Discussion Paper No. 830

ON THE DESIGN OF COMPLEX ORGANIZATIONS AND DISTRIBUTIVE ALGORITHMS*

DONALD G SAARI**

Northwestern University

April 1989

(*) This research was supported, in part, by NSF grant # IRI-8803505 and by a 1988-89 Guggenheim Fellowship
(**) Department of Mathematics, Northwestern University, Evanston, IL 60201

Abstract

To efficiently realize a specified goal in a distributive fashion, there needs to be an appropriate "division of labor." This is true for distributive algorithms that take advantage of the concurrent features of the new generation of computers. This is true in the design of a complex organization intended to realize a specified goal. The problem is to determine what is the appropriate division of labor. Here, a geometric characterization of all possible divisions of labor, or communication networks, is given. It is illustrated how this characterization can be used to design the communication networks.

There are striking similarities between the problem of designing distributive algorithms to take advantage of the concurrent and parallel features of the new generation of computers and the problem of designing an efficient organization to accomplish a specified goal. For both, the objective is to parcel the workload among the various participating units in an efficient, coordinated fashion. For instance, consider what is involved in creating a distributive algorithm. The main task is to determine what it is that each processor should compute and what partially computed information should be conveyed to which other processors. There is a similar problem for the design of an organization. Here responsibilities need to be assigned to the different departments and divisions; namely, the goal is to establish an organizational chart to determine the assignments and the reporting structure. Indeed, the design both of distributive algorithms and of organizations can be summarized with the coordinating questions of "who should do what?" and "who should say what to whom?"

For many situations, there exist algorithms and organizations that efficiently solve this division of labor problem. But, in general, the design of a system remains as an important open question. In all cases the purpose of an organization is to achieve a stated objective. So, the major obstacle is to understand how to start with the stated objectives and then extract from these goals the appropriate structures - structures that can be exploited to create the organization. The principal purpose of this paper is to attack this problem by developing a geometric characterization of this design problem. The geometric constructs introduced here expose the structures associated with the universal issues i) of determining *the kind of information* each unit needs to convey in order to achieve a stated objective and ii) of establishing *the reporting structure* of who reports what to whom. Because my emphasis is to introduce some of the underlying basic concepts, I treat here only a simplified model where I ignore the many other related problems. A more complete description is planned for elsewhere.

To state the problem in a simple setting, let the objective be given by the smooth function

1.1 $$F: R^{k(1)} \times \ldots \times R^{k(J)} \longrightarrow R$$

where $k(i)$, $i = 1,\ldots,J$, are positive integers. Think of each space $R^{k(i)}$ as representing the data available to the i^{th} unit (processor, department, individual, agent, etc.). The function F represents the specified objectives. In a computational problem, F may be a function that is to be evaluated where the relevant data is divided so that processor i can access only the data represented in $R^{K(i)}$, $i = 1,\ldots,n$. For a hypothetical organizational example, consider a firm trying to optimize profits coming from sales of a particular product. Let a vector in $R^{K(1)}$ represent data about potential markets, $R^{K(2)}$ represent data about costs and availability of raw materials needed for production, and $R^{K(3)}$ represent other technical variables. Let F represent either the optimal profits, or the output of the product that will achieve the maximal maximal profits with the current environment. The goal is to efficiently transfer information (or partial computations, partially constructed products, etc.) so that F is realized.

The objective function F specifies what is to be done - the goals. Once F is given, the object is to find the ways - the organizations - so that the task of realizing F is divided among the several cooperating units. To do this, I build upon the ideas of Abelson [1], Hurwicz [3] and others to model the flow of information among the units. The basic idea, which is a slight extension of Abelson's model, is simple and very natural. In the beginning, each unit has knowledge only of the data assigned to it; the i^{th} unit can only use the data from $R^{K(i)}$. This data must be processed in a manner that contributes toward realizing F. Represent this first step of computation by $g^1_i(x_i) = m^1_i$; $i = 1,\ldots,n$, $x_i \in R^{K(i)}$, $m^1_i \in R$. Namely, at the first stage (denoted by the superscripts on g and m), the i^{th} unit uses the available data x_i to compute the value m^1_i. Of course, the choice of g^1_i is intended so that the value m^1_i contributes toward determining the value of $F(x_1,\ldots,x_n)$. (In general it is not obvious how to define g^1_i; indeed, finding guidelines for an appropriate selection of these functions is major aspect of the *design* problem.) Let $\mathbf{m}^1 = (m^1_1,\ldots,m^1_n) \in R^n$ denote the vector of the first stage computations.

At the second stage, each unit can use not only its assigned data, but also the partial computations, or *messages* $\mathbf{m}^1$, transmitted at the first stage. This means that the computations at the second stage can be denoted by $g^2_i(x_i,\mathbf{m}^1) = m^2_i \in R$. The general situation at the α^{th} stage is that the i^{th} unit can use all of the partial computations, or messages, from the other units as well as the original data $\mathbf{x}_i$. Therefore the computation at this stage is represented by

1.2 $$g^{\alpha}_{i}(x_{i}, \mathbf{m}^{1}, \ldots, \mathbf{m}^{\alpha-1}) = m^{\alpha}_{i};$$

i.e., this computation is represented by a function

1.3 $$g^{\alpha}_{i}: R^{k(i)} \times (R^{J})^{\alpha-1} \longrightarrow R,$$

where $\mathbf{m}^{k} \in R^{J}$ is the vector of partial computations at the k^{th} step, $k = 1, \ldots \alpha-1$.

At some step it may be that certain units have nothing to contribute or do. This is the situation if, for instance, a particular unit cannot proceed with meaningful work until it receives certain messages from specified other units. The above modeling admits such circumstances by defining the particular function to be $g^{\alpha}_{i} \equiv 0$.

Suppose it takes β stages of partial computations to determine the value of F. I model this by assuming that all but one of the units complete their partial computations at the $(\beta-1)^{th}$ step. The remaining unit uses the messages of partial computations and its data to compute the value of F.[1] Namely, I assume there is a specific index s so that

1.4 $$g^{\beta}_{s}(x_{s}, \mathbf{m}^{1}, \ldots \mathbf{m}^{\beta-1}) = m^{\beta}_{s}, \quad g^{\beta}_{j} \equiv 0 \text{ for } j \neq s,$$

where

1.5 $$F(x_{1}, \ldots, x_{j}) = m^{\beta}_{s} = g^{\beta}_{s}(x_{s}, \mathbf{m}^{1}, \ldots, \mathbf{m}^{\beta-1}).$$

Because at certain stages some of the units may not be transmitting a message, the effective messages - the images of the g functions - form only a linear subspace of $(R^{J})^{\beta}$. Let M, the *message space*, denote this linear subspace.

With this model, the functions $\{g^{\alpha}_{i}\}$ specify what each unit must do, compute, and communicate at each stage. Because these functions determine "who says what to whom," I call a choice of smooth functions $\{g^{\alpha}_{i}\}$ that satisfies these conditions a ***communication network that realizes F.*** Furthermore, I call the issue of characterizing all possible communication networks that realize F the ***central design problem*** associated with F.

---

1. This approach, which follows Abelson, is reasonable for models of computation. An alternative model, with a slightly different supporting mathematical theory, is where each unit finishes its computations at the β^{th} stage. The final determination of F is based only on these messages. Thus, there is a function $h: M \longrightarrow R$ so that $h(\mathbf{m}^{1}, \ldots, \mathbf{m}^{\beta}) = F(x_{1}, \ldots, x_{j})$. Here h may correspond to the "auctioneer," the central authority, the team leader, or a central computer. This alternative approach more closely represents several models from economics.

By solving or characterizing the solution of the central problem, all sorts of information may be available about the communication network functions $\{g^\alpha_i\}$. This information can be used to analytically compare competing communication networks, to develop complexity measures, and so forth. As an immediate observation, note that the value of β serves as a crude measure of the "speed" of the communication network. This is because it indicates that F is realized in β steps. Thus, there may be many situations whereby smaller values of β imply a more valued communication network.

One can conceive of situations where efficiency, or mininimal cost is determined by how much information needs to be transferred among the units. This is particularly so should it be expensive, or time consuming to transmit messages (or partial products, etc.). When this is the case, measures of complexity can be developed to reflect this fact. To see how this is done, suppose a communication network $\{g^\alpha_i\}$ is given and consider the reporting issue of determining "who says what to whom?" The function g^α_i represents what the i^{th} unit does at the α^{th} stage, so the dependency of this function on the m variables determines who has to communicate what partial computations to this unit. Namely, if for any choice of $s<\alpha$ and $k \neq i$, the partial derivative of g^α_i with respect to m^s_k is not identically zero, then the k^{th} unit needs to communicate this value to the i^{th} unit before the α^{th} stage.

As a third issue, note that it is of value to understand the "kind of information" associated with a communication network. (This is particularly true for theoretical investigations of a communication network.) By "kind of information," I mean an equivalence class of data that gives rise to the same value of each partial computation. In other words, starting with the given data, at each step each unit computes the value of a message, m^α_i. It may be that with a different choice of data, all of the messages are precisely the same. (If so, then both data points give rise to the same value of F.) So, all data giving rise to the same messages define the same kind of information. Thus the "kind of information" associated with a communication network is characterized by the level sets of g^α_i.

Definition. Let $\Gamma = \{g^\alpha_i\}_{\alpha = 1,\ldots,\beta;\ i=1,\ldots,j}$ be a given communication network that realizes F. We say that $\mathbf{x}, \mathbf{x}' \in R^{k(1)}\times\ldots\times R^{k(j)}$ are "Γ equivalent" if the following holds: $g^1_i(\mathbf{x}_i) = g^1_i(\mathbf{x}')$ for all i. This requires the messages at the first stage to be the same. By induction, for all α, $g^{\alpha}_{i}(x_i, \mathbf{m}^1, \ldots \mathbf{m}^{\alpha-1}) = g^{\alpha}_{i}(x'_i, \mathbf{m}^1 \ldots, \mathbf{m}^{\alpha-1})$. An equivalence class of data is called a "Γ information set."

As indicated, many of the basic issues for the design of algorithms or organizational structures can be characterized in terms of the properties of a communication network $\{g^{\alpha}_{i}\}$. However, it is not at all clear how to start with an objective function F and then determine an associated, non-trivial communication network. It would be useful to determine structures that would assist in this design. This goal, of finding what such a construction depends upon, is the basic theme of this current paper. I characterize the communication networks in terms of certain geometric constructs. As I indicated earlier, the purpose of these geometric properties is to expose the hidden, implicit structures of F that govern the admissible communication networks. This approach involves solving several equations; equations that need not be particularly easy to solve. On the other hand, these equations do indicate what must be done to achieve such a network. As such, they form a most useful place to start.

While my goal is to characterize all possible communication networks, I would like to call attention to the several clever arguments used to find properties of all possible communication networks without solving the central problem. In particular, I point to the paper by Abelson [1], where, for i = 2 (i.e., only two units are allowed) he introduces a complexity measure, the total information transfer, that is based on counting the number of messages that are required to be conveyed between the processors. Thus, in terms of the above discussion, this measure is determined by counting the non-zero partial derivatives of the communication network functions, $\{g^{\alpha}_{i}\}$, with respect to the $\mathbf{m}$ variables. As such, with the efficiency assumptions introduced in the next section, a lower bound for this measure is [dim(M) - 1] where the (-1) term corresponds to m^{β}_{s} - a message that is not transferred. (For $j \geq 3$, this may not be a sharp lower bound because the same message may be transferred to several units.) Abelson finds a lower bound for all possible communication networks strictly in terms of the rank properties of the Hessian of the objective function F. By using more sophisticated mathematical approach based on concepts from differential geometry, P. Chen [2] improves upon Abelson's lower bound: Chen's theorem is based on the rank of a bordered Hessian. Again, Chen's improved lower bound depends only on the differential properties of F: he circumvents the more difficult issue of solving the central problem.

2. Single Shot Mechanisms.

In this section, some insight is obtained about the kinds of information admitted by a specified F. I do this by showing that a communication network for F can be viewed as being a special case of a different kind of network that realizes F - the single shot mechanism. An important advantage of relating the two problems is that in this way I can exploit existing results characterizing all possible single shot mechanisms. This characterization can be used to impose bounds on what is possible for the associated communication networks, as well as to characterize the possible "kinds of information" admitted by the possible networks. Then, in Section 3, a characterization of the central problem is provided.

