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HYPERSPACES OF TOPOLOGICAL VECTOR SPACES:
THEIR EMBEDDING IN TOPOLOGICAL VECTOR SPACES

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Prem Prakash and Murat E. Sertel

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We prove the following

G. HAUPTSATZ: Let \( L \) be a real (Hausdorff) topological vector space. The space \( K[L] \) of nonempty compact subsets of \( L \) forms a (Hausdorff) topological semivector space with singleton origin when \( K[L] \) is given the uniform (equivalently, the finite) hyperspace topology determined by \( L \). Then \( K[L] \) is locally compact iff \( L \) is so. Furthermore, \( KQ[L] \), the set of nonempty compact convex subsets of \( L \), is the largest pointwise convex subset of \( K[L] \) and is a cancellative topological semivector space. For any nonempty compact and convex set \( \mathbf{X} \subseteq L \), the collection \( KQ[\mathbf{X}] \subseteq KQ[L] \) is nonempty compact and convex. \( L \) is isomorphically embeddable in \( KQ[L] \) and, in turn, there is a smallest vector space \( \mathbf{L} \) in which \( KQ[L] \) is algebraically embeddable (as a cone). Furthermore, \( L \) can be given a vector topology \( \mathbf{T} \) such that the algebraic embedding of \( KQ[L] \) in \( L \) is an isomorphism, while \( L \) is, respectively, locally convex/
normable accordingly as \( \mathcal{L} \) is so; indeed, \( T \) can be so chosen that, when \( \mathcal{L} \) is normed, the embedding of \( \mathcal{L} \) in \( \mathcal{K}\mathcal{O}[\mathcal{L}] \) and that of \( \mathcal{K}\mathcal{O}[\mathcal{L}] \) in \( \mathcal{L} \) are both isometries.
REFERENCES


1. PRELIMINARIES

$\mathbb{R}$ denotes the set of real numbers with the usual topology, and $\mathbb{R}_+ = \{ \lambda \in \mathbb{R} \mid \lambda \geq 0 \}$. For any set $X$, $[X]$ denotes the set of nonempty subsets of $X$. When $X$ is a topological space, $K[X]$ denotes the set of compact nonempty subsets of $X$. When $X$ lies in a real vector space, $Q[X]$ denotes the set of convex nonempty subsets of $X$. Finally, when $X$ lies in a real topological vector space, $KQ[X] = K[X] \cap Q[X]$.

In topologizing hyperspaces (i.e., spaces of subsets), we will use the uniform topology, regarding which we adopt Michael [1] as standard reference. Let $X$ be a uniform space, and let $\{ E_{\alpha} \subseteq X \times X \mid \alpha \in A \}$ be a fundamental system of symmetric entourages of $X$. The uniform topology for $[X]$ is the topology generated by declaring $E_{\alpha}[A]$ is a nbhd of $A$ ($A \subseteq [X]$). By the uniform topology on a hyperspace $H[X] \subseteq [X]$ is meant the relative topology of $H[X]$ when $[X]$ carries the uniform topology.

1.0 DEFINITION [2]: Let $(S, \oplus)$ be a commutative semi-group and $\Upsilon: \mathbb{R}_+ \times S \to S$ a map such that, denoting $\Upsilon(\lambda, s) = \lambda s$, 


\[ \lambda(\mu s) = (\lambda \cdot \mu) s \]  
\[ ls = s \]  
\[ \lambda(s \otimes t) = \lambda s \otimes \lambda t \]  

(Left action)  
(Unitariness)  
(Homomorphism)

for all \( \lambda, \mu \in \mathbb{R} \) and \( s, t \in S \). We call \( S \) a semivector space. When \( S \) is a Hausdorff space and the operations \( \cdot \) and \( \otimes \) are both continuous, we call \( S \) a topological semivector space.

Thus, real vector spaces are all semivector spaces, so that the topological vector spaces we speak of are those with Hausdorff topology.
2. **SEMICOMPACT SPACES OF TOPOLOGICAL VECTOR SPACES**

Let \( L \) be a real vector space, and \( e \) its identity element. Now \([L]\) is a semivector space with identity \( (e) \) when \( A \odot B = \{a + b \mid a \in A, b \in B\} \) and \( \lambda A = \{\lambda a \mid a \in A\}, \lambda \in \mathbb{R} \). Furthermore, \( \mathbb{Q}[L] \subseteq [L] \) is also a semivector space and is pointwise convex, i.e., \( \{A\} \) is convex for each \( A \in \mathbb{Q}[L] \). In fact \( \mathbb{Q}[L] \) is the largest pointwise convex subset of \([L] \). If \( A \in [L] \) and \( \lambda A \odot \lambda' A \subseteq A \) for each \( \lambda = (1-\lambda') \in [0, 1] \), then \( A \subseteq L \) must be convex.

