary of S plays a prominent role, because it visibly separates S from its complement. In the special case of weak orders with S=H, the boundary of S is the "diagonal" in $\Re^2$, defined by

$$D=\{(\alpha,\beta)\in\Re^2 : \alpha=\beta\}.$$

[1]For a semiorder, $x(I\cup P)zPyQx$ implies zPx, $y(I\cup P)zPxQy$ implies zPy, and xPzIyPwIx implies yPz, three contradictions that exhaustively preclude xQyQx.

[2]In fact, Luce (1956) gives proof that Q is a weak order *if and only if* P is a semiorder.

We are interested in utility representations with similarly simple boundaries for preferences that are not weak orders.

Definition G4. A δ-**threshold** GNR (GUR) is a monotonic GNR (GUR) (v,S) such that

$$\text{boundary}(S)=\{(\alpha,\beta)\in\Re^2 : \alpha=\beta+\delta\}.$$

The 0-threshold GNRs are precisely the (u,H) classical utility representations of weak orders. δ-threshold GNRs with (strictly) positive δ, or in short **positive-threshold GNRs**, also exist for some, but not all, weak orders, as well as for some preferences which are not weak orders. The following characterization of positive-threshold GNRs is a famous old result.

Theorem 3.6 (Scott and Suppes, 1958).
A binary relation on a finite set has a positive-threshold GNR if and only if it is a semiorder.

That the Scott-Suppes theorem does not apply for infinite sets is apparent from the observation that the following axiom, which is trivially satisfied for finite sets but not so for infinite sets, is a necessary condition for P to have a positive threshold GNR.[1]

Axiom A1 For every $w\in A$ and every infinite sequence $x_1,x_2,...\in A$, if x_iPx_{i+1} for i=1,2,... then for some n wPx_n, and if $x_{i+1}Px_i$ for i=1,2,... then for some n x_nPw.

The Scott-Suppes theorem was first extended to denumerable sets by Manders (1981). Theorem 3.7 below is a simpler and more straightforward characterization (in the following sections we give further extensions of this important theorem to non-denumerable sets). Once again, the obvious necessary conditions are found to be also sufficient.

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[1] An example of a semiorder that does not satisfy axiom 1 was already used by Fishburn (1985) to demonstrate the limited domain of the Scott-Suppes theorem, but to the best of our knowledge the present study is the first time that the axiom is given an explicit statement.

<u>Theorem 3.7</u> (improved version of Manders, 1981)

For a binary relation P on a countable set, the following statements are equivalent.

(i) P is a semiorder which satisfies axiom A1.

(ii) P has a positive-threshold GNR.

(iii) P has a positive-threshold GUR.

<u>Remark.</u> If (v,S) is a GNR for an asymmetric binary relation, then clearly S must not intersect the diagonal D in $\Re^2$ (as defined above). It is interesting to note as a corollary to 3.7 that if a binary relation has a monotonic GNR (v,S) such that S is <u>bounded away from D</u> then it also has a positive-threshold GUR.

When (v,S) is a 0-threshold GNR (i.e. a classical utility representation) then S=H is open in $\Re^2$. In GNRs or GURs which have a positive threshold, or ones which are just bilaterally or unilaterally monotonic, S may be open, closed, or neither open nor closed.[1] In conformity with the classical utility representation, there has been a traditional tendency to use, whenever possible, representations where S is open. If (v,S) is a GNR for a binary relation P on a set A such that (v,S) satisfies upward monotonicity and S is open then we can set, for every $x \in A$,

$$\sigma(x)= \inf\{\alpha-v(x) : (\alpha,v(x))\in S\}\geq 0$$

and then $zPx \Leftrightarrow (v(z),v(x))\in S \Leftrightarrow v(x)+\sigma(x)<v(z)$. Alternatively, if (v,S) satisfies downward monotonicity and S is open we can similarly assign to every $x\in A$ some $\zeta(x)\geq 0$ such that $xPy \Leftrightarrow v(y)<v(x)-\zeta(x)$. In either case, we can assign to every $x\in A$ a closed real interval such that xPy if and only if the interval assigned to y fully "precedes" (as in Wiener's example above) the interval assigned to x. Indeed, Fishburn's celebrated results for interval orders are coined in precisely these terms.

Similarly, the traditional statement of the Scott-Suppes theorem corresponds to

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[1]A representation where S is an open set is sometimes termed **strong**, and one where S is a closed set is termed **strong***. The term **weak** then referes to representations where S is neither open nor closed.

a positive-threshold representation with S open. Our next and essentially last result for this section states that for countable sets this distinction is immaterial. Modifications are necessary for uncountable sets, and we will get to these in due course.

Theorem 3.8

For a binary relation P on a countable set A, the following statements are equivalent.

(i) P has a positive-threshold GUR.

(ii) There is a real valued function u on A such that $xPy \Leftrightarrow u(x)>u(y)+1$.

(iii) There is a real valued function u on A such that $xPy \Leftrightarrow u(x)\geq u(y)+1$.

Furthermore, in parts (ii) and (iii) one can add or omit the stipulation $xQy \Leftrightarrow u(x)>u(y)$.

An analogous result can be shown to apply for interval orders.

For completeness, we conclude this section with a restatement of a central result from classical utility theory. In content, this can be traced to Cantor (1915).

Theorem 3.9

An asymmetric binary relation on a countable set has a 0-threshold GNR if and only if it is a weak order.

We note that every 0-threshold GNR is necessarily a GUR.

4. LIMITED DISCRIMINATION PREFERENCE ORDERS

We have noted previously that when P is a weak order then I=E, so that the absence of stated preference between two alternatives can be safely interpreted as evidence of their equivalence. But in general the absence of a stated preference between two alternatives can be accompanied by indirectly revealed preference in either direction, and even in both directions simultaneously. Any contradiction in the revealed preferences (xQy and yQx) leaves the underlying standings of the alternatives undetermined. Indeed, this can happen even when P is a strict partial order (where PP implies P but II does not necessarily imply I), a prominent example being the Pareto aggregation of n weak orders (i.e. xPy if xP_jy for some $j \in \{1,...,n\}$ and [not yP_kx] for all k=1,...,n).[1] On the other hand, when P is a semiorder Q is a weak order, and gives a coherent specification of the relative standings of all alternatives. Here if xIy while not xEy the failure (of P) to give a clear preference statement cannot be due to undetermined relative standings, and it is most natural to attribute this failure to a limited ability of the decision-maker to discriminate between the two alternatives and identify the one which is inherently better.