The more general system is where all of the information is communicated among the different units in a single step. For this to be possible, the values of $\mathbf{m}$ need to be determined implicitly. Thus, rather than communicating a value (as is true for a communication network), the i^{th} unit communicates a *set* $\{\mathbf{m} | G^{a}_{i}(x_i, \mathbf{m}) = 0\}$. The actual message is the intersection of these sets, i = 1,...,n, in a message space M. Such systems occur quite naturally as part of the equilibrium analysis of a dynamical exchange of information that assumes the form $m'_i = G^{a}_{i}(x_i, \mathbf{m})$. The basic purpose of the dynamic given by this differential equation is to allow each unit to update its message based on its own characteristics, x_i, and the recent messages of the other units. The equilibrium state of the dynamic is where the G functions are all equal to zero. Notice that this modeling generalizes the common price dynamic story from economics where prices change according to the market pressures of supply and demand. For more detailed discussion of this and other interpretations, see Hurwicz [3].

Single Shot Problem: For a given objective function F, find smooth functions $G^{a}_{i}(x_i, \mathbf{m}): R^{k(i)} \times M \rightarrow M$, $M = R^m$, $a = 1,\dots,n_i$, $i = 1,\dots,n$; and a smooth function $h: M \rightarrow R$ so that with any value of $\mathbf{m}$ implicitly defined by

2.1 $G^{\alpha}_{i}(x_i,\mathbf{m}) = 0,$

we have that

2.2 $h(\mathbf{m}) = F(x_1,..,x_j)$. The triple $(\{G^{\alpha}_{i}\}, M, h)$ is called a single-shot mechanism that realizes F.

Thus the single shot mechanism corresponds to factoring a function F through another space, M, in a non-standard implicit form. Of course, the "kind of information" associated with a single shot mechanism $\{G^{\alpha}_{i}\}$ is defined in a similar way as the Γ information sets - it is given by the level sets of the G^{α}_{i} functions. The relationship between the single shot and the central problem is stated in the following formal statement.

Theorem 1. If a function F admits a communication network, then this network defines a single shot mechanism, $\{G^{\alpha}_{i}\}$, for F. The message space for both systems is the same. Moreover, an information set associated with this communication network is same information set associated with the defined mechanism $\{G^{\alpha}_{i}\}$.

The proof of this theorem is immediate. This is because the communication network function, Eq. 1.2, can be expressed in the implicit single shot form $G^{\alpha}_{i}(x_i,\mathbf{m}) = 0$; $\alpha = 1,..,\beta$; $i = 1,...,j$; where $\mathbf{m} = (\mathbf{m}^1, \mathbf{m}^2,...,\mathbf{m}^{\beta}) \in R^{\beta j} = M$ by defining $G^{\alpha}_{i}(x_i,\mathbf{m}) = g^{\alpha}_{i}(x_i,\mathbf{m}^1,...,\mathbf{m}^{\alpha-1}) - m^{\alpha}_{i}$. The assertions of the theorem now follow immediately. Chen's Theorem is based on a similar observation.

An advantage of Theorem 1 is that there exists two characterization of the single shot mechanisms (Hurwicz, Reiter, and Saari [4], and Saari [5]). For the purposes of this paper, I adopt the characterization in Saari [5,6] because it is more general and it appears to be computationally easier to use. According to Theorem 1, this characterization can be invoked to limit the possible choices of the communication networks. This is because the communication networks are those single shot mechanisms that satisfy an additional rank condition.[2]

---

2. These rank conditions are the obvious ones required to take the equation for the single shot mechanism and solve them to obtain a communication network.

In general there are infinitely many choices of $\{G^{\alpha}{}_i\}$ functions that give rise to the same information sets.[3] However, a given set $\{G^{\alpha}{}_i\}$ can be pared to a basic set by eliminating redundancies. This is the purpose of the following set of efficiency assumptions. In these conditions, treat $\{G^{\alpha}{}_i\}$ as a mapping from $R^{k(1)}x..R^{k(J)}xM$ into an Euclidean space that agrees with the number of $G^{\alpha}{}_i$ functions.

Efficiency Assumptions on $\{G^{\alpha}{}_i\}$.

a. The dimension of M agrees with the number of $\{G^{\alpha}{}_i\}$ functions.

Let $\mathbf{X} = (x_1,\ldots,x_j)$ and $\mathbf{m}$ represent variables in a zero set of $\{G^{\alpha}{}_i\}$.

b. At $(\mathbf{X},\mathbf{m})$ the Frechet derivative of $\{G^{\alpha}{}_i\}$ with respect to $\mathbf{m}$ is non-singular.

c. At $(\mathbf{X},\mathbf{m})$ the Frechet derivative of $\{G^{\alpha}{}_i\}$ with respect to $\mathbf{X}$ has maximal rank.

(The Frechet derivative can be viewed as being the Jacobian of $\{G^{\alpha}{}_i\}$ with respect to the indicated variables.

The characterization of the single-shot mechanisms for a given F are expressed in a differential form. The idea is that the zero sets of the $\{G^{\alpha}{}_i\}$ functions define level sets, or certain collections of related foliations of the space $R^{k(1)}x..xR^{k(J)}$. Thus, the leaves from the foliations correspond to the kinds of information. Foliations can be totally characterized in terms of their normal vectors. These vectors define the normal bundle. When these vectors are expressed in terms of differential one-forms, the normal bundle becomes an ideal of differential forms. The necessary integrability conditions on the normal bundle now are expressed in terms of a condition on the ideal; it must be a differential ideal. These concepts lead to the following statement. For a proof, a discussion of these terms, and more details, along with a partial history of this problem see Saari [5].

----

3. This is why I place more emphasis on the "kinds of information" than on the actual single shot mechanisms or communication networks. In fact, a useful equivalence relationship can be defined among the mechanisms (the communication networks) in terms of these level sets. In this manner, networks that seem to have little to do with each other can be shown to be equivalent.

Theorem 2. Let a smooth objective function F be given. The following are necessary and sufficient conditions that a smooth single shot mechanism $(\{G^{\alpha}_{i}\},M,h)$ for F exists in a neighborhood of $X \in R^{k(1)}x..xR^{k(J)}$ that satisfies the efficiency assumptions.

1. For each i, there is a differential ideal $I_i = \langle dF, w_{i,1}, \ldots, w_{i,s(i)}; \{dx_j\}_i \rangle$, $s(i)=n_i-1$, with $(\Sigma_{j \neq i} k_j)+n_i$ linearly independent one-forms. Here, each $w_{i,s}$ is a smooth one form and the set $\{dx_j\}_i = \{dx_k \mid x_k$ is a coordinate direction for a parameter not in $R^{k(i)}\}$.

2. The set $I = \cap_i I_i$ is a differential ideal with $n = \Sigma_i n_i$ linearly independent one-forms.

The resulting mechanism to realize F has a message space of dimension n where there are n_i functions relating the parameters of the i^{th} unit with the messages.

The proof that this is a sufficient condition follows from the Frobenius Theorem (see Saari [5]). That this is a necessary condition comes by re-expressing the gradients $\{\nabla G^{\alpha}_{i}\}$ in terms of differential forms $\{dG^{\alpha}_{i}\}$. These one-forms form a basis for the differential ideals, $\{I_i\}_{i=1,\ldots,j}$, I, that have the specified properties. The reason the one-forms $\{dx_j\}_i$ are in I_i is to capture the condition that the i^{th} unit has access only to data from $R^{k(i)}$. The requirement on the ideal I is to ensure that the the conveyed messages are compatible with one another in evaluating F.

To illustrate how Theorem 1 can be used, notice that a trivial single shot mechanism is a "parameter transfer" where one unit communicates the value of all of its parameters to the second unit. After these values are transferred, the second unit computes the value of F. Namely, if $F(x_1,x_2):R^k x R^k \to R$, then $\beta = k+1$, and $G^s_1 = x_s - m^s_1 = 0$ where x_s is the s^{th} component of x_1, $s = 1,\ldots,k$, while $G^1_2 = F((m^1_1,\ldots,m^k_1),x_2)-m^1_2 = 0$. The function h is the projection $h(\mathbf{m}) = m^1_2$. This single shot mechanism has a message space M with $\dim(M) = k+1$. The communication network associated with the parameter transfer is $g^s_2 \equiv 0$, $g^s_1 = m^s_1$ where m^s_1 is the s^{th} component of x_1, $s = 1,\ldots,k$, while $g^{\beta}_2 = F((m^1_1,\ldots,m^k_1),x_2)$ and $g^{k+1}_1 \equiv 0$. This communication network associated with the parameter transfer does not reflect the kind of benefits one expect from a system capable of concurrent or distributive action. After all, this system just transfers all of the work to another unit. Thus, such a communication network is one that is not overly efficient. Yet, suppose the only single shot mechanisms admitted by F are equivalent to a parameter transfer. It follows from Theorem 1 that all possible communication networks associated with F must be related to this undesired transfer method.

More generally, the class of all possible single-shot mechanisms that realize F restrict the kinds of communication networks that are associated with F. Thus, Theorems 1 and 2 form an important first step toward determining what kinds of networks are possible. In Saari [5,6], several examples of F are analyzed to characterize the associated single shot mechanisms. One example is repeated here to illustrate Theorem 2.

Example 1. Let $F:R^2 \times R^2 \rightarrow R$ be defined by $F(\mathbf{x},\mathbf{y}) = \Sigma_i x_i y_i$. I show that this function admits only a parameter transfer. To do this, I first consider I_1. This set must contain $dF = \Sigma_i y_i dx_i + \Sigma_i x_i dy_i = d_x F + d_y F$. It also contains dy_1 and dy_2, as well as all possible linear combinations of these three one-forms where the coefficients are smooth functions of **x** and **y**. As the second summation in dF, $d_y F$, can be expressed as combinations of dy_1 and dy_2, this part of dF can be eliminated. Thus, these forms can be reduced to the set $\{d_x F = \Sigma_i y_i dx_i, dy_1, dy_2\}$. If I_1 were to admit any other linearly independent one-form, then a basis for I_1 would be $\langle dx_1, dx_2, dy_1, dy_2 \rangle$. The foliation identified with this ideal is given by the intersection of the level sets of x_i, y_i, $i = 1,2$. In other words, the messages are equivalent to the first unit transmitting the value of x to the second unit. This means that the accompanying mechanism is (equivalent to) a parameter transfer. Hence, assume that $I_1 = \langle d_x F = \Sigma_i y_i dx_i, dy_1, dy_2 \rangle$. A similar argument shows that to avoid a parameter transfer of the y values, $I_2 = \langle d_y = \Sigma_i x_i dy_i, dx_1, dx_2 \rangle$. Consequently, $I = \langle \Sigma_i x_i dy_i, \Sigma_i y_i dx_i \rangle$.

It remains to determine whether I_1, I_2, and I are differential ideals. Trivially, I_1 and I_2 are differential ideals. One way to show this is to note that $r_1 = (\Sigma_i y_i dx_i) \wedge dy_1 \wedge dy_2$ is a three-form. A necessary and sufficient condition for I_1 to be a differential ideal is that $dw \wedge r_1 \equiv 0$ where w is any one form from I_1. But, $dw \wedge r$ is a five-form in a four dimensional space, so it must be identically zero.

An alternative argument proving that I_1 is a differential ideal uses the fact that this is so iff there is an associated foliation identified with I_1. This foliation is given by the intersection of the level sets (in $R^2 \times R^2$) of F.

$f_1 = y_1$, and $f_2 = y_2$. A similar argument proves that I_2 also is a differential ideal.

The final step is to show that I is *not* a differential ideal. First, $r = (\Sigma_i y_i dx_i) \wedge (\Sigma_i x_i dy_i) \neq 0$ and $d(\Sigma_i x_i dy_i) = \Sigma_i dx_i \wedge dy_i$. A necessary and sufficient condition for I to be a differential ideal is that $d(\Sigma_i x_i dy_i) \wedge r \equiv 0$. However, a direct computation proves that $d(\Sigma_i x_i dy_i) \wedge r \neq 0$. Because I is not a differential ideal, there does not exist a single shot mechanism with $n_1 = n_2 = 1$. This means that any single shot mechanism associated with F must involve adding another independent one-form either to I_1 or to I_2, and, hence, to I. Suppose this one form is added to I_1. As shown above, the addition of this independent one-form makes $I_1 = \langle dx_1, dx_2, dy_1, dy_2 \rangle$. In turn, this means that the kind of information associated with the mechanism is equivalent to a parameter transfer of the x values to the other unit. Namely, for this choice of F, all single shot mechanisms are equivalent to the parameter transfer mechanism.