From here on, \( L \) will always be a topological vector space.

Now \( K[L] \subseteq [L] \) is a semivector subspace and \( K\mathbb{Q}[L] \) is the largest pointwise convex semivector subspace of \( K[L] \). Also, the origin \( 0[L] = 0K[L] = 0\mathbb{Q}[L] = 0K\mathbb{Q}[L] = \{\{e\}\} \) is singleton. N.B.: The uniform topology on \( K[L] \) coincides with the finite topology (1.1, pp. 153, and 3.3 pp. 160, of [1]).

2.1 **PROPOSITION:** (1) \( K[L] \) is a topological semivector space, locally compact iff \( L \) is. (2) The map \( \hat{\delta} : x \mapsto \{x\} \) \( (x \in L) \) isomorphically embeds \( L \) into the topological semivector subspace \( K\mathbb{Q}[L] \subseteq K[L] \).
Proof: (ad (1)): \( K[L] \) is Hausdorff as \( L \) is (see 4.9.8, pp. 164 of [1]), and will be locally compact iff \( L \) is locally compact (see 4.9.12, pp. 14 of [1]). This leaves only the continuity of the operations \( \oplus \) and \( \otimes \) of \( K[L] \) to show. The continuity of vector addition \( \oplus: L \times L \to L \) implies the continuity of the map \( \hat{\oplus}: \{L \times L\} \to \{L\} \) defined by \( \hat{\oplus}(P) = (a \oplus b | a, b \in P) \) (see 5.9.1, pp. 169 of [1]). Thus, the restriction of \( \hat{\oplus} \) to the space \( B = \{C \times D | C, D \in K[L]\} \subseteq K[L \times L] \) of compact boxes is also continuous. Furthermore, the Cartesian product \( \pi(C, D) = C \times D \) is continuous on \( K[L] \times K[L] \to B \) (see Theorem 3 of [3]). Now \( \oplus \) is simply the composition \( \oplus = \hat{\oplus} \circ \lambda(L) \times K[L] \to K[L] \), and so is continuous. Similarly, the continuity of scalar multiplication \( R_+ \times L \to L \) implies that of scalar multiplication \( \otimes: R_+ \times K[L] \to K[L] \).

(ad (2)): From (1) it follows that the space \( KQ[L] \subseteq K[L] \) is a topological semivector space. Now the map \( \lambda \) is a homeomorphism (2, pp. 155 of [1]) and is easily checked to be a homomorphism. \( \Diamond \)

2.3 Proposition: \( KQ[L] \) is cancellative (i.e., \( A \oplus B = A \oplus C \implies B = C \)) and \( A \otimes B \subseteq A \otimes C \implies B \subseteq C \)

\((A, B, C \in KQ[L])\).
Proof: From 2.1(2) and above, $KQ[L]$ is a pointwise convex (Hausdorff) topological semivector space with singleton origin, hence, by Theorem 2.11 of [2], cancellative. Let $A$, $B$, $C \in KQ[L]$ and $A \otimes B \otimes C$. Supposing $b \in B \setminus C$, we have $A \otimes ((b) \cup C) = A \otimes C$ and cancelling $A$ gives $(b) \cup C = C$, a contradiction. Hence, $B \setminus C = \emptyset$, implying $B \subseteq C$.

2.3 THEOREM: If $X \subseteq L$ is nonempty compact and convex, then $KQ[X] \subseteq KQ[L]$ is (nonempty) compact and convex, or

$$x \in KQ[L] \Rightarrow KQ[x] \in KQ[KQ[L]].$$

Proof: Let $X \subseteq L$ be nonempty compact and convex. The uniform topology which the (uniform space) $X$ determines for $K[X]$ yields $K[X]$ compact Hausdorff, since $X$ is compact Hausdorff (see 3.3, pp. 160, and 4.9.12, pp. 164, of [1]). Furthermore, $K[X]$ inherits the same topology as a subspace of $K[L]$ as it receives "from $X$ (see 5.2.3 and 5.2.3', p. 167 of [1]), so that $K[X] \subseteq K[L]$ is compact Hausdorff.