Generalized utility representations reflect this notion of limited discrimination power by specifying for each alternative the upper and lower "just noticeable difference" in utility terms. For δ-threshold GURs these minimal differences are all the same (the "threshold constant" δ). Of course, the distinction between constant and variable thresholds may well be spurious. One prominent example is Weber's law, which states that for a class of real-valued stimuli the just-noticeable-difference is proportional to the stimulus. This can be directly represented by a variable-threshold GUR, but with a logarithmic representation of the stimuli the discrimination threshold becomes constant.

Another important point relates to spurious distinctions between 0-threshold and positive-threshold representations. When a weak order satisfies axiom A1 (as does the natural ordering of the integers), it can be given representations that have either zero threshold or positive threshold. Therefore, a positive-threshold representation does not, per-se, reflect limited powers of discrimination. We shall let the term **limited discrimination preference order** stand for preference relations that have generalized utility representations but where P does not fully

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[1]For strict partial orders xQy and yQx can occur simultaneously *only* if xIy, whereas in general this can occur also with xPy (or yPx).

coincide with Q. Such preferences do not have 0-threshold (classical) utility representations.

One question that immediately suggests itself is when can a comparison of two preference relations indicate that one of them exhibits "stronger discrimination power" than the other. The essence of the concept that we are looking for is that the two relations under consideration differ only in their power to identify differences between (the relevant aspects of) some pairs of alternatives, but not in the underlying attitudes that the respective preferences reflect. Such a comparability (if one is established) must obviously involve only a partial ranking of preference relations because, e.g., if xP^1y and yP^2x then the discrepancy between P^1 and P^2 cannot be attributed to different powers of discrimination. We shall first indicate how seemingly intuitive definitions of the notion of "stronger discriminating power" can easily fail, and then suggest a definition that seems to work.

Basic intuition might suggest that P^1 can be said to have stronger discriminating power than P^2 if P^2 is a (proper) subset of P^1: the two are never in explicit conflict, but there are some pairs of alternatives for which P^1 can give a clear statement of preference but P^2 cannot. Consider, however, the following example. Suppose that $A=\{x,y,z\}$, and let $P^1=\{(x,y)\}$, $P^2=\{(x,y),(y,z),(x,z)\}$ and $P^3=\{(x,y),(z,y),(z,x)\}$. Here P^1 is a subset of P^2 and at the same time a subset of P^3, yet P^1 cannot be positively identified as a version of P^2 with lower powers of discrimination and at the same time be positively identified as a version of P^3 with lower powers of discrimination, because P^2 and P^3 are in clear conflict.

A similar difficulty is encountered in the next example, but here it appears in the opposite direction and in a more subtle guise. For $A=\{x,y,z\}$ as above, let $P^1=\{(x,z),(y,z)\}$, $P^2=\{(x,z)\}$, and $P^3=\{(y,z)\}$. Here both P^2 and P^3 are subsets of P^1, but they cannot be considered lower discrimination versions of P^1, because the indirectly revealed preferences associated with P^2 and P^3 are in conflict, with xQ^2y and yQ^3x (note that P^1, P^2, and P^3 are all semiorders, and recall that for semiorders the implicit preferences are weak orders and thus asymmetric). This example can be made more paradoxical by denoting the alternatives by numbers rather than letters. Letting x=6 y=4 and z=0, it appears as if both P^1 and P^2 seek the higher-valued alternatives, but while P^1 is able to discriminate between alternatives whose values differ by more than 3, P^2 can discriminate only between pairs that differ by more than, say, 5. The more "neutral" representation by letters highlights the symmetry between P^2 and P^3 and indicates that the above interpretation is premature.

The above examples show that a comparison of the "discriminating power" expressed by two binary relations can be meaningful only if it can be ascertained that the two relations are guided by the same underlying standing of the alternatives. This leads to the following suggested criterion for (partially) ordering binary relations by their presumed power to discriminate between "similar" alternatives.

Definition D4.1 A binary relation P^1 has **stronger discriminating power** than a binary relation P^2 if $P^2 \subseteq P^1$, $P^2 \neq P^1$, and $Q^1 = Q^2$.

5. NON-DENUMERABLE SETS.

This section is devoted to numerical representations of preferences over arbitrary sets which are not necessarily countable. This discussion is inherently more technical. The special case of lotteries has additional structure which permits more direct characterizations, and this is dealt with in the next section.

A binary relation P on an uncountable set may fail to admit *any* GNR, let alone one satisfying monotonicity of one kind or another. For example, suppose that A is the collection of all subsets of the real line $\Re$, and let xPy if $x \subseteq y$. Here if (v,S) is a GNR for P then v must be a one-to-one mapping of A into $\Re$ - an impossiblity which is summarized by the statement that the collection of all subsets of $\Re$ has higher

cardinality than that of $\Re$, denoted by $\aleph$ (Aleph). On the other hand, high cardinality of A does not *per-se* inhibit a numerical representation: if there are very many alternatives that are equivalent to each other then it is always possible, and in fact desirable, to assign to all of them the same numerical value. To concentrate on the more relevant aspects, we shall in what follows simply consider all alternatives which are equivalent to each other as one element (an "equivalence class") in a modified (or "reduced") set of alternatives, for which preferences are defined in the obvious way. The modified set is marked by the property that no two distinct elements are equivalent, and we shall refer to pairs (A,P) satisfying this property, i.e. $xEy \Rightarrow x=y$, as **irreducible**. Every **related set**, i.e. pair (A,P), has a unique **irreducible form**.[1]

Observation 5.1

(a) A related set (A,P) has a GNR if and only if its irreducible form has a GNR. Every GNR (v,S) for the reduced form can be trivially extended to apply for (A,P), satisfying $v(x)=v(y)$ whenever xEy..

(b) An irreducible related set (A,P) has a GNR if and only if the cardinality of A does not exceed Aleph.