It now follows from Theorem 1 that the communication networks for this scalar product are equivalent to networks of the following form: Let $g^s{}_1 = x_s = m^s{}_1$, $g^s{}_2 \equiv 0$, $s = 1,2$, $g^3{}_2 = \Sigma_i m^i{}_1 y_i$, $g^3{}_1 \equiv 0$.

3. Characterization of the Communications Networks.

The characterization of communication networks also can be expressed in terms of differential ideals, except several more ideals are required. These additional ideals account for the rank conditions needed to ensure that the equations for a single shot mechanism can be solved to determine the associated communication network. Again, for any F, there are an infinite number of associated communication networks, so the first task is to eliminate certain redundancies. As in the previous section, this is done by imposing efficiency assumptions. In these conditions, consider only the non-constant functions in $\{g^\alpha{}_i\}$ and treat the remaining functions as defining a mapping.

Efficiency Assumptions on a Communication Network $\{g^\alpha{}_i\}$.

a. The dimension of M agrees with the number of non-constant $\{g^\alpha{}_i\}$ functions.

Let $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_j)$ and $\mathbf{m}$ represent variables in a zero set of $\{g^\alpha{}_i\}$.

b. At $(\mathbf{X}, \mathbf{m})$ the Frechet derivative of the non-constant $\{g^\alpha{}_i\}$ with respect to $\mathbf{m}$ is non-singular.

c. At $(\lambda, \mathbf{m})$ the Frechet derivative of the non-constant $\{g^{\alpha}_i\}$ with respect to λ has maximal rank.

Theorem 3. Let a smooth objective function F be given. The following are necessary and sufficient conditions that a smooth communication network $\{g^{\alpha}_i\}$ that satisfies the efficiency assumptions exists in a neighborhood of $\lambda \in R^{k(1)}\times..\times R^{k(j)}$.

1. For each i, there is a differential ideal $I^1_i = \langle w^1_i,;\{dx_j\}_i\rangle$ where the one-form $w^1_i \in T^*R^{k(i)}$; i.e., it is a linear combination of the differentials of the coordinate functions in $R^{k(i)}$ where the scalar functions are smooth functions from $R^{k(i)}$ to R.

2. By induction, for each i and each α satisfying $1<\alpha<\beta$, there is a one form w^{α}_i so that $I^{\alpha}_i = \langle w^{\alpha}_i, I^{\alpha-1}_i\rangle$ is a differential ideal. Secondly, for all i with the exception of an index s, $I^{\beta}_i = I^{\beta-1}_i$. In the exceptional case of i = s, there can be a one-form w^{β}_s so that $I^{\beta}_s = \langle w^{\beta}_s, I^{\beta-1}_s\rangle$ is a differential ideal. For all i, $dF \in I^{\beta}_i$.

3. For all i and all α satisfying $1<\alpha<\beta$ and for $\alpha = \beta$ when i = s, all of the ideals $J^{\alpha}_i = I^{\alpha}_i \cap (\cap_{k\neq i} I^{\alpha-1}_k)$ are differential ideals.

The resulting communication network takes β steps and the dimension of the message space corresponds to the dimension of J^{β}_s.

The proof of this theorem will appear elsewhere. Some of the connections between Theorems 2 and 3 are that i) the ideal J^{β}_s from Theorem 3 plays the role of the ideal I in Theorem 2 while ii) the ideals I^{β}_i from Theorem 3 correspond to the ideals I_i from Theorem 2. The remaining ideals correspond to the added conditions required to ensure that a single shot mechanism can be expressed in the form of a communication network. Notice that the conditions on the ideals for the first stage, I^1_i, amount to choosing a one-form w^1_i to be a functional multiple of $dg^1_i(x_i)$. It is not obvious how to choose the functions $g^1_i(x_i)$. Therefore it is iinteresting to note, as illustrated in the following examples, that this choice is partially governed by the conditions on the J^2_i ideals as well as the other conditions from Theorem 3. While the resulting set of equations may be difficult to solve, this approach does provide additional structure to understand how to decompose F into an organizational format. Finally, notice that because the dimension of J^{β}_s agrees with the dimension of

M, the structure of J^{B}_{s} provides valued information about Abelson's total information transfer.

The differences between Theorem 2 and 3, as well as an indication how to use these results, is illustrated with the following examples. The first one, Example 2, shows that not all single shot mechanisms are related to a communication network.

Example 2. Let $F:R^2 \times R^2 \dashrightarrow R$ be defined as $F(\mathbf{x},\mathbf{y}) = [x_1y_2+x_2]/[1-x_1y_1]$. I show that F admits a (1,1) single shot mechanism; that is, there is a single shot mechanism with $n_1=n_2=1$. This conclusion is by no means obvious. What is even less obvious is how to decompose F into the appropriate messages from the two units. Therefore, it is worth noting how the structures of the ideals lead to the resulting mechanism.

If F admits a (1,1) mechanism, then I_1 must be $\langle d_xF;dy_1,dy_2\rangle$ and $I_2 = \langle d_yF;dx_1,dx_2\rangle$ where, as in Example 1, d_xF and d_yF are, respectively, the part of dF that has only dx_j differentials and dy_j differentials. If $w_1 = (1-x_1y_1)^2d_xF$ and $w_2 = [(1-x_1y_1)^2/x_1]d_yF$, then $I_1 = \langle w_1,dy_1,dy_2\rangle$, $I_2 = \langle w_2,dx_1,dx_2\rangle$, $I = \langle w_1,w_2\rangle$, $w_1 = (y_2+x_2y_1)dx_1 + (1 - x_1y_1)dx_2$, and $w_2 = (x_1y_2+x_2)dy_1 + (1 - x_1y_1)dy_2$. By using argument similar to those found in Example 1, it follows that I_1 and I_2 are differential ideals. Thus it suffices to show that I is a differential ideal.

The ideal I is a differential ideal with dimension two iff $r = w_1 \wedge w_2 \neq 0$ and both $dw_1 \wedge r$ and $dw_2 \wedge r$ are identically zero. But, because $d(d_xF) = -d(d_yF)$, it follows that I is a differential ideal if $r \neq 0$ and $dw_1 \wedge r \equiv 0$. A computation shows that $r = (y_2+x_2y_1)(x_1y_2+x_2)dx_1 \wedge dy_1 + (y_2+x_2y_1)(1-x_1y_1)dx_1 \wedge dy_2 + (1-x_1y_1)(x_1y_2+x_2)dx_2 \wedge dy_1 + (1-x_1y_1)^2dx_2 \wedge dy_2$ and $dw_1 = -dx_1 \wedge dy_2 - 2y_1dx_1 \wedge dx_2 - x_2dx_1 \wedge dy_1 + x_1dx_2 \wedge dy_1$. It is clear that $r \not\equiv 0$. A direct computation proves that $dw_1 \wedge r \equiv 0$. This establishes that I is a differential ideal, so it also follows (from Theorem 2) that there does exist a (1,1) single shot mechanism that realizes F.

By following the scheme described in Saari [5,6], the single shot mechanism given by the G^a_i functions can be determined. One choice is

3.1 $$G^1_1 = x_1m_1 + x_2 - m_2 = 0,$$
$$G^1_2 = y_1m_2 + y_2 - m_1 = 0.$$

In other words, for this single shot mechanism, each unit transmits a line. In $M = R^2$, these two lines intersect in a unique point; this point is the equilibrium value of $\mathbf{m} = (m_1, m_2)$. The function $h: M = R^2 \rightarrow R$ is given by $h(\mathbf{m}) = m_2$.

Now consider all of the communication networks associated with F. Each choice of a network specifies the particular unit that is charged with computing the value of F at the β^{th} step. Secondly, to start the computation process, at least one of the two units must make an initial partial computation; i.e., at least one $dg^1_i \neq 0$. This requires either I^1_1 or I^1_2 to have a one-form in addition to the one-forms corresponding to the other units coordinate functions. This one-form characterizes the initial computation step. So, assume that the second unit is to determine the value of F and that I^1_1 has an independent one-form other than dy_1 and dy_2. (All other cases have a similar argument.) The integrability conditions force this one-form in I^1_1 to be a scalar function multiple of the differential of a function $g^1_1(\mathbf{x})$. It follows immediately from the form of $d_x F$ that there does not exist a function $\tau(\mathbf{x},\mathbf{y})$ so that the coefficients of $\tau(\mathbf{x},\mathbf{y})d_x F$ are strictly functions of $\mathbf{x}$. Consequently, both $d_x F$ and $dg^1_1(\mathbf{x})$ are in $I^{\beta-1}_1$, and they are linearly independent. This forces $I^{\beta-1}_1 = \langle dx_1, dx_2; dy_1, dy_2 \rangle$. In turn, this means that the kind of information associated with any communication network must be equivalent to a parameter transfer, so $\beta = 3$. One such network is $g^1_1(\mathbf{x}) = x_i = m^i_1$, $g^i_2(\mathbf{y}) \equiv 0$; $i = 1,2$; while $g^3_2 = F((m^1_1, m^2_1), \mathbf{y})$, and $g^3_1 \equiv 0$. In other words, even though the above single shot mechanism provides a distributive way to code information about F that results in a saving over the parameter transfer, such economies do not extend or exist for any of the communication networks associated with F.

The total information transfer is 2; the first unit transfers m^1_1 and m^2_1 to the second unit. That it is impossible to find a communication network that improves upon the above constructed one for F follows either from the above analysis or from Chen's theorem. Chen's result shows that the lower bound for information transfer for this choice of F is 2.

Example 3. Abelson uses the following function F to illustrate certain features of a communication process. Chen uses the same F to illustrate that his lower bound (of 3) improves upon Abelson's. I use this F to illustrate how Theorem 3 can be used to determine a communication network.

Let $F: R^n \times R^n \rightarrow R$ be $F(\mathbf{x},\mathbf{y}) = \Sigma_s x_s (y_1)^s + \Sigma_s y_s (x_1)^s$. A direct computation shows that $dF = \Sigma_s (y_1)^s dx_s + (\Sigma_s s y_s (x_1)^{s-1}) dx_1 +$

$\Sigma_s(x_1)^s dy_s$ $(\Sigma_s s(y_1)^{s-1}x_s)dy_1$. At the first stage, $I^1_1 = \langle dg^1_1(\mathbf{x}); dy_1,\ldots,dy_n\rangle$ and $I^2_1 = \langle dg^1_2(\mathbf{y}); dx_1,\ldots,dx_n\rangle$. The choice of the functions g^1_i is not obvious. What is interesting is that the choice of the functions is determined at the second step by the structure of the ideals given in Theorem 3.

It is clear that there must be at least one more stage. If not, then to satisfy condition 2 of Theorem 3, either d_xF must be a scalar (function) multiple of $dg^1_1(\mathbf{x})$, or d_yF must be a scalar multiple of $dg^1_2(\mathbf{y})$. Because of the mixed x_i, y_j form of the coordinate functions in these two differentials, neither is possible. If only one additional stage is required before the value of F can be computed by, say, unit 2, then $I^2_1 = \langle dg^1_1, dF; dy_1,\ldots,dy_n\rangle$ and $J^2_1 = \langle dg^1_1, dg^1_2, d_xF\rangle$.

The ideal I^2_1 is a differential ideal because it describes the foliation given by the intersection of the level sets of g^1_1, F, and $f_s(\mathbf{x},\mathbf{y}) = y_s$, $s = 1,\ldots,n$. On the other hand, J^2_1 is a differential ideal iff $d(d_xF)\wedge r \equiv 0$ where $r = dg^1_1\wedge dg^1_2\wedge d_xF \neq 0$. (This is because $d(dg^1_i) \equiv 0$ for $i = 1,2$.) As $d(d_xF) = -(\Sigma s(y_1)^{s-1}dx_s)\wedge dy_1 + (\Sigma s(x_1)^{s-1}dy_s)\wedge dx_1$, it is easy to see that a necessary and sufficient condition for J^2_1 to be a differential ideal is that $dg^1_1(\mathbf{x})$ and $dg^1_2(\mathbf{y})$ are, respectively, scalar function multiples of dx_1 and dy_1. From this, following the scheme described in Saari [5], a communication network can be constructed. Namely, $g^1_1(\mathbf{x}) = x_1 = m^1_1$, $g^1_2(\mathbf{y}) = y_1 = m^1_2$, $g^2_1 = \Sigma_s x_s(m^1_2)^s = m^2_1$, $g^2_2 \equiv 0$, and $g^3_2 = m^2_1 + \Sigma_s y_s(m^1_1)^s = m^3_2 = F(\mathbf{x},\mathbf{y})$.