Now $KQ[X] \subseteq K[X]$ is clearly nonempty and convex, since $X$ is so. This leaves only to show that $KQ[X] \subseteq K[X]$ is closed. To that end, let $F$ be a converging filterbase in $KQ[X]$. Since $K[X]$ is compact Hausdorff, the limit point,
say \( Q \) is unique and \( Q \in K[x] \). We show that \( Q \) is also convex.

For each \( \lambda \in [0, 1] \), denote \( \lambda' = (1-\lambda) \) and define the map \( \Omega_\lambda \) on \( K[x] \) through \( \Omega_\lambda(p) = \lambda p + \lambda' p \) (\( p \in K[x] \)).

By 2.1, \( \Omega_\lambda \) for each \( \lambda \in [0, 1] \) is continuous, so that \( \Omega_\lambda(K[x]) \subset K[x] \); as \( X \) is convex, we actually have \( \Omega_\lambda(K[x]) \subset K[x] \). Furthermore, for each \( \lambda \in [0, 1] \), the restriction of \( \Omega_\lambda \) to \( K\mathbb{C}[x] \) is nothing but the identity map of \( K\mathbb{C}_0[x] \). Also, given a \( p \in K[x] \), if \( \Omega_\lambda(p) \subset p \)
for each \( \lambda \in [0, 1] \), then \( p \in K\mathbb{C}[x] \). Take any \( \lambda \in [0, 1] \).

We show that \( \Omega_\lambda(Q) = Q \). Let \( V \subset K[x] \) be any nbd of \( \Omega_\lambda(Q) \in K[x] \). As \( \Omega_\lambda \) is continuous, there is a nbd \( U \subset K[x] \) of \( Q \in K[x] \) such that \( \Omega_\lambda(U) \subset V \). As \( F \) converges to \( Q \), there is some \( \mathcal{W} \subset F \) with \( W \subset U \). But \( W \subset K\mathbb{C}[x] \), so that \( \mathcal{W} = \Omega_\lambda(W) \subset \Omega_\lambda(U) \subset V \). This shows that \( F \) converges to \( \Omega_\lambda(Q) \); and, the limit point being unique, \( \Omega_\lambda(Q) = Q \). Then, \( Q \in K\mathbb{C}[x] \), showing that \( K\mathbb{C}[x] \) is closed and completing the proof. \( \diamond \)
3. EMBEDDING $K\mathbb{Q}[L]$ IN A TOPOLOGICAL VECTOR SPACE

A subset of the semivector space $[L]$, to be embeddable in a vector space, must clearly be pointwise convex and cancellative. Now the largest pointwise convex set in $[L]$ is $\mathbb{Q}[L]$, but clearly $\mathbb{Q}[L]$ fails to be cancellative and is, therefore, not embeddable in a vector space. On the other hand, we have just extended the operations of $L$ to $K\mathbb{Q}[L]$ (see 2.1), and this is a topological semivector space which is both pointwise convex and cancellative (2.2). In standard fashion (see also 2.9 of [2]) we embed it in

The Real Vector Space $L$: Denoting $S = K\mathbb{Q}[L] \times K\mathbb{Q}[L]$, equip $S$ with coordinatewise addition $(A, B) \oplus (C, D)$ and define the equivalence relation $G \subseteq S$ through $(A, B) G (C, D) \iff A \oplus D = B \oplus C$, so that $G$ is a semigroup congruence and the quotient $L = S/G$ is a group. Denote the equivalence class of $(A, B)$ by $[A, B]$, and define scalar multiplication $\psi : \mathbb{R} \times L \to L$ by setting $\psi(\lambda, [A, B]) = [\lambda A, \lambda B]$ if $\lambda \geq 0$ and $\psi(\lambda, [A, B]) = [\lambda A, \lambda B]$ if $\lambda \leq 0$. Now $L$ is a real vector space and the map $\psi$ which sends each $A \in K\mathbb{Q}[L]$ to the equivalence class $[2A, A] \in L$ is an algebraic isomorphism embedding $K\mathbb{Q}[L]$ into $L$. Evidently, $L$ is, up to an isomorphism, the
smallest vector space in which $KQ[L]$ may be algebraically embedded. \textit{N.B.} Clearly, $[A, A] = [B, B]$ for all $A, B \in KQ[L]$, and this equivalence class is the identity element of $L$.