If the irreducible form of a related set (A,P) has sufficiently low cardinality, then it has *some* GNR (v,S). When the irreducible form is countable then all the results of section 3 apply, and the degree of monotonicity that the GNR can be made to have depends only on the degree of "consistency" in the stated preferences (from suborders up to weak orders). On the other hand, when the irreducible form is uncountable then the existence of a GNR and the consistency of the preferences are not sufficient to guarantee that the GNR can be made to have *any* of the desirable monotonicity properties. A famous counterexample from classical utility theory, due

[1]Formally, the irreducible form, say $(\underline{A},\underline{P})$, of an arbitrary related set (A,P) is defined as follows. $\underline{A}=A|E$, i.e. $\underline{A}$ is the collection of E-equivalent classes in A, defined by $\underline{A}=\{\underline{x} \subseteq A\text{: } \underline{x}\neq\phi, \text{ and } x\in\underline{x} \text{ implies } [y\in\underline{x} \text{ if and only if } yEx]\}$, and then $\underline{P}$ is defined on $\underline{A}$ by $\underline{P}=\{(\underline{x},\underline{y})\in\underline{A}^2\text{: } xPy \text{ for some } x\in\underline{x},\ y\in\underline{y}\}$.

to Debreu (1954), is the lexicographic ordering of the real plane $\Re^2$, viz. $(x_1,x_2)P(y_1,y_2)$ if $x_1>y_1$ or $[x_1=y_1$ and $x_2>y_2]$, which is a weak order on a set of cardinality Aleph that does not have a classical utility representation (nor indeed any representation with $S\subseteq H$). The key property, which will be seen below to be central for all numerical representations, is "separability", defined as follows.

Definition P4

A binary relation B on a set A is **separable** if A contains a countable subset C such that whenever y and z are two (distinct) elements in A but not in C and yBz there is some $x_i \in C$ satisfying yBx_i and x_iBz (the set C is said to be "B-dense in A").

The classical condition for the representability of non-denumerable weak orders is again traceable to Cantor (1915):

Theorem. A weak order has a 0-threshold GNR, equivalently has a GUR, if and only if its irreducible form is separable.

In contrast to the above, separability of the stated preferences is neither necessary nor sufficient for the representation of an imperfectly ordered set by a GNR with a lower degree of monotonicity. Here there is a crucial difference between complete representation by a GNR (v,S) and partial representation by a numerical function v alone. Separability is sufficient for a partial representation, as stated in the next theorem.

Theorem 5.2 (Richter, 1966).

If the irreducible form of a suborder P on a set A is separable, then there is a real valued function $v:A\rightarrow\Re$ such that $v(x)>v(y)$ whenever xPy.

But separability is *not* sufficient for a complete representation by a GNR (v,S) satisfying $S\subseteq H$, as demonstrated by the following example. Suppose that $A=\Re\times\{0,1\}$ and

$(x,i)P(y,j)$ if $x>y$ and $[i=j \text{ or } x-y\neq 1]$. This irreducible suborder is clearly separable, and indeed $v(x,i)=x$ satisfies the partial representation condition of theorem 5.2. But a GNR (v,S) must also satisfy $v(x,0)\neq v(x,1)$ for all $x\in\mathfrak{R}$ (because the two are not equivalent), and if $S\subseteq H$ then also $v(x,i)>v(y,j)$ whenever $x>y$, which is impossible.[1]

On the other hand, separability of the stated preferences is clearly not *necessary,* not even for a complete representation by a GNR. This is evident from the semiorder where xPy applies whenever $x-y>1$, which is not separable but has a trivial numerical representation with $S\subseteq H$. The following counterpart to theorem 5.2 gives a necessary condition for representations that exhibit any degree of monotonicity.

Definition P5

An asymmetric binary relation P on a set A is **weakly separable** if A contains a countable subset C such that whenever y and z are two (distinct) elements in A but not in C and yPz there is some $x_i\in C$ satisfying $[yPx_i \text{ or } yIx_i]$ and $[x_iPz \text{ or } x_iIz]$.

Theorem 5.3

If there is a function $v:A\rightarrow\mathfrak{R}$ such that $v(x)>v(y)$ whenever xPy then P is weakly separable.

Since the weak separability in theorem 5.3 is a *necessary* condition, it is also, afortiory, a necessary condition for the existence of a GNR with $S\subseteq H$.

For the representation of semiorders, there is a separability condition that is both necessary and sufficient. Unlike the previous results, this involves not the separability of the stated preferences P, but rather the separability of the revealed preferences Q.

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[1]The intervals $[\min\{v(x,0),v(x,1)\},\max\{v(x,0),v(x,1)\}]$, $x\in\mathfrak{R}$, would have to be an uncountable collection of disjoint intervals of positive length, which is impossible.

Theorem 5.4.

(a) (Version of Fishburn, 1985) An irreducible semiorder P has a monotonic GNR (alternatively GUR) if and only if Q is separable.

(b) An irreducible semiorder P satisfying axiom A1 has a positive-threshold representation if and only if Q is separable.

Separability of the revealed preferences Q does not, however, resolve the representability of less well structured preferences. The example following theorem 5.2 shows a suborder that does not have a GNR (v,S) with $S \subseteq H$, even though Q (as well as P) is separable. For unilaterally monotonic representation of interval orders, it is necessary and sufficient that *both* the relation PI and the relation IP (rather than just their union Q) be separable (Fishburn, 1985).

We conclude this section with a discussion of conditions for P to have a GNR (v,S) where S is an open set in $\Re^2$, or such that S is closed, or possibly representations of both kinds, as in theorem 3.8 above. This depends on appropriately defined notions of continuity of the preference relation. A number of equally plausible formulations will do, and we elect to use the following definitions:

Definition P6.

a. An alternative $x \in A$ is a **P-gap-edge-point** if there is $y \in A$ such that yPx and not yPQx.

b. An alternative $x \in A$ is an **I-upper-edge-point** if there is $y \in A$ such that yIx and not yIQx.

Theorem 5.5

(a) An irreducible semiorder P satisfying A1 has a GNR (v,S) where S is open if and only if Q is separable and the set of P-gap-edge-points is countable.

(b) An irreducible semiorder P satisfying A1 has a GNR (v,S) where S is closed if and only if Q is separable and the set of I-upper-edge-points is countable.

We note that in both parts of theorem 5.5 "GNR" may be replaced by "GUR".

6. DECISION UNDER UNCERTAINTY

This section is devoted to the special case in which the set of alternatives A is a space of lotteries, i.e. the space of finite-support probability distributions over some given set of outcomes. We are interested in representations that maintain the attractive structure of "expected utility" even though preferences are only semiordered. This context is of sufficient special interest to merit special treatment.

Let us assume, then, that A is the mixture space of finite-support distributions over an arbitrary set of outcomes $\mathbf{O}$. That is, an alternative $x \in A$ is defined by a tuple $\langle (p_1,\mathbf{o}_1),\ldots,(p_n,\mathbf{o}_n)\rangle$ where n is some positive integer, and for $i=1,\ldots,n$ $\mathbf{o}_i \in \mathbf{O}$ and $p_i \in (0,1]$, with $\Sigma p_i=1$. For $x,y \in A$ and $\alpha \in (0,1)$, the "mixture" $\alpha x+(1-\alpha)y$ gives an element in A, with the probabilities of outcomes being defined by the appropriate convex combinations of those under x and under y. As before, the binary relation P represents stated preferences over A, and Q represents the associated revealed preferences.