Example 4. A very simple example is $F: R^k \times R^k \to R$ given by $F(\mathbf{x},\mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$ where f and g are smooth functions. An obvious communication network is $m^1_1 = f(\mathbf{x})$ and $F(\mathbf{x},\mathbf{y}) = m^2_2 = m^1_1 g(\mathbf{y})$. I show how this network arises out of Theorem 3. First of all note that to minimize the value of B, the goal is to choose communication functions that will permit dF to be in an ideal as soon as possible. Therfore, we check to see if it is possible for $dF \in I^1_1$. This is true because $d_xF = g(\mathbf{y})df(\mathbf{x})$, so it is in the ideal $\langle df(\mathbf{x}); dy_1,\ldots,dy_k\rangle$. The described message system follows immediately.

Example 5. As a final example, I consider $F: R^n \times R^n \to R$ that is given by the scalar product; $F(\mathbf{x},\mathbf{y}) = \Sigma_s x_s y_s$. According to both Abelson's and Chen's Theorems, the total information transfer must be at least n - the same as for a parameter transfer. However, a parameter transfer requires $B = n + 1$.

Therefore, it is worth questioning whether F admits communication networks other than the parameter transfer that permit $B < n + 1$. The best one can do is if at each stage, each unit transfers a message to the other unit. If this transfer is done efficiently, then the network would require $(B-1) = n/2$. (Recall, there is no transfer of information at the B^{th} step; this is the stage where the value of F is computed.) To find efficient networks is easy. However, I use this simple choice of F to illustrate how the structures of Theorem 3 help to design communication networks. (The analysis also shows what other methods are, or are not possible.) Because I am using F to illustrate the use of the above theorems, my description is phrased in a general fashion so that one can extend the notions to other choices of F.

At the first stage, $I^1_1 = \langle dg^1_1(\mathbf{x});dy_1,..,dy_n\rangle$ and $I^1_2 = \langle dg^1_2(\mathbf{y});dx_1,..,dx_n\rangle$. As true with the earlier examples given above, while the choice of the functions $\{g^1_k\}$ is not obvious, assistance for the choice of these functions is provided by the structure of J^α_i for $\alpha\geq2$. I will show how this happens in different ways. For my first choice, I consider what manner of conditions for the ideals lead to the following kind of communication network: At the first stage, the first unit communicates the value of x_1 while the second communicates the value of y_n. At the second stage, the first unit computes and transmits the value of $x_n y_n$ (based on the message it received) while the second unit transmits the value of $x_1 y_1$. The process continues.

To see how the above kind of network arises, consider what happens should a one-form $w^2_1(\mathbf{x},\mathbf{y}) = \Sigma\theta_i(\mathbf{x},\mathbf{y})dx_i$ be added to I^2_1 where at least one of the θ_i functions does depend on the $\mathbf{y}$ variable. The first condition is that I^2_1 is a differential ideal. This involves showing that $dw^2_1\wedge r \equiv 0$ where r is the $(n+2)$-form $dg^1_1\wedge w^2_1\wedge[dy_1\wedge\ldots\wedge dy_n]$. The dw^2_1 term can be expressed as $d_x w^2_1 + d_y w^2_1$ where the first terms comes from the partial derivatives of the x variables while the second comes from the partial derivatives with respect to the y variables. The bracketed term in r annihilates the $d_y w^2_1$ contribution, so all that remains is that $d_x w^2_1\wedge w^2_1\wedge dg^1_1 \equiv 0$. This is guaranteed for w^2_1 being the x-part of the differential of any function $H(\mathbf{x},\mathbf{y})$; i.e., $w^2_1 = d_x H(\mathbf{x},\mathbf{y})$. Assume this is the case where, of course, the choice of H is to be determined.

The second part of the $\alpha = 2$ stage is to show that $J^2_1 = \langle d_x H(\mathbf{x},\mathbf{y}), dg^1_1(\mathbf{x}), dg^1_2(\mathbf{y})\rangle$ is a differential ideal. The only thing that needs to be done here is to show that $d(d_x H)\wedge[d_x H\wedge dg^1_1(\mathbf{x})\wedge dg^1_2(\mathbf{y})] \equiv 0$. As I have already shown above that $d_x w^2{}_1 \wedge w^2{}_1 \wedge dg^1{}_1 \equiv 0$, so it remains to show that $d_y w^2{}_1 \wedge dg^1{}_1 \wedge dg^1{}_2 \wedge w^2{}_1 \equiv 0$. But $d_y w^2{}_1$ is in the space spanned by $\{dx_i \wedge dy_j\}$. Another basis can be given by the wedge product of the $\{dx_i\}$ terms with the orthogonal basis $\{dg^1{}_2(\mathbf{y}), \tau_i(\mathbf{y})\}$. Therefore $d_y w^2{}_1$ can be expressed as a linear (with scalar functions as coefficients) combination of $\{dg^1{}_2(\mathbf{y}) \wedge dx_i, \tau_j(\mathbf{y}) \wedge dx_j\}$. If $d_y w^2$ admits any terms of the form $\tau_i(\mathbf{y}) \wedge dx_j$ (but not of the form $\tau_i(\mathbf{y}) \wedge dg^1{}_1$ or $\tau_i(\mathbf{y}) \wedge d_x H(\mathbf{x},\mathbf{y})$) then the differential ideal condition will not be satisfied. This means that the "y" part of $H(\mathbf{x},\mathbf{y})$ must depend upon the message $g^2{}_1 = m^1{}_2$. One choice is if $g^2{}_1 = y_n$, then $H(\mathbf{x},\mathbf{y}) = x_n y_n$. (There are many other choices, such as counterproductive choices of $x_1 y_n$. However, such choices are quickly excluded at the β^{th} step when dF must be in all ideals. Indeed, the object in the design of the $g^a{}_i$ functions is to include dF in each of the ideals as quickly, or as efficiently as possible. This role of dF is illustrated with the next design of a network.)

It is very easy to determine that the above kind of network is not very efficient. The inefficiencies are created by adding one forms to $I^2{}_i$ that depend on the other unit's variables. Therefore, it is worth questioning what happens if the one-forms added at each stage are designed to avoid the other unit's variables for as long as possible. Namely, suppose for each $a < s$, $w^a{}_1$ depends only on $\mathbf{x}$ while $w^a{}_2$ depends only on the $\mathbf{y}$ variables. Because none of the added one-forms involve any of the other unit's variables, it is only necessary to show that $I^a{}_i$ is a differential ideal; the fact that $J^a{}_i$ is a differential ideal follows immediately. Moreover, the choice of the one-forms and the statement that each $I^a{}_i$ is a differential ideal guarantees that there are communication functions $g^a{}_1(\mathbf{x})$ and $g^a{}_2(\mathbf{y})$. The important fact is that these functions do not depend upon the communicated messages; they depend only upon the data available to each unit.

Suppose the s^{th} stage is the last step of the exchange of information; that is, $\beta - 1 = s$. This requires $I^s{}_1 = \langle d_x F, dg^1{}_1(\mathbf{x}), \ldots, dg^{s-1}{}_1(\mathbf{x}); dy_1, \ldots, dy_n \rangle$ and $J^s{}_1 = \langle d_x F, dg^1{}_1(\mathbf{x}), \ldots, dg^{s-1}{}_1(\mathbf{x}); dg^1{}_2(\mathbf{y}), \ldots, dg^{s-1}{}_2(\mathbf{y}) \rangle$. Again, $I^s{}_1$ is a differential ideal because it corresponds to the foliation given by the level sets of F, $\{g^a{}_1(\mathbf{x})\}_{a = 1, \ldots, s-1}$, $f_i = y_i$. The only part to verify is that $J^s{}_i$ is a differential ideal. This computation just involves showing that $d(d_x F) \wedge r \equiv 0$ where r is the wedge product of the basis one-forms defining $J^s{}_1$. As above, $d(d_x F)$ has two parts determined by the two sets of basis $\{dx_i \wedge dx_j\}_{i<j}$ and

$\{dx_i \wedge dy_j\}$. Denote them as $d_{xx}F$ and $d_{xy}F$.

The basic condition now becomes $d_{xx}F \wedge r + d_{xy}F \wedge r \equiv 0$. The first term is identically zero because I^s_1 is a differential ideal. (One could either use the fact that mixed partial derivatives are equal, or the fact that because I^s_1 is a differential ideal, $\{(d_{xx}F+d_{xy}F) \wedge d_xF \wedge dg^1_1(x) \wedge \ldots \wedge dg^{s-i}_1(x)\} \wedge \{dy_1 \wedge \ldots \wedge dy_n\} \equiv 0$. The last bracketed expression has the effect of annihilating all terms in the first bracket that involve a dy_i. The remaining terms have no dy_j forms, so $\{d_{xx}F \wedge d_xF \wedge dg^1_1(x) \wedge \ldots \wedge dg^{s-i}_1(x)\} \equiv 0$. But, this expression is part of the $d_{xx}F \wedge r$ computation.) Thus, it remains to show that $d_{xy}F \wedge r \equiv 0$.

To show when $d_{xy}F \wedge r \equiv 0$, note that the basis for the two-forms of this mixed type can be divided into four parts. First, take the space generated by the $\{dx_i\}$ and find another basis specified in two orthogonal parts - $P^1_1 = \{dg^{\alpha}_i(x)\}$ and $P^2_1 = \{t_{i,1}\}$. Likewise, do the same for the space generated by $\{dy_i\}$ where the division is $P^1_2 = \{dg^{\alpha}_2(y)\}$ and $P^2_2 = \{t_{i,2}\}$. The n^2 terms in the basis for the mixed two-forms is given by the wedge products of one-forms from one set with the other. Thus, any components of $d_{xy}F$ with a term in either P^1_i is annihilated. The $d_{xy}F$ terms that frustrate satisfying the differential condition are those expressed as a wedge product of forms from P^2_1 and P^2_2. By assumption (that the process now is complete and dF is in the last ideal) this cannot happen. Therefore, the $\{g^{\alpha}_i\}$ functions are to be chosed to avoid the possibility of $d_{xy}F$ having any terms in the product of the P^2_i spaces. Moreover, the choice of the g^{α}_i's should be made so that all of this is true for as small of a value of β as possible. As $d_{xy}F = \Sigma_i dx_i \wedge dy_i$, it is clear that all of this holds if the choice of the $\{dg^{\alpha}_1\}$ is such that it includes dx_i for half of the indices while the choice of the $\{dg^{\alpha}_2\}$ includes dy_j for the other half of the indices.

A communication network that satisfies the above conditions is $g^s_1(\mathbf{x}) = x_s = m^s_1$, $g^s_2(\mathbf{y}) = y_{n+1-s} = m^s_2$, $s = 1,\ldots,n/2 = \beta - 2$, $g^{\beta-1}_1(\mathbf{x},\mathbf{m}) = \Sigma_s x_{n+1-s} m^s_2 = m^{\beta-1}_1$, $g^{\beta-1}_2 \equiv 0$, $g^{\beta}_2(\mathbf{x},\mathbf{m}) = (\Sigma y_s m^s_1) + m^{\beta-1}_1$.

References.

1. Abelson, H., Lower bounds on information transfer in distributed computations, Jour. Assoc. for Computing Machinery, 27 (1980), 384-392.

2. Chen, F., Northwestern University PhD. Dissertation, June, 1989.

3. Hurwicz, L., Optimality and informational efficiency in resource allocation processes, in Arrow, Suppes, Karlin, ed. "Mathematical Methods in the Social Sciences" Stanford Univ. Press, 1960.

4. Hurwicz, L., S. Reiter, and D. G. Saari, On constructing mechanisms with message spaces of minimal dimensions for smooth performance functions, preprint, Conference Seminar on Decentralization, University of Minn, April, 1978.

5. Saari, D. G., A method for constructing message systems for smooth performance functions, Jour Econ Theory **33** (1984), 249-274.

6. Saari, D. G., On the types of information and mechanism design, Jour of Computational and Applied Mathematics **22** (1988), 231-242.