We now take a fundamental system $U = \{ U_a | a \in A \}$ of symmetric open nbds of the identity $e$ in $L$, and for $L$ we define

The Topology $\mathcal{T}$: For each $a \in A$, declare $U_a = \{ [A, B] \in L \mid B \leq A \leq U_a, A \leq B \leq U_a \}$ to be an open nbhd of the identity element $[A, A]$ of $L$; and, for each $[P, Q] \in L$, declare $[P, Q] \in U_a$ to be an open nbhd of $[P, Q]$. (We check that, if $[A, B] \in U_a$ and $(C, D) \in [A, B]$, then $D \leq C \leq U_a$ and $C \leq D \leq U_a$. As $(C, D) \in [A, B]$, we have $A \leq D = B \leq C$, while $A \leq C \leq D \leq U_a$, so that $B \leq C \leq B \leq D \leq U_a$, from which 2.2 implies $C \leq D \leq U_a$, similarly, $D \leq C \leq U_a$.)

3.1 THEOREM: (1) $L$ equipped with the topology $\mathcal{T}$ is a topological vector space, and (2) $g$ embeds $KQ[L]$ isomorphically in $L$.

Proof: (ad (1)): To see that the family $W = \{ U_a | a \in A \}$ is a local base for a Hausdorff vector topology on $L$, we note
that each $W_i$ is symmetric, and check that:

(i) For each pair $a, b \in A$, there is a $r \in A$ such that $W_r \subset W_a \cap W_b$; Choose $r \in A$ such that $U_r \subset U_a \cap U_b$.

(ii) For each $a \in A$, there is a $b \in A$ such that $W_b \subset W_a$; Choose $b \in A$ such that $U_b \subset U_a$.

(iii) For each $a \in A$, there is a $b \in A$ such that $\lambda W_b \subset W_a$ for each scalar $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$; Choose $b \in A$ such that $\lambda U_b \subset U_a$ for each $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.

(iv) Given any $[A, B] \in L$ and $a \in A$, there is $\lambda \in \mathbb{R}$ such that $[A, B] \subset \lambda W_a$. Taking any $b \in B$, for each $a \in A$ find $\lambda_a \in \mathbb{R}$ such that $a \in \lambda_a U_a \cap \{b\}$. Then, for each $a \in A$, $\lambda_a \in \mathbb{R}$ and, so $\lambda A \subset U_a \cap \{b\}$ is an open cover of $A$ and, since $A \subset L$ is compact, there is a finite subcover $\{\lambda_{a(i)} U_a \cap \{b\} \mid i = 1, \ldots, m\}$. Defining $\lambda_A = \max \{\lambda_{a(i)} \mid i = 1, \ldots, m\}$, now $A \subset \lambda A \subset U_a \cap \{b\}$. Finding $\lambda_A$ in similar fashion and setting $\lambda = \max \{\lambda_A, \lambda_B\}$ we see that $[A, B] \subset \lambda W_a$.

(v) $\mathbb{W}_A = \{[A, A]\}$ (where $[A, A]$ is the identity element of $L$); $[A, A] \subset \mathbb{W}_A$, since $[A, A] \subset U_a$ for each $a \in A$.

On the other hand, if $B, C \in \mathbb{W}_L$ are distinct, then there is an $b \in A$ such that $B \neq b \cdot U_B$ or $C \neq b \cdot U_B$, so that $[B, C] \notin \mathbb{W}_B$ and $[B, C] \notin \mathbb{W}_A$.

(ad (2)): Having already seen that $g$ is an algebraic isomorphism, all we need to check here is that $g$ is continuous and open. A basic open nbhd of an element $p \in K_0[L]$ is
of the form \( U_a(P) = \{ Q \in KQ[L] \mid P \subset Q \subset U_a, Q \subset P \subset U_a \} \) 
\((a \in A)\). A basic open nbd of \( g(P) = \{2P, P\} \in L \) according to the subspace topology of \( g(KQ[L]) \) determined by \( T \) is of the form \( U'_a(P) = \{2P, P\} \times U_a \cap g(KQ[L]) \) \((a \in A)\).

What we actually show now is the formula \( g(U_a(P)) = U'_a(P) \).