"Expected utility" representations are characterized by axioms that relate the structure of preferences to the special structure of the mixture space. We start with two standard axioms, commonly used for the characterization of the classical VonNeuman-Morgenstern expected utility representation of weak orders (Fishburn 1970a). Although we have not to this point restricted the class of preferences under consideration, these axioms will be used only when P is a semiorder, so that one may think of Q as a weak order.

<u>L1 (Q-independence)</u>: For all $x,y,z \in A$ and $\alpha \in (0,1)$, $xQy \Leftrightarrow [\alpha x+(1-\alpha)z]Q[\alpha y+(1-\alpha)z]$.

<u>L2 (Q-continuity)</u>: If $xQyQz$ there are $\alpha,\beta \in (0,1)$ for which

$[\alpha x+(1-\alpha)z]QyQ[\beta x+(1-\beta)z]$.

The key property for positive-threshold expected-utility representations is given by the next axiom.

L3 (spread-dependent indifference): For all $x,y \in A$ and $\alpha,\alpha',\beta,\beta' \in (0,1)$,

if $\alpha'-\alpha=\beta'-\beta$ then $[\alpha x+(1-\alpha)y]I[\beta x+(1-\beta)y] \Leftrightarrow [\alpha' x+(1-\alpha')y]I[\beta' x+(1-\beta')y]$.

Finally, we note that A is uncountable and introduce two alternative continuity axioms that are associated with alternative topological properties of S. Again, the axioms make use of the special structure of the mixture space.

L4 (open preference): If xPy then for every $z \in A$ there is some $\alpha \in (0,1)$ such that

$[\alpha x+(1-\alpha)z]Py$ and $xP[\alpha y+(1-\alpha)z]$.

L5 (open indifference): If xIy then for every $z \in A$ there is some $\alpha \in (0,1)$ such that

$[\alpha x+(1-\alpha)z]Iy$.

The above axioms lead to the following theorems on the existence of nicely structured expected utility representations for (some) imperfectly ordered preferences over alternatives with uncertain outcomes.

Theorem 6.1 The following statements are equivalent:

(i) P is a semiorder satisfying A1 and L1-L4.

(ii) There is a function $u:\mathbf{O}\rightarrow\Re$ such that

$xPy \Leftrightarrow E[u(x)]-E[u(y)] > 1$ and $xQy \Leftrightarrow E[u(x)] > E[u(y)]$.

Theorem 6.2 The following statements are equivalent:

(i) P is a semiorder satisfying A1, L1-L3, and L5.

(ii) There is a function $u:\mathbf{O}\rightarrow\Re$ such that

$xPy \Leftrightarrow E[u(x)]-E[u(y)] \geq 1$ and $xQy \Leftrightarrow E[u(x)] > E[u(y)]$.

Note that A1 cannot be omitted from statement (i) in either of these theorems. For instance, if P=Q and all other axioms but A1 are satisfied statement (ii) (of either theorem 6.1 or theorem 6.2) is false. However, in the presence of L3 one can replace A1 by the condition that $P \neq Q$. Our choice of the formulation that uses A1 was dictated by the desire to emphasize a feature common to this case and to the more general theorems stated earlier.

It is also worthy of note that in the representations (of non-trivial preferences) axiomatized here there must be two alternatives, say x and y, such that $u(x)-u(y) \geq 2$. For instance, suppose that $O=\{a,b\}$ with xPy if and only if $E[u(x)]>E[u(y)]+1$, and suppose further that (contrary to the above) $u(a)=1.5$ and $u(b)=0$. Then for $\alpha,\beta \in (1/3,2/3)$ $[\alpha a+(1-\alpha)b]E[\beta a+(1-\beta)b]$, whereas $E[u(.)]$ cannot be constant over that interval. This also shows that the example must violate axiom L1.

7. SOME COMMENTS ON UNIQUENESS.

One of the natural questions that arise once the existence of numerical representations is established is their uniqueness. It is interesting to investigate the extent to which one can change a given numerical representation, and identify the features shared by all numerical representations of a given preference relation.

We start with the following question: suppose a preference relation P has a GUR (u,S^o) where $S^o=\{(\alpha,\beta) : \alpha-\beta>1\}$, to what extent is the generalized utility function u unique? To identify the permissible transformations of u, consider any strictly increasing function $f:[0,1)\to[0,1)$, and let $T_f:\Re\to\Re$ be defined by

$$T_f(\alpha)=\lfloor\alpha\rfloor+f(\alpha-\lfloor\alpha\rfloor)$$

where $\lfloor\alpha\rfloor$ denotes the largest integer not exeeding α. Then clearly $(T_f u,S^o)$ is also a GUR for P. It is not difficult to see that when (range u)=$\Re$ the class of transformations T_f described above, in conjunction with arbitrary choices of origin, exhausts all the transformations that can be applied to the utility repesentation u while maintaining S^o. Defining similarly $S^c=\{(\alpha,\beta): \alpha-\beta\geq 1\}$, the same analysis applies to closed fixed threshold representations (u,S^c).

However, we also wish to investigate to what extent S^o (alt. S^c) is unique (beyond the obviously permissible changes of scale). By the previous analysis one might tend to expect that some properties must be shared by all equivalent representations. In particular, if (v,S) is equivalent to (u,S^o) as a GUR for P then we might expect S to be monotonic, bounded away from the diagonal D in $\mathfrak{R}^2$, and open (or closed for S^c). But in fact none of these properties is strictly necessary: S need only be monotonic in its intersection with (range $v)^2$, it does not have to be bounded away from the diagonal for P to satisfy A1, and rather than being open (alt. closed) it may include (alt. exclude) a countable set of points from the intersection of its boundary with (range $v)^2$ and any subset of the boundary not in (range $v)^2$. Given that these properties are not necessary, we may still wonder whether in some sense they are sufficient. The answer is a qualified yes, depending on the precise sense in which the equivalence is sought, as follows.

<u>Theorem 7.1</u>

(a) Let (v,S) be a GUR for a preference relation P. If S is monotonic, bounded away from the diagonal D in $\mathfrak{R}^2$, and open (alt. closed), then there is some strictly increasing transformation $T:\mathfrak{R}\to\mathfrak{R}$ such that (Tv,S^o) (alt. (Tv,S^c)) is a GUR for P.