Discussion Paper No. 830

ON THE DESIGN OF COMPLEX ORGANIZATIONS AND DISTRIBUTIVE ALGORITHMS*

by Donald G. Saari
Department of Mathematics
Northwestern University
Evanston, Illinois 60208-2730

Abstract

To efficiently realize a specified goal in a distributive fashion, there needs to be an appropriate "division of labor." This is true for distributive algorithms that take advantage of the concurrent features of the new generation of computers. This is true in the design of a complex organization intended to realize a specified goal. The problem is to determine what is the appropriate division of labor. Here, a geometric characterization of all possible divisions of labor, or communication networks, is given. It is illustrated how this characterization can be used to design the communication networks.

---

*This research was supported, in part, by NSF grant # IRI-8803505 and by a 1988-89 Guggenheim Felloswhip.

There are striking similarities between the problem of designing distributive algorithms to take advantage of the concurrent and parallel features of the new generation of computers and the problem of designing an efficient organization to accomplish a specified goal. For both, the objective is to parcel the workload among the various participating units in an efficient, coordinated fashion. For instance, consider what is involved in creating a distributative algorithm. The main task is to determine what it is that each processor should compute and what partially computed information should be conveyed to which other processors. There is a similar problem for the design of an organization. Here responsibilities need to be assigned to the different departments and divisions; namely, the goal is to establish an organizational chart to determine the assignments and the reporting structure. Indeed, the design both of distributive algorithms and of organizations can be summarized with the coordinating questions of "who should do what?" and "who should say what to whom?"

For many situations, there exist algorithms and organizations that efficiently solve this division of labor problem. But, in general, the design of a system remains as an important open question. In all cases the purpose of an organization is to achieve a stated objective. So, the major obstacle is to understand how to start with the stated objectives and then extract from these goals the appropriate structures - structures that can be exploited to create the organization. The principal purpose of this paper is to attack this problem by developing a geometric characterization of this design problem. The geometric constructs introduced here expose the structures associated with the universal issues i) of determining *the kind of information* each unit needs to convey in order to achieve a stated objective and ii) of establishing *the reporting structure* of who reports what to whom. Because my emphasis is to introduce some of the underlying basic concepts, I treat here only a simplified model where I ignore the many other related problems. A more complete description is planned for elsewhere.

To state the problem in a simple setting, let the objective be given by the smooth function

1.1 $$F: R^{k(1)} \times \ldots \times R^{k(j)} \longrightarrow R$$

where $k(i)$, $i = 1,..,j$, are positive integers. Think of each space $R^{k(i)}$ as representing the data available to the i^{th} unit (processor, department, individual, agent, etc.). The function F represents the specified objectives. In a computational problem, F may be a function that is to be evaluated where the relevant data is divided so that processor i can access only the data represented in $R^{k(i)}$, $i = 1,..,j$. For a hypothetical organizational example, consider a firm trying to optimize profits coming from sales of a particular product. Let a vector in $R^{k(1)}$ represent data about potential markets, $R^{k(2)}$ represent data about costs and availability of raw materials needed for production, and $R^{k(3)}$ represent other technical variables. Let F represent either the optimal profits, or the output of the product that will achieve the maximal maximal profits with the current environment. The goal is to efficiently transfer information (or partial computations, partially constructed products, etc.) so that F is realized.

The objective function F specifies what is to be done - the goals. Once F is given, the object is to find the ways - the organizations - so that the task of realizing F is divided among the several cooperating units. To do this, I build upon the ideas of Abelson [1], Hurwicz [3] and others to model the flow of information among the units. The basic idea, which is a slight extension of Abelson's model, is simple and very natural. In the beginning, each unit has knowledge only of the data assigned to it; the i^{th} unit can only use the data from $R^{k(i)}$. This data must be processed in a manner that contributes toward realizing F. Represent this first step of computation by $g^1{}_i(x_i) = m^1{}_i$; $i = 1,..,j$, $x_i \in R^{k(i)}$, $m^1{}_i \in R$. Namely, at the first stage (denoted by the superscripts on g and m), the i^{th} unit uses the available data x_i to compute the value $m^1{}_i$. Of course, the choice of $g^i{}_1$ is intended so that the value $m^1{}_i$ contributes toward determining the value of $F(x_1,..,x_j)$. (In general it is not obvious how to define $g^1{}_i$; indeed, finding guidelines for an appropriate selection of these functions is major aspect of the *design* problem.) Let $\mathbf{m}^1 = (m^1{}_1,...,m^1{}_j) \in R^j$ denote the vector of the first stage computations.

At the second stage, each unit can use not only its assigned data, but also the partial computations, or *messages* $\mathbf{m}^1$, transmitted at the first stage. This means that the computations at the second stage can be denoted by $g^2{}_i(x_i,\mathbf{m}^1) = m^2{}_i \in R$. The general situation at the α^{th} stage is that the i^{th} unit can use all of the partial computations, or messages, from the other units as well as the original data x_i. Therefore the computation at this stage is represented by

1.2 $g^{\alpha}_{i}(x_i, \mathbf{m}^1, \ldots, \mathbf{m}^{\alpha-1}) = m^{\alpha}_{i}$;

i.e., this computation is represented by a function

1.3 $g^{\alpha}_{i}: R^{k(i)} \times (R^j)^{\alpha-1} \longrightarrow R$,

where $\mathbf{m}^k \in R^j$ is the vector of partial computations at the k^{th} step, $k = 1, \ldots, \alpha-1$.

At some step it may be that certain units have nothing to contribute or do. This is the situation if, for instance, a particular unit cannot proceed with meaningful work until it receives certain messages from specified other units. The above modeling admits such circumstances by defining the particular function to be $g^{\alpha}_{i} \equiv 0$.

Suppose it takes ß stages of partial computations to determine the value of F. I model this by assuming that all but one of the units complete their partial computations at the $(\beta-1)^{th}$ step. The remaining unit uses the messages of partial computations and its data to compute the value of F.[1] Namely, I assume there is a specific index s so that

1.4 $g^{\beta}_{s}(x_s, \mathbf{m}^1, \ldots, \mathbf{m}^{\beta-1}) = m^{\beta}_{s}$, $g^{\beta}_{j} \equiv 0$ for $j \neq s$,

where

1.5 $F(x_1, \ldots, x_j) = m^{\beta}_{s} = g^{\beta}_{s}(x_s, \mathbf{m}^1, \ldots, \mathbf{m}^{\beta-1})$.

Because at certain stages some of the units may not be transmitting a message, the effective messages - the images of the g functions - form only a linear subspace of $(R^j)^{\beta}$. Let M, the *message space*, denote this linear subspace.

With this model, the functions $\{g^{\alpha}_{i}\}$ specify what each unit must do, compute, and communicate at each stage. Because these functions determine "who says what to whom," I call a choice of smooth functions $\{g^{\alpha}_{i}\}$ that satisfies these conditions a ***communication network that realizes F.*** Furthermore, I call the issue of characterizing all possible communication networks that realize F the ***central design problem*** associated with F.

----

1. This approach, which follows Abelson, is reasonable for models of computation. An alternative model, with a slightly different supporting mathematical theory, is where each unit finishes its computations at the β^{th} stage. The final determination of F is based only on these messages. Thus, there is a function $h: M \longrightarrow R$ so that $h(\mathbf{m}^1, \ldots, \mathbf{m}^{\beta}) = F(x_1, \ldots, x_j)$. Here h may correspond to the "auctioneer," the central authority, the team leader, or a neutral computer. This alternative approach more closely represents several models from economics.

By solving or characterizing the solution of the central problem, all sorts of information may be available about the communication network functions $\{g^{\alpha}_{i}\}$. This information can be used to analytically compare competing communication networks, to develop complexity measures, and so forth. As an immediate observation, note that the value of β serves as a crude measure of the "speed" of the communication network. This is because it indicates that F is realized in β steps. Thus, there may be many situations whereby smaller values of β imply a more valued communication network.

One can conceive of situations where efficiency, or mininimal cost is determined by how much information needs to be transferred among the units. This is particularly so should it be expensive, or time consuming to transmit messages (or partial products, etc.). When this is the case, measures of complexity can be developed to reflect this fact. To see how this is done, suppose a communication network $\{g^{\alpha}_{i}\}$ is given and consider the reporting issue of determining "who says what to whom?" The function g^{α}_{i} represents what the i^{th} unit does at the α^{th} stage, so the dependency of this function on the m variables determines who has to communicate what partial computations to this unit. Namely, if for any choice of $s<\alpha$ and $k \neq i$, the partial derivative of g^{α}_{i} with respect to m^{s}_{k} is not identically zero, then the k^{th} unit needs to communicate this value to the i^{th} unit before the α^{th} stage.

As a third issue, note that it is of value to understand the "kind of information" associated with a communication network. (This is particularly true for theoretical investigations of a communication network.) By "kind of information," I mean an equivalence class of data that gives rise to the same value of each partial computation. In other words, starting with the given data, at each step each unit computes the value of a message, m^{α}_{i}. It may be that with a different choice of data, all of the messages are precisely the same. (If so, then both data points give rise to the same value of F.) So, all data giving rise to the same messages define the same kind of information. Thus the "kind of information" associated with a communication network is characterized by the level sets of g^{α}_{i}.

Definition. Let $\Gamma = \{g^{\alpha}_{i}\}_{\alpha = 1,\dots,\beta;\ i=1,\dots,j}$ be a given communication network that realizes F. We say that $x, x' \in R^{k(1)}x..xR^{k(j)}$ are "Γ equivalent" if the following holds: $g^{1}_{i}(x_{i}) = g^{1}_{i}(x')$ for all i. This requires the messages at the first stage to be the same. By induction, for all α, $g^{\alpha}_{i}(x_i, \mathbf{m}^1, .., \mathbf{m}^{\alpha-1}) = g^{\alpha}_{i}(x'_i, \mathbf{m}^1, .., \mathbf{m}^{\alpha-1})$. An equivalence class of data is called a "Γ information set."

As indicated, many of the basic issues for the design of algorithms or organizational structures can be characterized in terms of the properties of a communication network $\{g^{\alpha}_{i}\}$. However, it is not at all clear how to start with an objective function F and then determine an associated, non-trivial communication network. It would be useful to determine structures that would assist in this design. This goal, of finding what such a construction depends upon, is the basic theme of this current paper. I characterize the communication networks in terms of certain geometric constructs. As I indicated earlier, the purpose of these geometric properties is to expose the hidden, implicit structures of F that govern the admissible communication networks. This approach involves solving several equations; equations that need not be particularly easy to solve. On the other hand, these equations do indicate what must be done to achieve such a network. As such, they form a most useful place to start.

While my goal is to characterize all possible communication networks, I would like to call attention to the several clever arguments used to find properties of all possible communication networks without solving the central problem. In particular, I point to the paper by Abelson [1], where, for j = 2 (i.e., only two units are allowed) he introduces a complexity measure, the total information transfer, that is based on counting the number of messages that are required to be conveyed between the processors. Thus, in terms of the above discussion, this measure is determined by counting the non-zero partial derivatives of the communication network functions, $\{g^{\alpha}_{i}\}$, with respect to the $\mathbf{m}$ variables. As such, with the efficiency assumptions introduced in the next section, a lower bound for this measure is [dim(M) - 1] where the (-1) term corresponds to m^{β}_{s} - a message that is not transferred. (For $j \geq 3$, this may not be a sharp lower bound because the same message may be transferred to several units.) Abelson finds a lower bound for all possible communication networks strictly in terms of the rank properties of the Hessian of the objective function F. By using more sophisticated mathematical approach based on concepts from differential geometry, P. Chen [2] improves upon Abelson's lower bound; Chen's theorem is based on the rank of a bordered Hessian. Again, Chen's improved lower bound depends only on the differential properties of F; he circumvents the more difficult issue of solving the central problem.

2. Single Shot Mechanisms.

In this section, some insight is obtained about the kinds of information admitted by a specified F. I do this by showing that a communication network for F can be viewed as being a special case of a different kind of network that realizes F - the single shot mechanism. An important advantage of relating the two problems is that in this way I can exploit existing results characterizing all possible single shot mechanisms. This characterization can be used to impose bounds on what is possible for the associated communication networks, as well as to characterize the possible "kinds of information" admitted by the possible networks. Then, in Section 3, a characterization of the central problem is provided.