Let \([2Q, Q] \in g(U_a(P))\), so that \(P \subset Q \subset U_a\) and \(Q \subset P \subset U_a\). Let \([A, B] = [2Q, Q] \circ [P, 2P] = [2Q \circ P, Q \circ 2P]\), so that \(A \circ Q \circ 2P = B \circ 2Q \circ P, \text{i.e., } A \circ P = B \circ Q\). As \(A \circ P \subset A \circ Q \circ U_a\), we have \(B \circ Q \subset A \circ Q \circ U_a\), and 2.2 then yields \(B \subset A \circ U_a\). Similarly, \(A \subset B \circ U_a\), so that \([A, B] \in U_a\) and \([2Q, Q] = [2P, P] \circ [A, B] \in U'_a(P), \text{i.e., } g(U_a(P)) \subset U'_a(P)\).

Now let \([2P, P] \circ [2A, A] = [2(P \circ A), P \circ A] \in U'_a(P)\), so that \(2A \subset A \circ U_a\) and \(A \subset 2A \circ U_a\). Then \(P \circ 2A \subset P \circ A \circ U_a\) and \(P \circ A \subset P \circ 2A \circ U_a\), so that 2.2 gives \(P \circ A \subset P \circ U_a\) and \(P \circ A \subset P \circ U_a, \text{i.e., } P \circ A \subset U_a(P)\) and \([2(P \circ A), P \circ A] \in g(U_a(P))\). Thus, \(g(U_a(P)) \subset U'_a(P)\), and we conclude that \(g(U_a(P)) = U'_a(P)\), completing the proof.  

3.2 THEOREM: \(L\) with the topology \(T\) is locally convex iff 
\(L\) is locally convex.

Proof: "Only if" follows from the conjunction of 2.1(2) and 3.1(2). To see "if," assume \(L\) to be locally convex. W.l.o.g., we may assume that, for each \(a \in A, U_a\) is convex, circled, and radial at \(e\) and that, for each nonzero \(\lambda \in \mathbb{R}, \lambda U_a \subset U\).
Let \( a \in A \). It is straightforward to check that (i) \( \mathcal{U}_a \) is circled and (ii) for each nonzero \( \lambda \in \mathbb{R} \), \( \lambda \mathcal{U}_a \subseteq \mathcal{U}_a \). To check that (iii) \( \mathcal{U}_a \) is convex, let \([A, B], [C, D] \in \mathcal{U}_a\) and \( \lambda = (1-\lambda') \in [0, 1] \). Now \( \lambda [A, B] \oplus \lambda' [C, D] = [\lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D] \); and, since \( \mathcal{U}_a \) is convex, we have \( \lambda U_a \oplus \lambda' U_a = U_a \).

Now \([A, B], [C, D] \in \mathcal{U}_a\) says \( A \subseteq B \oplus U_a \) and \( C \subseteq D \oplus U_a \), so that \( \lambda A \oplus \lambda' C \subseteq \lambda B \oplus \lambda' D \oplus \lambda U_a \oplus \lambda' U_a \). Similarly, \( \lambda B \oplus \lambda' D \subseteq \lambda A \oplus \lambda' C \oplus \lambda U_a \oplus \lambda' U_a \). Thus, \( \lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D \in \mathcal{U}_a \), showing that \( \mathcal{U}_a \) is convex. This in conjunction with (iv) in the proof of 3.1 (1) implies that (iv) \( \mathcal{U}_a \) is radial at the identity element \([A, A] \) of \( L \). Thus, \( \mathcal{U} \) is a local base for a (unique) locally convex topology in \( L \).

### 3.3 THEOREM

(Rådström [4]): (1) \( L \) with the topology \( T \) is normable iff \( L \) is normable, and (2) if \( L \) is normed, \( L \) admits a norm for which \( \phi \) and \( \varphi \) are isometries.

**Proof:** (ad (1)): "Only if" is obvious from the conjunction of 3.1(2) and 3.1(2). To see "if," assume that \( L \) is normed by a norm \( \phi \), so that \( V = \{x \in L | \phi(x) < 1\} = U_a \) for some \( a \in A \). Thus, \( \mathcal{U}_0 = \{(A, B) \in L | A \subseteq B \oplus V, B \subseteq A \oplus V\} \in \mathcal{U} \).

Since \( V \) is radial at the origin, circled, convex and bounded, one easily checks (see also the proof of 3.1(1)) that \( \mathcal{U}_a \) has these properties too, so that (the Hausdorff space) \( L \) is normable, proving (1).
(ad (2)): In fact, the Minkowski functional $p^*$ of $W$ is a norm for $L$ and, computing that $p^*[2p, p]$ $= \sup_p p(p)$ for each $p \in K(L)$, one easily sees $g$ and $g$ to be isometries. ∅
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