(b) Let (u,S^o) (alt. (u,S^c)) be a GUR for a preference relation P, and let $S\subseteq\mathfrak{R}^2$ be monotonic, bounded away from the diagonal D in $\mathfrak{R}^2$, and open (alt. closed). If the boundary of S is the graph of a strictly increasing, unbounded and continuous function on $\mathfrak{R}$ then there is a strictly increasing transformation $T:\mathfrak{R}\to\mathfrak{R}$ such that (Tu,S) is a GUR for P.

To see that the additional qualification in part (b) of thoerem 7.1 is indeed necessary, let $A=\mathfrak{R}$ with xPy whenever $x-y>1$, and consider $S=\{(\alpha,\beta): \alpha>1 \text{ and } \beta<0\}$, which is monotonic, bounded away from the diagonal, and open. But if (v,S) is a GNR for P then $xP(x-2)$ implies $v(x)>1$ and $(x+2)Px$ implies $v(x)<0$, a contradiction which proves our point. Another counterexample with a different slant involves the same related set (A,P) as above and $S'=\{(\alpha,\beta) : \alpha-\beta>1$ and $[\alpha>3$ or $\beta<1]\}$. If (v,S') is a GUR for P then v must be strictly increasing and (since P satisfies A1) unbounded both from below and from above. Hence there are $a,b\in\Re$ such that

$$a=\sup\{x:v(x)<1\}=\inf\{x:v(x)>1\} \quad \text{and} \quad b=\sup\{x:v(x)<3\}=\inf\{x:v(x)>3\}.$$

Furthermore, $b=a+1$, since $x>b$ and $y<a$ imply xPy and $x-y>1$, while $x<b$ and $y>a$ imply $v(x)<3$ and $v(y)>1$ hence $(v(x),v(y))\notin S$ and $x-y<1$. Then $(b+0.1)-(a+0.5)=0.6<1$ implies $v(a+0.5)\geq v(b+0.1)-1>2$ while $(a+0.5)-(a-0.1)=0.6<1$ implies $v(a+0.5)\leq v(a-0.1)+1<2$, once again a contradiction.

Beyond the sheer interest in uniqueness, part (a) of theorem 7.1 is clearly motivated by the intuitive appeal of S^o and S^c. But the motivation underlying part (b) may perhaps seem somewhat dubious: if the preference P is already known to be representable by the highly attractive S^o (or S^c), why search for other representations with apparantly less attractive S? The answer is that the intuitive appeal of alternative representations depends very much on the specific context. An obvious example is Weber's law, where the proportional-threshold representation in terms of the stimuli is by no means less attractive than the constant-threshold representation in terms of the logarithms.

Much has been said and written in classical utility theory about the distinction between the "ordinal" utility representations that are amenable to any strictly increasing transformation, and the "cardinal" utility representations of preferences over lotteries, where in order to maintain the mathematical expectations structure only positive *linear* transformations are permissible. As we have noted here, the fixed-threshold representations of limited discrimination preference orders exhibit some intermediate degree of "cardinality": even though they do not admit *all* strictly increasing transformations, they are not restricted to linear transformations alone. One can perhaps say that positive-threshold representations admit arbitrary monotonically increasing transformations "in the small", but only linear positive transformations "in the large". The range of admissible transformations "in the large" is reduced further to the choice of origin alone if one wishes to maintain the discrimination threshold as the unit of measurment for the "utility scale". In contrast to the classical utility representations, where the choice of scale is totally arbitrary, the discrimination threshold of limited discrimination preference orders offers a natural choice of a unit of measurement which is indicated by the preference pattern itself. Indeed, our constructive proof of existence of positive-threshold utility representations for semiorders satisfying axiom A1 makes direct use of this unit of measurement.

Of course, the whole idea of numerical representation, and especially the numerical representations that exhibit a degree of monotonicity, is primarily an appeal to the intuition: from a strictly logical viewpoint any definition of the preference relation will do. The research on the existence of attractive numerical representations and on the extent that alternative representations are interchangeable can be interpreted as an effort to bring the formal analysis and its intuitive interpretation closer together.

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Appendix: Proofs of the Theorems

Theorem 3.2(a): The "only if" part is immediate. For the "if" part, note that the transitive closure, P^o, of a suborder P is a strict partial order (i.e., asymmetric and transitive), set $\{v(x), x \in A\}$ sequentially so that $v(x) > v(y)$ whenever xP^oy and $v(x) = v(y)$ if and only if xEy, and then set $S = \{(v(x),v(y)) \in \mathbb{R}^2: (x,y) \in P\}$. //

Theorem 3.3: It is easy to see that either (ii) or (iii) imply that P is an interval order. To show the converse implication, assume P is an interval order and use Theorems 6 and 8 in Fishburn (1985, pp. 28-29) to conclude that there are $b,e\colon A \to \mathbb{R}$ such that (1) $b(x) \leq e(x)$ and (2) xPy iff $b(x) > e(y)$.

To prove that (ii) holds, set $\{v(x), x \in A\}$ sequentially to satisfy $v(x) > v(y)$ if and only if $[b(x) > b(y)$ or $(b(x) = b(y)$ and $e(x) > e(y))]$. Then define $S = \{(\alpha, v(y)) | \alpha \geq v(x)$ for some $x \in A$ such that xPy$\}$.

The proof for (iii) is symmetric. //

Theorem 3.4: The "only if" part is straightforward. For the "if" part use Theorems 7 and 8 in Fishburn (1985, p. 29) and construct v and S as in (either part of) the proof of Theorem 3.3 above. //

Theorem 3.2(a)*: First note that xP*yPz implies xPz (for otherwise yQx) and xPyP*z also implies xPz (for otherwise zQy), i.e., $P^*P \cup PP^* \subset P$. With the transitivity of P* (established in Section 2) it follows that when P is a suborder then $P \cup P^*$ is also a suborder, and we continue as in the proof of

3.2(a).