The more general system is where all of the information is communicated among the different units in a single step. For this to be possible, the values of $\mathbf{m}$ need to be determined implicitly. Thus, rather than communicating a value (as is true for a communication network), the i^{th} unit communicates a *set* $\{\mathbf{m} \mid G^{\alpha}_{i}(x_i, \mathbf{m}) = 0\}$. The actual message is the intersection of these sets, $i = 1,..,j$, in a message space M. Such systems occur quite naturally as part of the equilibrium analysis of a dynamical exchange of information that assumes the form $\mathbf{m}'_i = G^{\alpha}_{i}(\mathbf{x}_i, \mathbf{m})$. The basic purpose of the dynamic given by this differential equation is to allow each unit to update its message based on its own characteristics, $\mathbf{x}_i$, and the recent messages of the other units. The equilibrium state of the dynamic is where the G functions are all equal to zero. Notice that this modeling generalizes the common price dynamic story from economics where prices change according to the market pressures of supply and demand. For more detailed discussion of this and other interpretations, see Hurwicz [3].

Single Shot Problem: For a given objective function F, find smooth functions $G^{\alpha}_{i}(x_i, \mathbf{m}): R^{k(i)} \times M \to M$, $M = R^m$, $\alpha = 1,..,n_i$, $i = 1,..,j$; and a smooth function $h: M \to R$ so that with any value of $\mathbf{m}$ implicitly defined by

2.1 $G^{a}_{i}(x_i, m) = 0$,

we have that

2.2 $h(m) = F(x_1,..,x_j)$. The triple $(\{G^{a}_{i}\}, M, h)$ is called a single-shot mechanism that realizes F.

Thus the single shot mechanism corresponds to factoring a function F through another space, M, in a non-standard implicit form. Of course, the "kind of information" associated with a single shot mechanism $\{G^{a}_{i}\}$ is defined in a similar way as the Γ information sets - it is given by the level sets of the G^{a}_{i} functions. The relationship between the single shot and the central problem is stated in the following formal statement.

Theorem 1. If a function F admits a communication network, then this network defines a single shot mechanism, $\{G^{a}_{i}\}$, for F. The message space for both systems is the same. Moreover, an information set associated with this communication network is same information set associated with the defined mechanism $\{G^{a}_{i}\}$.

The proof of this theorem is immediate. This is because the communication network function, Eq. 1.2, can be expressed in the implicit single shot form $G^{a}_{i}(x_i, m) = 0$; $a = 1,..,\beta$; $i = 1,..,j$; where $m = (m^1, m^2,...,m^\beta) \in R^{\beta j} = M$ by defining $G^{a}_{i}(x_i, m) = g^{a}_{i}(x_i, m^1,...,m^{a-1}) - m^{a}_{i}$. The assertions of the theorem now follow immediately. Chen's Theorem is based on a similar observation.

An advantage of Theorem 1 is that there exists two characterization of the single shot mechanisms (Hurwicz, Reiter, and Saari [4], and Saari [5]). For the purposes of this paper, I adopt the characterization in Saari [5,6] because it is more general and it appears to be computationally easier to use. According to Theorem 1, this characterization can be invoked to limit the possible choices of the communication networks. This is because the communication networks are those single shot mechanisms that satisfy an additional rank condition.[2]

----

2. These rank conditions are the obvious ones required to take the equation for the single shot mechanism and solve them to obtain a communication network.

In general there are infinitely many choices of $\{G^{\alpha}_{i}\}$ functions that give rise to the same information sets.[3] However, a given set $\{G^{\alpha}_{i}\}$ can be pared to a basic set by eliminating redundancies. This is the purpose of the following set of efficiency assumptions. In these conditions, treat $\{G^{\alpha}_{i}\}$ as a mapping from $R^{k(1)}x..R^{k(j)}xM$ into an Euclidean space that agrees with the number of G^{α}_{i} functions.

Efficiency Assumptions on $\{G^{\alpha}_{i}\}$.

a. The dimension of M agrees with the number of $\{G^{\alpha}_{i}\}$ functions.

Let $X = (x_1,..,x_j)$ and ■ represent variables in a zero set of $\{G^{\alpha}_{i}\}$.

b. At (X,■) the Frechet derivative of $\{G^{\alpha}_{i}\}$ with respect to ■ is non-singular.

c. At (X,■) the Frechet derivative of $\{G^{\alpha}_{i}\}$ with respect to X has maximal rank.

(The Frechet derivative can be viewed as being the Jacobian of $\{G^{\alpha}_{i}\}$ with respect to the indicated variables.

The characterization of the single-shot mechanisms for a given F are expressed in a differential form. The idea is that the zero sets of the $\{G^{\alpha}_{i}\}$ functions define level sets, or certain collections of related foliations of the space $R^{k(1)}x..xR^{k(j)}$. Thus, the leaves from the foliations correspond to the kinds of information. Foliations can be totally characterized in terms of their normal vectors. These vectors define the normal bundle. When these vectors are expressed in terms of differential one-forms, the normal bundle becomes an ideal of differential forms. The necessary integrability conditions on the normal bundle now are expressed in terms of a condition on the ideal; it must be a differential ideal. These concepts lead to the following statement. For a proof, a discussion of these terms, and more details, along with a partial history of this problem see Saari [5].

----

3. This is why I place more emphasis on the "kinds of information" than on the actual single shot mechanisms or communication networks. In fact, a useful equivalence relationship can be defined among the mechanisms (the communication networks) in terms of these level sets. In this manner, networks that seem to have little to do with each other can be shown to be equivalent.

Theorem 2. Let a smooth objective function F be given. The following are necessary and sufficient conditions that a smooth single shot mechanism $(\{G^{\alpha}_i\},M,h)$ for F exists in a neighborhood of $X \in R^{k(1)}x..xR^{k(j)}$ that satisfies the efficiency assumptions.

1. For each i, there is a differential ideal $I_i = \langle dF, w_{i,1}, \ldots, w_{i,s(i)}; [dx_j]_i \rangle$, $s(i)=n_i-1$, with $(\Sigma_{j\neq i}k_j)+n_i$ linearly independent one-forms. Here, each $w_{i,s}$ is a smooth one form and the set $[dx_j]_i = \{dx_k \mid x_k$ is a coordinate direction for a parameter not in $R^{k(i)}\}$.

2. The set $I = \cap_i I_i$ is a differential ideal with $n = \Sigma_i n_i$ linearly independent one-forms.

The resulting mechanism to realize F has a message space of dimension n where there are n_i functions relating the parameters of the i^{th} unit with the messages.

The proof that this is a sufficient condition follows from the Frobenius Theorem (see Saari [5]). That this is a necessary condition comes by re-expressing the gradients $\{\nabla G^{\alpha}_i\}$ in terms of differential forms $\{dG^{\alpha}_i\}$. These one-forms form a basis for the differential ideals, $\{I_i\}_{i=1,\ldots,j}$, I, that have the specified properties. The reason the one-forms $[dx_j]_i$ are in I_i is to capture the condition that the i^{th} unit has access only to data from $R^{k(i)}$. The requirement on the ideal I is to ensure that the the conveyed messages are compatible with one another in evaluating F.

To illustrate how Theorem 1 can be used, notice that a trivial single shot mechanism is a "parameter transfer" where one unit communicates the value of all of its parameters to the second unit. After these values are transferred, the second unit computes the value of F. Namely, if $F(x_1,x_2):R^k x R^k \to R$, then $\beta = k+1$, and $G^s_1 = x_s - m^s_1 = 0$ where x_s is the s^{th} component of x_1, $s = 1,..,k$, while $G^1_2 = F((m^1_1,\ldots,m^k_1),x_2)-m^1_2 = 0$. the function h is the projection $h(m) = m^1_2$. This single shot mechanism has a message space M with $\dim(M) = k+1$. The communication network associated with the parameter transfer is $g^s_2 \equiv 0$, $g^s_1 = m^s_1$ where m^s_1 is the s^{th} component of x_1, $s = 1,..,k$, while $g^{\beta}_2 = F((m^1_1,\ldots,m^k_1),x_2)$ and $g^{k+1}_1 \equiv 0$. This communication network associated with the parameter transfer does not reflect the kind of benefits one expect from a system capable of concurrent or distributive action. After all, this system just transfers all of the work to another unit. Thus, such a communication network is one that is not overly efficient. Yet, suppose the only single shot mechanisms admitted by F are equivalent to a parameter transfer. It follows from Theorem 1 that all possible communication networks associated with F must be related to this undesired transfer method.

More generally, the class of all possible single-shot mechanisms that realize F restrict the kinds of communication networks that are associated with F. Thus, Theorems 1 and 2 form an important first step toward determining what kinds of networks are possible. In Saari [5,6], several examples of F are analyzed to characterize the associated single shot mechanisms. One example is repeated here to illustrate Theorem 2.

Example 1. Let $F: R^2 x R^2 \rightarrow R$ be defined by $F(\mathbf{x},\mathbf{y}) = \Sigma_i x_i y_i$. I show that this function admits only a parameter transfer. To do this, I first consider I_1. This set must contain $dF = \Sigma_i y_i dx_i + \Sigma_i x_i dy_i = d_x F + d_y F$. It also contains dy_1 and dy_2, as well as all possible linear combinations of these three one-forms where the coefficients are smooth functions of $\mathbf{x}$ and $\mathbf{y}$. As the second summation in dF, $d_y F$, can be expressed as combinations of dy_1 and dy_2, this part of dF can be eliminated. Thus, these forms can be reduced to the set $\{d_x F = \Sigma_i y_i dx_i, dy_1, dy_2\}$. If I_1 were to admit any other linearly independent one-form, then a basis for I_1 would be $\langle dx_1, dx_2, dy_1, dy_2\rangle$. The foliation identified with this ideal is given by the intersection of the level sets of x_i, y_i, $i = 1,2$. In other words, the messages are equivalent to the first unit transmitting the value of $\mathbf{x}$ to the second unit. This means that the accompanying mechanism is (equivalent to) a parameter transfer. Hence, assume that $I_1 = \langle d_x F = \Sigma_i y_i dx_i,$ $dy_1, dy_2\rangle$. A similar argument shows that to avoid a parameter transfer of the $\mathbf{y}$ values, $I_2 = \langle d_y = \Sigma_i x_i dy_i, dx_1, dx_2\rangle$. Consequently, $I = \langle \Sigma_i x_i dy_i, \Sigma_i y_i dx_i\rangle$.

It remains to determine whether I_1, I_2, and I are differential ideals. Trivially, I_1 and I_2 are differential ideals. One way to show this is to note that $r_1 = (\Sigma_i y_i dx_i) \wedge dy_1 \wedge dy_2$ is a three-form. A necessary and sufficient condition for I_1 to be a differential ideal is that $dw \wedge r_1 \equiv 0$ where w is any one form from I_1. But, $dw \wedge r$ is a five-form in a four dimensional space, so it must be identically zero.

An alternative argument proving that I_1 is a differential ideal uses the fact that this is so iff there is an associated foliation identified with I_1. This foliation is given by the intersection of the level sets (in $R^2 x R^2$) of F,

$f_1 = y_1$, and $f_2 = y_2$. A similar argument proves that I_2 also is a differential ideal.

The final step is to show that I is *not* a differential ideal. First, $r = (\Sigma_i y_i dx_i) \wedge (\Sigma_i x_i dy_i) \neq 0$ and $d(\Sigma_i x_i dy_i) = \Sigma_i dx_i \wedge dy_i$. A necessary and sufficient condition for I to be a differential ideal is that $d(\Sigma_i x_i dy_i) \wedge r \equiv 0$. However, a direct computation proves that $d(\Sigma_i x_i dy_i) \wedge r \neq 0$. Because I is not a differential ideal, there does not exist a single shot mechanism with $n_1 = n_2 = 1$. This means that any single shot mechanism associated with F must involve adding another independent one-form either to I_1 or to I_2, and, hence, to I. Suppose this one form is added to I_1. As shown above, the addition of this independent one-form makes $I_1 = \langle dx_1, dx_2, dy_1, dy_2 \rangle$. In turn, this means that the kind of information associated with the mechanism is equivalent to a parameter transfer of the **x** values to the other unit. Namely, for this choice of F, all single shot mechanisms are equivalent to the parameter transfer mechanism.

It now follows from Theorem 1 that the communication networks for this scalar product are equivalent to networks of the following form: Let ${g^s}_1 = x_s = {m^s}_1$, ${g^s}_2 \equiv 0$, $s = 1,2$, ${g^3}_2 = \Sigma_i {m^i}_1 y_i$, ${g^3}_1 \equiv 0$.

3. Characterization of the Communications Networks.

The characterization of communication networks also can be expressed in terms of differential ideals, except several more ideals are required. These additional ideals account for the rank conditions needed to ensure that the equations for a single shot mechanism can be solved to determine the associated communication network. Again, for any F, there are an infinite number of associated communication networks, so the first task is to eliminate certain redundancies. As in the previous section, this is done by imposing efficiency assumptions. In these conditions, consider only the non-constant functions in $\{{g^a}_i\}$ and treat the remaining functions as defining a mapping.