<u>Theorem 3.3*</u>: We will show that the construction of Theorem 3.3 satisfies the additional requirements. First, note that if xEy one may, w.l.o.g., assign the same b and e values to x and y, whence $v(x) = v(y)$. Next, to show that $v(x) > v(y)$ when xP^*y, consider first the upward monotonicity construction. Knowing that xQy, there is a z such that xPz and not yPz, or such that zPy and not zPx. In the first case $b(x) > b(y)$ and $v(x) > v(y)$. In the second case we obtain $e(x) > e(y)$ and we still have to prove that $b(x) \geq b(y)$. However, a close look at the construction of the function b (see Thoerem 3 on pp. 23-4 of Fishburn (1985)) shows that $b(x) < b(y)$ would have implied the existence of an alternative z with $b(x) < e(z) < b(y)$, which would have meant yQx. But this conclusion is contradictory to xP^*y, whence we conclude that $b(x) \geq b(y)$ and (with $e(x) > e(y)$) we obtain $v(x) > v(y)$.

The proof for the downward monotonicity construction is symmetric. //

<u>Theorem 3.5</u>: Using the same theorems in Fishburn as in 3.4, we only note that his proof (via Theorems 3 and 4 on pp. 23-25) guarantees that the GNR is also a GUR. //

<u>Theorem 3.7</u>: The issue of substance here is to show that (i) implies (ii) and (iii). This proof requires a sequence of intermediate steps. In what follows, the more self-evident statements will be given without detailed proof.

We first define $Q^0 = Q \cup E$ and note that if P is a semi-order, then

$P \supset PQ^0$ and $P \supset Q^0P$. For $x, y \in A$ we define a set

$$M_x^+(y) = \{n \in N \mid \exists\, z_1, z_2, \ldots, z_{n-1} \in A \text{ such that } yPz_1Pz_2P \ldots P_{n-1}Qx\}$$

Define also

$$N_x^+(y) = \begin{cases} 0 & \text{if } M_x^+(y) = \emptyset, \\ \max M_x^+(y) & \text{if } M_x^+(y) \text{ is nonempty and finite,} \\ \infty & \text{if } M_x^+(y) \text{ is infinite.} \end{cases}$$

We observe that, for all $x, y \in A$, $N_x^+(y) > 0$ iff yQx.

Lemma 1: Assume that $x, y, z \in A$ satisfy $N_x^+(z) = n_1 > 0$ and $N_z^+(y) = n_2 > 0$. Then

$$n_1 + n_2 - 1 \leq N_x^+(y) \leq n_1 + n_2.$$

Proof: The left side inequality is implied by $P \supset QP$. As for the right side one, if $N_x^+(y) = k > n_1 + n_2$, there are $w_1, \ldots, w_{k-1}$ such that $yPw_1P \ldots Pw_{k-1}Qx$. Consider w_{n_2} and z. If $w_{n_2}Qz$, then $N_2^+(y) \geq n_2 + 1$, which is false. Hence $zQ^0w_{n_2}$ and zPw_{n_2+1} follows which, in turn, implies that $N_x^+(z) \geq k - n_2 > n_1$, a contradiction. //

Lemma 2: For all $x, y \in X$, $N_x^+(y)$ is finite.

Proof: Assume the contrary, and let $x, y \in A$ satisfy $N_x^+(y) = \infty$. Let $z \in A$ satisfy $yPzPx$ (of course, such exists). By Lemma 1, either $N_x^+(z) = \infty$ or $N_z^+(y) = \infty$ (or both). In the first case, denote $y_1 = z$ and consider x and y_1. In the second, let $x_1 = z$ and continue with x_1 and y. Arguing inductively, one obtains either a monotonically P-decreasing sequence (y_i) which is bounded from below (by x) or an increasing one (x_i) which is bounded from above (by y), or both. At any rate, this is a contradiction of axiom A1. //

Lemma 3: If $x, y, z \in A$ satisfy $N_x^+(y) > 0$ and zPy, then $N_x^+(z) \geq N_x^+(y) + 1$.

Lemma 4: Suppose that $N_x^+(y) = n > 0$ and that $yPz_1Pz_2P \ldots Pz_{n-1}Qx$. Then $N_x^+(z_i) = n - i$ for $1 \leq i \leq n - 1$.

Lemma 5: Suppose that xIy. Then f or all $z \in A$, $|N_z^+(x) - N_z^+(y)| \leq 1$.

Proof: Assume the contrary, e.g., xIy and $N_z^+(x) \geq N_z^+(y) + 2$. Suppose $xPw, Pw_2P \ldots Pw_{n-1}Qz$ where $n = N_z^+(x)$. Then $N_z^+(w_2) = n - 2$, and (w_2Py or w_2Iy) follows, whence xPy, a contradiction. //

We now proceed to define a new set of functions, to be denoted by $\{N_x^-\}_{x \in A}$. First define, for $x, y \in A$, $M_x^-(y) = \{n \in N | \exists\, z_1, z_2, \ldots, z_{n-1} \in A$ such that $yPz_1Pz_2P \ldots Pz_{n-1}Px\}$ and

$$N_x^-(y) = \begin{cases} 0 & \text{if } M_x^-(y) = \emptyset, \\ \max M_x^-(y) & \text{if } M_x^-(y) \text{ is nonempty and finite,} \\ \infty & \text{if } M_x^-(y) \text{ is infinite.} \end{cases}$$

Obviously, $N_x^-(y) > 0$ iff yPx for all $x,y \in A$.

Lemma 6: For all $x,y \in A$

$$N_x^-(y) \le N_x^+(y) \le N_x^-(y) + 1.$$

The functions $\{N_x^-\}_{x\in A}$ have similar properties to those of $\{N_x^+\}_{x\in A}$:

Lemma 7: Assume that $x,y,z \in A$ satisfy $N_x^-(z) = n_1 > 0$ and $N_z^-(y) = n_2 > 0$. Then

$$n_1 + n_2 \le N_x^-(y) \le n_1 + n_2 + 1.$$

Proof: Similar to Lemma 5. //

Lemma 8: For all $x,y \in A$, $N_x^-(y)$ is finite.

Proof: Like Lemma 2 or using Lemma 6. //

Lemma 9: Assume $N_x^-(y) > 0$. Then for z such that zPy $N_x^-(z) \ge N_x^-(y) + 1$ and for z such that xPz $N_z^-(y) \geq N_x^-(y) + 1$.

Lemma 10: Suppose that $N_x^-(y) = n > 0$ and that $yPz_1Pz_2 \ldots Pz_{n-1}Qx$. Then $N_x^-(z_i) = n - i$ for $i \leq i \leq n - 1$.

Lemma 11: If xIy, then $|N_z^-(x) - N_z^-(y)| \leq 1$, $\forall\, z \in A$.