Efficiency Assumptions on a Communication Network $\{{g^a}_i\}$.

a. The dimension of M agrees with the number of non-constant $\{{g^a}_i\}$ functions.

Let $X = (x_1, .., x_j)$ and $\mathbf{m}$ represent variables in a zero set of $\{{g^a}_i\}$.

b. At $(X, \mathbf{m})$ the Frechet derivative of the non-constant $\{{g^a}_i\}$ with respect to $\mathbf{m}$ is non-singular.

c. At (X,m) the Frechet derivative of the non-constant $\{g^{\alpha}{}_i\}$ with respect to X has maximal rank.

Theorem 3. Let a smooth objective function F be given. The following are necessary and sufficient conditions that a smooth communication network $\{g^{\alpha}{}_i\}$ that satisfies the efficiency assumptions exists in a neighborhood of $X \in R^{k(1)}x..xR^{k(j)}$.

1. For each i, there is a differential ideal $I^1{}_i = \langle w^i{}_i,;[dx_j]_i\rangle$ where the one-form $w^1{}_i \in T^*R^{k(i)}$; i.e., it is a linear combination of the differentials of the coordinate functions in $R^{k(i)}$ where the scalar functions are smooth functions from $R^{k(i)}$ to R.

2. By induction, for each i and each α satisfying $1<\alpha<\beta$, there is a one form $w^{\alpha}{}_i$ so that $I^{\alpha}{}_i = \langle w^{\alpha}{}_i, I^{\alpha-1}{}_i\rangle$ is a differential ideal. Secondly, for all i with the exception of an index s, $I^{\beta}{}_i = I^{\beta-1}{}_i$. In the exceptional case of i = s, there can be a one-form $w^{\beta}{}_s$ so that $I^{\beta}{}_s = \langle w^{\beta}{}_s, I^{\beta-1}{}_s\rangle$ is a differential ideal. For all i, $dF \in I^{\beta}{}_i$.

3. For all i and all α satisfying $1<\alpha<\beta$ and for $\alpha = \beta$ when i = s, all of the ideals $J^{\alpha}{}_i = I^{\alpha}{}_i \cap (\cap_{k\neq i} I^{\alpha-1}{}_k)$ are differential ideals.

The resulting communication network takes β steps and the dimension of the message space corresponds to the dimension of $J^{\beta}{}_s$.

The proof of this theorem will appear elsewhere. Some of the connections between Theorems 2 and 3 are that i) the ideal $J^{\beta}{}_s$ from Theorem 3 plays the role of the ideal I in Theorem 2 while ii) the ideals $I^{\beta}{}_i$ from Theorem 3 correspond to the ideals I_i from Theorem 2. The remaining ideals correspond to the added conditions required to ensure that a single shot mechanism can be expressed in the form of a communication network. Notice that the conditions on the ideals for the first stage, $I^1{}_i$, amount to choosing a one-form $w^1{}_i$ to be a functional multiple of $dg^1{}_i(x_i)$. It is not obvious how to choose the functions $g^1{}_i(x_i)$. Therefore it is iinteresting to note, as illustrated in the following examples, that this choice is partially governed by the conditions on the $J^2{}_i$ ideals as well as the other conditions from Theorem 3. While the resulting set of equations may be difficult to solve, this approach does provide additional structure to understand how to decompose F into an organizational format. Finally, notice that because the dimension of $J^{\beta}{}_s$ agrees with the dimension of

M, the structure of J^{β}_{s} provides valued information about Abelson's total information transfer.

The differences between Theorem 2 and 3, as well as an indication how to use these results, is illustrated with the following examples. The first one, Example 2, shows that not all single shot mechanisms are related to a communication network.

Example 2. Let $F:R^2 x R^2 \dashrightarrow R$ be defined as $F(x,\mathbf{y}) = [x_1y_2+x_2]/[1-x_1y_1]$. I show that F admits a (1,1) single shot mechanism; that is, there is a single shot mechanism with $n_1=n_2=1$. This conclusion is by no means obvious. What is even less obvious is how to decompose F into the appropriate messages from the two units. Therefore, it is worth noting how the structures of the ideals lead to the resulting mechanism.

If F admits a (1,1) mechanism, then I_1 must be $\langle d_xF;dy_1,dy_2\rangle$ and I_2 = $\langle d_yF;dx_1,dx_2\rangle$ where, as in Example 1, d_xF and d_yF are, respectively, the part of dF that has only dx_j differentials and dy_j differentials. If $w_1 = (1-x_1y_1)^2d_xF$ and $w_2 = [(1-x_1y_1)^2/x_1]d_yF$, then $I_1 = \langle w_1,dy_1,dy_2\rangle$, $I_2 = \langle w_2,dx_1,dx_2\rangle$, $I = \langle w_1,w_2\rangle$, $w_1 = (y_2+x_2y_1)dx_1 + (1 - x_1y_1)dx_2$, and $w_2 = (x_1y_2+x_2)dy_1 + (1 - x_1y_1)dy_2$. By using argument similar to those found in Example 1, it follows that I_1 and I_2 are differential ideals. Thus it suffices to show that I is a differential ideal.

The ideal I is a differential ideal with dimension two iff $r = w_1 \wedge w_2 \neq 0$ and both $dw_1 \wedge r$ and $dw_2 \wedge r$ are identically zero. But, because $d(d_xF) = -d(d_yF)$, it follows that I is a differential ideal if $r \neq 0$ and $dw_1 \wedge r \equiv 0$. A computation shows that $r = (y_2+x_2y_1)(x_1y_2+x_2)dx_1 \wedge dy_1 + (y_2+x_2y_1)(1-x_1y_1)dx_1 \wedge dy_2 + (1-x_1y_1)(x_1y_2+x_2)dx_2 \wedge dy_1 + (1-x_1y_1)^2dx_2 \wedge dy_2$ and $dw_1 = -dx_1 \wedge dy_2 - 2y_1dx_1 \wedge dx_2 - x_2dx_1 \wedge dy_1 + x_1dx_2 \wedge dy_1$. It is clear that $r \not\equiv 0$. A direct computation proves that $dw_1 \wedge r \equiv 0$. This establishes that I is a differential ideal, so it also follows (from Theorem 2) that there does exist a (1,1) single shot mechanism that realizes F.

By following the scheme described in Saari [5,6], the single shot mechanism given by the G^{α}_{i} functions can be determined. One choice is

3.1 $$G^1_1 = x_1m_1 + x_2 - m_2 = 0,$$
$$G^1_2 = y_1m_2 + y_2 - m_1 = 0.$$

In other words, for this single shot mechanism, each unit transmits a line. In $M = R^2$, these two lines intersect in a unique point; this point is the equilibrium value of $\mathbf{m} = (m_1, m_2)$. The function $h: M = R^2 \to R$ is given by $h(\mathbf{m}) = m_2$.

Now consider all of the communication networks associated with F. Each choice of a network specifies the particular unit that is charged with computing the value of F at the β^{th} step. Secondly, to start the computation process, at least one of the two units must make an initial partial computation; i.e., at least one $dg^1{}_i \not\equiv 0$. This requires either $I^1{}_1$ or $I^1{}_2$ to have a one-form in addition to the one-forms corresponding to the other units coordinate functions. This one-form characterizes the initial computation step. So, assume that the second unit is to determine the value of F and that $I^1{}_1$ has an independent one-form other than dy_1 and dy_2. (All other cases have a similar argument.) The integrability conditions force this one-form in $I^1{}_1$ to be a scalar function multiple of the differential of a function $g^1{}_1(\mathbf{x})$. It follows immediately from the form of $d_{\mathbf{x}}F$ that there does not exist a function $\tau(\mathbf{x},\mathbf{y})$ so that the coefficients of $\tau(\mathbf{x},\mathbf{y})d_{\mathbf{x}}F$ are strictly functions of $\mathbf{x}$. Consequently, both $d_{\mathbf{x}}F$ and $dg^1{}_1(\mathbf{x})$ are in $I^{\beta-1}{}_1$, and they are linearly independent. This forces $I^{\beta-1}{}_1 = \langle dx_1, dx_2; dy_1, dy_2\rangle$. In turn, this means that the kind of information associated with any communication network must be equivalent to a parameter transfer, so $\beta = 3$. One such network is $g^i{}_1(\mathbf{x}) = x_i = m^i{}_1$, $g^i{}_2(\mathbf{y}) \equiv 0$; $i = 1,2$; while $g^3{}_2 = F((m^1{}_1, m^2{}_1), \mathbf{y})$, and $g^3{}_1 \equiv 0$. In other words, even though the above single shot mechanism provides a distributive way to code information about F that results in a saving over the parameter transfer, such economies do not extend or exist for any of the communication networks associated with F.

The total information transfer is 2; the first unit transfers $m^1{}_1$ and $m^2{}_i$ to the second unit. That it is impossible to find a communication network that improves upon the above constructed one for F follows either from the above analysis or from Chen's theorem. Chen's result shows that the lower bound for information transfer for this choice of F is 2.

Example 3. Abelson uses the following function F to illustrate certain features of a communication process. Chen uses the same F to illustrate that his lower bound (of 3) improves upon Abelson's. I use this F to illustrate how Theorem 3 can be used to determine a communication network.

Let $F: R^n \times R^n \to R$ be $F(\mathbf{x},\mathbf{y}) = \Sigma_s x_s (y_1)^s + \Sigma_s y_s (x_1)^s$. A direct computation shows that $dF = \Sigma_s (y_1)^s dx_s + (\Sigma_s s y_s (x_1)^{s-1}) dx_1 +$

$\Sigma_s(x_1)^s dy_s$ $(\Sigma_s s(y_1)^{s-1}x_s)dy_1$. At the first stage, $I^1{}_1 = \langle dg^1{}_1(\mathbf{x});dy_1,..,dy_n\rangle$ and $I^2{}_1 = \langle dg^1{}_2(\mathbf{y}); dx_1,..,dx_n\rangle$. The choice of the functions $g^1{}_i$ is not obvious. What is interesting is that the choice of the functions is determined at the second step by the structure of the ideals given in Theorem 3.

It is clear that there must be at least one more stage. If not, then to satisfy condition 2 of Theorem 3, either d_xF must be a scalar (function) multiple of $dg^1{}_1(\mathbf{x})$, or d_yF must be a scalar multiple of $dg^1{}_2(\mathbf{y})$. Because of the mixed x_i, y_j form of the coordinate functions in these two differentials, neither is possible. If only one additional stage is required before the value of F can be computed by, say, unit 2, then $I^2{}_1 = \langle dg^1{}_1, dF; dy_1,..,dy_n\rangle$ and $J^2{}_1 = \langle dg^1{}_1, dg^1{}_2, d_xF\rangle$.

The ideal $I^2{}_1$ is a differential ideal because it describes the foliation given by the intersection of the level sets of $g^1{}_1$, F, and $f_s(\mathbf{x},\mathbf{y}) = y_s$, $s = 1,..,n$. On the other hand, $J^2{}_1$ is a differential ideal iff $d(d_xF)\wedge r \equiv 0$ where $r = dg^1{}_1\wedge dg^1{}_2\wedge d_xF \not\equiv 0$. (This is because $d(dg^1{}_i) \equiv 0$ for $i = 1,2$.) As $d(d_xF) = -(\Sigma s(y_1)^{s-1}dx_s)\wedge dy_1 + (\Sigma s(x_1)^{s-1}dy_s)\wedge dx_1$, it is easy to see that a necessary and sufficient condition for $J^2{}_1$ to be a differential ideal is that $dg^1{}_1(\mathbf{x})$ and $dg^1{}_2(\mathbf{y})$ are, respectively, scalar function multiples of dx_1 and dy_1. From this, following the scheme described in Saari [5], a communication network can be constructed. Namely, $g^1{}_1(\mathbf{x}) = x_1 = m^1{}_1$, $g^1{}_2(\mathbf{y}) = y_1 = m^1{}_2$, $g^2{}_1 = \Sigma_s x_s(m^1{}_2)^s = m^2{}_1$, $g^2{}_2 \equiv 0$, and $g^3{}_2 = m^2{}_1 + \Sigma_s y_s(m^1{}_1)^s = m^3{}_2 = F(x,\mathbf{y})$.

Example 4. A very simple example is $F:R^k xR^k \rightarrow R$ given by $F(\mathbf{x},\mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$ where f and g are smooth functions. An obvious communication network is $m^1{}_1 = f(\mathbf{x})$ and $F(\mathbf{x},\mathbf{y}) = m^2{}_2 = m^1{}_1 g(\mathbf{y})$. I show how this network arises out of Theorem 3. First of all note that to minimize the value of ß, the goal is to choose communication functions that will permit dF to be in an ideal as soon as possible. Therfore, we check to see if it is possible for $dF \in I^1{}_1$. This is true because $d_xF = g(\mathbf{y})df(\mathbf{x})$, so it is in the ideal $\langle df(\mathbf{x});dy_1,..,dy_k\rangle$. The described message system follows immediately.

Example 5. As a final example, I consider $F:R^n xR^n \rightarrow R$ that is given by the scalar product; $F(\mathbf{x},\mathbf{y}) = \Sigma_s x_s y_s$. According to both Abelson's and Chen's Theorems, the total information transfer must be at least n - the same as for a parameter transfer. However, a parameter transfer requires ß = n + 1.

Therefore, it is worth questioning whether F admits communication networks other than the parameter transfer that permit $\beta < n + 1$. The best one can do is if at each stage, each unit transfers a message to the other unit. If this transfer is done efficiently, then the network would require $(\beta-1) = n/2$. (Recall, there is no transfer of information at the β^{th} step; this is the stage where the value of F is computed.) To find efficient networks is easy. However, I use this simple choice of F to illustrate how the structures of Theorem 3 help to design communication networks. (The analysis also shows what other methods are, or are not possible.) Because I am using F to illustrate the use of the above theorems, my description is phrased in a general fashion so that one can extend the notions to other choices of F.

At the first stage, $I^1{}_1 = \langle dg^1{}_1(\mathbf{x}); dy_1,..,dy_n\rangle$ and $I^1{}_2 = \langle dg^1{}_2(\mathbf{y}); dx_1,..,dx_n\rangle$. As true with the earlier examples given above, while the choice of the functions $\{g^1{}_k\}$ is not obvious, assistance for the choice of these functions is provided by the structure of $J^{\alpha}{}_i$ for $\alpha\geq 2$. I will show how this happens in different ways. For my first choice, I consider what manner of conditions for the ideals lead to the following kind of communication network: At the first stage, the first unit communicates the value of x_1 while the second communicates the value of y_n. At the second stage, the first unit computes and transmits the value of $x_n y_n$ (based on the message it received) while the second unit transmits the value of $x_1 y_1$. The process continues.

To see how the above kind of network arises, consider what happens should a one-form $w^2{}_1(\mathbf{x},\mathbf{y}) = \Sigma\theta_i(x,y)dx_i$ be added to $I^2{}_1$ where at least one of the θ_i functions does depend on the $\mathbf{y}$ variable. The first condition is that $I^2{}_1$ is a differential ideal. This involves showing that $dw^2{}_1 \wedge r \equiv 0$ where r is the (n+2)-form $dg^1{}_1 \wedge w^2{}_1 \wedge [dy_1 \wedge \ldots \wedge dy_n]$. The $dw^2{}_1$ term can be expressed as $d_x w^2{}_1 + d_y w^2{}_1$ where the first terms comes from the partial derivatives of the x variables while the second comes from the partial derivatives with respect to the y variables. The bracketed term in r annihilates the $d_y w^2{}_1$ contribution, so all that remains is that $d_x w^2{}_1 \wedge w^2{}_1 \wedge dg^1{}_1 \equiv 0$. This is guaranteed for $w^2{}_1$ being the x-part of the differential of any function $H(\mathbf{x},\mathbf{y})$; i.e., $w^2{}_1 = d_x H(\mathbf{x},\mathbf{y})$. Assume this is the case where, of course, the choice of H is to be determined.

The second part of the $\alpha = 2$ stage is to show that $J^2{}_1 = \langle d_x H(\mathbf{x},\mathbf{y}), dg^1{}_1(\mathbf{x}), dg^1{}_2(\mathbf{y})\rangle$ is a differential ideal. The only thing that needs to be done here is to show that $d(d_x H) \wedge [d_x H \wedge dg^1{}_1(\mathbf{x}) \wedge dg^1{}_2(\mathbf{y})] \equiv 0$. As I have already shown above that $d_x w^2{}_1 \wedge w^2{}_1 \wedge dg^1{}_1 \equiv 0$, so it remains to show that $d_y w^2{}_1 \wedge dg^1{}_1 \wedge dg^1{}_2 \wedge w^2{}_1 \equiv 0$. But $d_y w^2{}_1$ is in the space spanned by $\{dx_i \wedge dy_j\}$. Another basis can be given by the wedge product of the $\{dx_i\}$ terms with the orthogonal basis $\{dg^1{}_2(\mathbf{y}), \tau_i(\mathbf{y})\}$. Therefore $d_y w^2{}_1$ can be expressed as a linear (with scalar functions as coefficients) combination of $\{dg^1{}_2(\mathbf{y}) \wedge dx_i, \tau_i(\mathbf{y}) \wedge dx_j\}$. If $d_y w^2$ admits any terms of the form $\tau_i(\mathbf{y}) \wedge dx_j$ (but not of the form $\tau_i(\mathbf{y}) \wedge dg^1{}_1$ or $\tau_i(\mathbf{y}) \wedge d_x H(\mathbf{x},\mathbf{y})$) then the differential ideal condition will not be satisfied. This means that the "y" part of $H(\mathbf{x},\mathbf{y})$ must depend upon the message $g^2{}_1 = \mathbf{m}^1{}_2$. One choice is if $g^2{}_1 = y_n$, then $H(\mathbf{x},\mathbf{y}) = x_n y_n$. (There are many other choices, such as counterproductive choices of $x_1 y_n$. However, such choices are quickly excluded at the β^{th} step when dF must be in all ideals. Indeed, the object in the design of the $g^\alpha{}_i$ functions is to include dF in each of the ideals as quickly, or as efficiently as possible. This role of dF is illustrated with the next design of a network.)

It is very easy to determine that the above kind of network is not very efficient. The inefficiencies are created by adding one forms to $I^2{}_i$ that depend on the other unit's variables. Therefore, it is worth questioning what happens if the one-forms added at each stage are designed to avoid the other unit's variables for as long as possible. Namely, suppose for each $\alpha < s$, $w^\alpha{}_1$ depends only on $\mathbf{x}$ while $w^\alpha{}_2$ depends only on the $\mathbf{y}$ variables. Because none of the added one-forms involve any of the other unit's variables, it is only necessary to show that $I^\alpha{}_i$ is a differential ideal; the fact that $J^\alpha{}_i$ is a differential ideal follows immediately. Moreover, the choice of the one-forms and the statement that each $I^\alpha{}_i$ is a differential ideal guarantees that there are communication functions $g^\alpha{}_1(\mathbf{x})$ and $g^\alpha{}_2(\mathbf{y})$. The important fact is that these functions do not depend upon the communicated messages; they depend only upon the data available to each unit.

Suppose the s^{th} stage is the last step of the exchange of information; that is, $\beta-1 = s$. This requires $I^s{}_1 = \langle d_x F, dg^1{}_1(\mathbf{x}), \ldots, dg^{s-1}{}_1(\mathbf{x}); dy_1, \ldots, dy_n \rangle$ and $J^s{}_1 = \langle d_x F, dg^1{}_1(\mathbf{x}), \ldots, dg^{s-1}{}_1(\mathbf{x}); dg^1{}_2(\mathbf{y}), \ldots, dg^{s-1}{}_2(\mathbf{y}) \rangle$. Again, $I^s{}_1$ is a differential ideal because it corresponds to the foliation given by the level sets of F, $\{g^\alpha{}_1(\mathbf{x})\}_{\alpha = 1,\ldots,s-1}$, $f_i = y_i$. The only part to verify is that $J^s{}_1$ is a differential ideal. This computation just involves showing that $d(d_x F) \wedge r \equiv 0$ where r is the wedge product of the basis one-forms defining $J^s{}_1$. As above, $d(d_x F)$ has two parts determined by the two sets of basis $\{dx_i \wedge dx_j\}_{i<j}$ and

$\{dx_i \wedge dy_j\}$. Denote them as $d_{xx}F$ and $d_{xy}F$.

The basic condition now becomes $d_{xx}F \wedge r + d_{xy}F \wedge r \equiv 0$. The first term is identically zero because $I^s{}_1$ is a differential ideal. (One could either use the fact that mixed partial derivatives are equal, or the fact that because $I^s{}_1$ is a differential ideal, $[(d_{xx}F+d_{xy}F) \wedge d_xF \wedge dg^1{}_1(\mathbf{x}) \wedge \ldots \wedge dg^{s-1}{}_1(\mathbf{x})] \wedge [dy_1 \wedge \ldots \wedge dy_n] \equiv 0$. The last bracketed expression has the effect of annihilating all terms in the first bracket that involve a dy_i. The remaining terms have no dy_i forms, so $[d_{xx}F \wedge d_xF \wedge dg^1{}_1(\mathbf{x}) \wedge \ldots \wedge dg^{s-1}{}_1(\mathbf{x})] \equiv 0$. But, this expression is part of the $d_{xx}F \wedge r$ computation.) Thus, it remains to show that $d_{xy}F \wedge r \equiv 0$.

To show when $d_{xy}F \wedge r \equiv 0$, note that the basis for the two-forms of this mixed type can be divided into four parts. First, take the space generated by the $\{dx_i\}$ and find another basis specified in two orthogonal parts - $P^1{}_1 = \{dg^\alpha{}_1(\mathbf{x})\}$ and $P^2{}_1 = \{\tau_{i,1}\}$. Likewise, do the same for the space generated by $\{dy_i\}$ where the division is $P^1{}_2 = \{dg^\alpha{}_2(\mathbf{y})\}$ and $P^2{}_2 = \{\tau_{i,2}\}$. The n^2 terms in the basis for the mixed two-forms is given by the wedge products of one-forms from one set with the other. Thus, any components of $d_{xy}F$ with a term in either $P^1{}_j$ is annihilated. The $d_{xy}F$ terms that frustrate satisfying the differential condition are those expressed as a wedge product of forms from $P^2{}_1$ and $P^2{}_2$. By assumption (that the process now is complete and dF is in the last ideal) this cannot happen. Therefore, the $\{g^\alpha{}_i\}$ functions are to be chosed to avoid the possibility of $d_{xy}F$ having any terms in the product of the $P^2{}_i$ spaces. Moreover, the choice of the $g^\alpha{}_i$'s should be made so that all of this is true for as small of a value of β as possible. As $d_{xy}F = \Sigma_i dx_i \wedge dy_i$, it is clear that all of this holds if the choice of the $\{dg^\alpha{}_1\}$ is such that it includes dx_i for half of the indices while the choice of the $\{dg^\alpha{}_2\}$ includes dy_j for the other half of the indices.

A communication network that satisfies the above conditions is $g^s{}_1(\mathbf{x}) = x_s = m^s{}_1$, $g^s{}_2(\mathbf{y}) = y_{n+1-s} = m^s{}_2$, $s = 1,..,n/2 = \beta - 2$, $g^{\beta-1}{}_1(\mathbf{x},\mathbf{m}) = \Sigma_s x_{n+1-s} m^s{}_2 = m^{\beta-1}{}_1$, $g^{\beta-1}{}_2 \equiv 0$, $g^\beta{}_2(\mathbf{x},\mathbf{m}) = (\Sigma y_s m^s{}_1) + m^{\beta-1}{}_1$.

References.

1. Abelson, H., Lower bounds on information transfer in distributed computations, Jour. Assoc. for Computing Machinery, 27 (1980), 384-392.

2. Chen, P., Northwestern University PhD. Dissertation, June, 1989.

3. Hurwicz, L., Optimality and informational efficiency in resource allocation processes, in Arrow, Suppes, Karlin, ed. "Mathematical Methods in the Social Sciences" Stanford Univ. Press, 1960.

4. Hurwicz, L., S. Reiter, and D. G. Saari, On constructing mechanisms with message spaces of minimal dimensions for smooth performance functions, preprint. Conference Seminar on Decentralization, University of Minn. April, 1978.

5. Saari, D. G., A method for constructing message systems for smooth performance functions, Jour Econ Theory **33** (1984), 249-274.

6. Saari, D. G., On the types of information and mechanism design, Jour of Computational and Applied Mathematics **22** (1988), 231-242.