We can finally define the functions we are really after. For $x \in A$ define N_x: $A \to Z$ by

$$N_x(y) = \begin{cases} N_x^+(y) & \text{if } yQx \\ 0 & \text{if } yEx \\ -N_y^-(x) & \text{if } xQy \end{cases}$$

The following lemmata are quite straightforward. The somewhat tedious proofs (based on a case-by-case study) are omitted.

Lemma 12: If xPy, then $N_z(x) \geq N_z(y) + 1$, $\forall\, z \in A$.

Lemma 13: If xIy, then $|N_z(x) - N_z(y)| \leq 1$, $\forall\, z \in A$.

Lemma 14: For all $x, y \in A$, $0 \leq N_x(y) + N_y(x) \leq 1$.

Lemma 15: For all $x, y, z \in A$

$$|N_z(y) - N_z(x) - N_x(y)| \leq 3.$$

We will now define a real function on A as an average of all functions $\{N_x(\bullet)\}_{x \in A}$. We need a measurable structure which will be provided by the real line. We note here that Q is separable. (This is, of course, trivial since A is countable, but for later adaptations of the proof it is worthy of note that the countability of A is used here for the first time and that the separability of Q is all we need.) We may therefore assume that there exists v: $A \to \mathbb{R}$ such that xQy iff $v(x) > v(y)$.

Let C = conv (range (v)) (i.e., C is a (not necessarily finite) interval in $\mathbb{R}$). For $\alpha \in C$ define N^α: $A \to Z$ by

$$N^\alpha(y) = \min \{N_x(y) | v(x) \leq \alpha\}.$$

Note that $N^\alpha(\bullet)$ is well defined, finite and monotonically nondecreasing. Moreover, for each $y \in A$, $N^\alpha(y)$ is a nonincreasing function of α--hence, measurable. Let μ be a probability measure on C with the following properties:

(i) $\mu(I) > 0$ for every positive-length interval I;

(ii) For some $x \in A$, the integral

$$\int_C N^\alpha(x) d\mu$$

is well-defined and finite.

Note that such measures do exist.

Lemma 16: For every $y \in A$,

$$u(y) \equiv \int_C N^{\alpha}(y)d\mu$$

is well-defined and finite.

Proof: By Lemma 15, $u(y)$ and $u(x)$ converge or diverge together for every $x,y \in A$. //

Lemma 17: For all $x,y \in A$,

$$xPy \Rightarrow u(x) \geq u(y) + 1$$

$$xIy \Rightarrow |u(x) - u(y)| \leq 1.$$

Lemma 18: For all $x,y \in A$,

$$xQy \Leftrightarrow u(x) > u(y).$$

Proof: Assume xQy, whence $v(x) > v(y)$. For all $\alpha \in (v(y),v(x))$, $N^{\alpha}(x) \geq 1 > 0 \geq N^{\alpha}(y)$ whence $u(x) > u(y)$. On the other hand, xEy surely implies $u(x) = u(y)$. //

Lemmata 17 and 18 in fact complete the proof of Theorem 3.7. //

Theorem 3.8: (ii) and (iii) surely imply (i). We avoid proving the converse implications at this point because (in conjunction with Theorem 3.7) they follow immediately from Theorem 5.5 below. //

Theorem 5.3: Suppose $v: A \to \mathbb{R}$ satisfies $v(x) > v(y)$ for all $x, y \in A$ such that xPy. It is sufficient to show that there exists a countable $C \subseteq A$ such that $v(x) > v(y)$ implies the existence of $z \in C$ with $v(x) \geq v(z) \geq v(y)$. For every pair of rational numbers, $a < b$, if range $(v) \cap (a,b) \neq \emptyset$, choose $x \in A$ with $v(x) \in (a,b)$, and denote the set of alternatives thus chosen by C_0. Next define a "hole" to be a positive-length interval which does not intersect the range of v. Note that every hole is contained in a maximal hole, and that distinct maximal holes are disjoint, hence there are only countably many maximal holes. Let $\{H_1\}_i$ be an enumeration of those maximal holes which are finite open intervals. For each H_i, there are $x_i, y_i \in A$ such that $H_i = (v(x_i), v(y_i))$. Define $C = C_0 \cup \bigcup_{i \geq 1} \{x_i, y_i\}$, and note that it satisfies the requirement of weak separability. //

Theorem 5.4

a. For the "if" part, assume Q is separable and let $v: A \to \mathbb{R}$ represent it (i.e., $v(x) > v(y) \iff xQy$). Then define $S = \{(\alpha,\beta) \mid \alpha \geq v(x),\ \beta \leq v(y)$ for some $x, y \in A$ such that $xPy\}$, and note that (v,S) is a monotonic GUR of P. For the "only if" part, we recall the remark following Theorem 3.5, and conclude that if (v,S) is a monotonic GNR of an irreducible preference P, then it is also a monotonic GUR and v represents Q, whence Q is separable. //

b. The "only if" part follows from part (a). The "if" part is proved similarly to Theorem 3.7. //

Theorem 5.5

a. First assume that P has a positive-threshold GNR where S is open. This means that there exists a function $u: A \to \mathbb{R}$ satisfying $xPy \iff u(x) > u(y) + 1$ for all $x,y \in A$. It is easily seen that this implies that P is a semi-order satisfying A1, that u represents Q and that Q is separable. To see that there are only countably many P-gap-edge-points, let x be one and let y satisfy yPx where there is no z for which yPzQx holds. Hence, range $(u) \cap (u(x),u(y) - 1) = \emptyset$. This means that the u-value of every P-gap-edge-point alternative is the left endpoint of a positive-length interval not intersecting range (u). Since P is irreducible, there are only countably many P-gap-edge-points.

We will now prove the converse. Assume, then, that P is an irreducible semi-order satisfying A1, with countably many P-gap-edge-points, and that Q is separable. For $x \in A$ we say that x is P-regular if it is not a P-gap-edge-point, and that it is P-regular of order 2 if for every $y \in A$ satisfying yPx there are $z,w \in A$ such that yPzQwQx. Let C_i be the set of all alternatives which are not P-regular of order i ($i = 1,2$).

Lemma 1: C_2 is countable.

Proof: It suffices to show that there are only countably many alternatives x which are P-regular but not P-regular of order 2. Let x be such an alternative. Then there is $y \in A$ such that yPx, but $yPzQwQx$ does not hold for any $(z,w) \in A^2$. By P-regularity, there is $t \in A$ such that $yPtQx$. The alternative, t, itself has to be P-gap-edge-point. Furthermore, there is no $w \in A$ satisfying $tQwQx$. This means that there exists a 1-1 function from $C_2 \setminus C_1$ to C_1, and the conclusion follows. //

(Note that by the same method one can show that C_i is countable for all $i \geq 1$, hence also $\bigcup_{i \geq 1} C_i$ is countable.)

We will now extend the set of alternatives and the relations defined on it as follows. For every $x \in C_2$ let us introduce two new alternatives, denoted $\bar{x}$ and $\bar{\bar{x}}$. Let $\bar{A} = A \cup \{\bar{x} | x \in C_2\} \cup \{\bar{\bar{x}} | x \in C_2\}$. We define $\bar{P}$ and $\bar{Q}$ on $\bar{A}$ as follows:

(i) For all $x,y \in A$, let $x\bar{P}y \iff xPy$ and $x\bar{Q}y \iff xQy$.

(ii) For $x \in C_2$ and $y \in A$, $y \neq x$, let

$\bar{x}\bar{P}y \iff \bar{\bar{x}}\bar{P}y \iff xPy$; $y\bar{P}\bar{x} \iff y\bar{P}\bar{\bar{x}} \iff yPx$

$\bar{x}\bar{Q}y \iff \bar{\bar{x}}\bar{Q}y \iff xQy$; $y\bar{Q}\bar{x} \iff y\bar{Q}\bar{\bar{x}} \iff yQx$.

(iii) For $x,y \in C_2$, $x \neq y$, let

$\bar{x}\bar{P}\bar{y} \iff \bar{x}\bar{P}\bar{\bar{y}} \iff \bar{\bar{x}}\bar{P}\bar{y} \iff \bar{\bar{x}}\bar{P}\bar{\bar{y}} \iff xPy$

$\bar{x}\bar{Q}\bar{y} \iff \bar{x}\bar{Q}\bar{\bar{y}} \iff \bar{\bar{x}}\bar{Q}\bar{y} \iff \bar{\bar{x}}\bar{Q}\bar{\bar{y}} \iff xQy$.

And finally,

(iv) For $x \in C_2$, let $\bar{\bar{x}}Q\bar{x}Qx$.

For $x,y \in \bar{A}$, let $x\bar{I}y$ if neither $x\bar{P}y$ nor $y\bar{P}x$. The following needs no proof:

Lemma 2

(a) $\bar{P}$ is an irreducible semi-order satisfying A1;

(b) $\bar{Q}$ is a separable weak order;

(c) $\bar{Q} \supset \bar{P}\bar{I} \cup \bar{I}\bar{P}$; $\bar{P} \supset \bar{Q}\bar{P} \cup \bar{P}\bar{Q}$.

(Note that $\bar{Q}$ is not necessarily the indirect preference relation corresponding to $\bar{P}$. I.e., $\bar{Q}$ may be strictly larger than $\bar{P}\bar{I} \cup \bar{I}\bar{P}$.)

Next we note that for binary relations $\bar{P}$ and $\bar{Q}$ which satisfy conditions (a)-(c) of Lemma 2, all the lemmata in the proof of Theorem 3.7 hold. Consider the function $u: \bar{A} \to \mathbb{R}$ constructed in the proof. For all $x,y \in \bar{A}$ we have (Lemma 17)

$$xPy \Rightarrow u(x) \geq u(y) + 1$$

and

$$xIy \Rightarrow |u(x) - u(y)| \leq 1.$$

We now have to show that for $x,y \in A$, xPy implies $u(x) > u(y) + 1$. However, for such $x,y \in A$ there are $z,w \in \bar{A}$ such that $x\bar{P}z\bar{Q}w\bar{Q}y$. Consider $\alpha \in (v(y),v(w))$ (where v represents $\bar{Q}$ as in the proof). For such a value α, $N^{\alpha}(x) \geq 2$ where $N^{\alpha}(y) \leq 0$. As $v(y) > v(w)$, $u(x) > u(y) + 1$. Hence P has a positive-threshold GUR with an open set S, and the proof is complete. //

b. Both parts of the statement are proved similarly to their counterparts in (a) above. //

<u>Theorem 6.1</u>: It is trivial that (ii) implies (i). Assume, then that (i) holds, whence there is a $u: X \to \mathbb{R}$ such that $xQy \iff Eu(x) > Eu(y)$ for all $x,y \in A$. (One may use Fishburn's (1970a) formulation of the von Neumann-Morgenstern theorem, applied to Q.) Suppose first that there are Q-maximal and Q-minimal elements $x^*, x_* \in X$. For each $y \in A$ there is a unique $\alpha(y) \in [0,1]$ such that $y\ E\ [\alpha x^* + (1 - \alpha)x_*]$, and the conclusion follows from L3 and L4 applied to $\{\alpha x^* + (1 - \alpha)x_* | \alpha \in [0,1]\}$. Next, if range (u) has no sup or inf, choose a sequence $\{(x_i^*, x_{*i})\}_{i\geq 1}$ such that range $(u) \subset \cup_{i\geq 1} (u(x_{*i}), u(x_i^*))$ and for each $i > 1$ construct u_i as an extension of u_{i-1}. //

<u>Theorem 6.2</u>: As 6.1 above. //

<u>Theorem 7.1</u>: (a) Theorem 5.5 implies that P has a GUR (u, S^o)(alt., (u, S^c)). Because both u and v represent Q, there exist a strictly increasing $T: \mathbb{R} \to \mathbb{R}$ with $u = Tv$.

(b) By the provision of the theorem, there exists $u: A \to \mathbb{R}$ such that xPy iff $u(x) > u(y) + 1$ ($u(x) \geq u(y) + 1$) and xQy iff $u(x) > u(y)$. Assume without loss of generality that range $(u) = \mathbb{R}$ (otherwise, extend A, P and u). Therefore, the function v we are looking for is a monotone transformation of u. Choose an arbitrary $x_0 \in A$ and let $v(x_0) = u(x_0)$. Let x_i satisfy $u(x_i) - u(x_0) = i$ for $i \in Z$. The value $v(x_i)$ is determined by the boundary of S. (I.e., $v(x_1) = \inf \{\alpha | (\alpha, v(x_0)) \in S\}$, and so forth). Define $v(x)$ for $\{x | u(x_0) < u(x) < u(x_1)\}$ so that v is some (strictly) monotonic continuous function of u on $[u(x_0), u(x_1)]$, and extend v to all A according to S as above. Since bd(S) is the graph of a strictly increasing, unbounded and continuous function (defined on all the reals), this procedure generates a well-defined v (which is continuous in u) such that (v,S) is a GUR of P. //