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HYPERSPACES OF TOPOLOGICAL VECTOR SPACES:
THEIR EMBEDDING IN TOPOLOGICAL VECTOR SPACES

by

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We prove the following

O. HAUPTSATZ: Let L be a real (Hausdorff) topological vector space. The space $K[L]$ of nonempty compact subsets of L forms a (Hausdorff) topological semivector space with singleton origin when $K[L]$ is given the uniform (equivalently, the finite) hyperspace topology determined by L . Then $K[L]$ is locally compact iff L is so. Furthermore, $KQ[L]$, the set of nonempty compact convex subsets of L , is the largest pointwise convex subset of $K[L]$ and is a cancellative topological semivector space. For any nonempty compact and convex set $X \subset L$, the collection $KQ[X] \subset KQ[L]$ is nonempty compact and convex. L is isomorphically embeddable in $KQ[L]$ and, in turn, there is a smallest vector space L in which $KQ[L]$ is algebraically embeddable (as a cone). Furthermore, L can be given a vector topology T such that the algebraic embedding of $KQ[L]$ in L is an isomorphism, while L is, respectively, locally convex/

normable accordingly as L is so; indeed, T can be so
chosen that, when L is normed, the embedding of L in
 $KQ[L]$ and that of $KQ[L]$ in L are both isometries.

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1. PRELIMINARIES

\mathbb{R} denotes the set of real numbers with the usual topology, and $\mathbb{R}_+ = \{\lambda \in \mathbb{R} \mid \lambda \geq 0\}$. For any set X , $[X]$ denotes the set of nonempty subsets of X . When X is a topological space, $K[X]$ denotes the set of compact nonempty subsets of X . When X lies in a real vector space, $Q[X]$ denotes the set of convex nonempty subsets of X . Finally, when X lies in a real topological vector space, $KQ[X] = K[X] \cap Q[X]$.

In topologizing hyperspaces (i.e., spaces of subsets), we will use the uniform topology, regarding which we adopt Michael [1] as standard reference. Let X be a uniform space, and let $\{E_\alpha \subset X \times X \mid \alpha \in A\}$ be a fundamental system of symmetric entourages of X . The uniform topology for $[X]$ is the topology generated by declaring $E_\alpha[A]$ $= \{B \in [X] \mid B \subset E_\alpha(A) \text{ and } A \subset E_\alpha(B)\}$ for each $\alpha \in A$ to be a nbd of A ($A \in [X]$). By the uniform topology on a hyperspace $H[X] \subset [X]$ is meant the relative topology of $H[X]$ when $[X]$ carries the uniform topology.

1.0 DEFINITION [2]: Let (S, \oplus) be a commutative semi-group and $\Psi: \mathbb{R}_+ \times S \rightarrow S$ a map such that, denoting $\Psi(\lambda, s) = \lambda s$,

$$\begin{aligned} \lambda(\mu s) &= (\lambda \cdot \mu) s && \text{(left action)} \\ 1s &= s && \text{(unitariness)} \\ \lambda(s \oplus t) &= \lambda s \oplus \lambda t && \text{(homomorphism)} \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}_+$ and $s, t \in S$. We call S a semivector space. When S is a Hausdorff space and the operations \oplus and Ψ are both continuous, we call S a topological semivector space.

Thus, real vector spaces are all semivector spaces, so that the topological vector spaces we speak of are those with Hausdorff topology.

2. SEMIVECTOR HYPERSPACES OF TOPOLOGICAL VECTOR SPACES

Let L be a real vector space, and e its identity element. Now $[L]$ is a semivector space with identity $\{e\}$ when $A \oplus B = \{a + b \mid a \in A, b \in B\}$ and $\lambda A = \{\lambda a \mid a \in A\}$, where $+$ stands for vector addition in L ($A, B \in [L]$, $\lambda \in \mathbb{R}_+$). Furthermore, $Q[L] \subset [L]$ is also a semivector space and is pointwise convex, i.e., $\{A\}$ is convex for each $A \in Q[L]$. In fact $Q[L]$ is the largest pointwise convex subset of $[L]$: If $A \in [L]$ and $\lambda A \oplus \lambda' A \subset A$ for each $\lambda = (1 - \lambda') \in [0, 1]$, then $A \subset L$ must be convex.

From here on, L will always be a topological vector space.

Now $K[L] \subset [L]$ is a semivector subspace and $KQ[L]$ is the largest pointwise convex semivector subspace of $K[L]$. Also, the origin $o[L] = oK[L] = oQ[L] = oKQ[L] = \{\{e\}\}$ is singleton. N.B.: The uniform topology on $K[L]$ coincides with the finite topology (1.1, pp. 153, and 3.3 pp. 160, of [1]).

2.1 PROPOSITION: (1) $K[L]$ is a topological semivector space, locally compact iff L is. (2) The map $\{ : x \mapsto \{x\}$ ($x \in L$) isomorphically embeds L into the topological semivector subspace $KQ[L] \subset K[L]$.

Proof: (ad (1)): $K[L]$ is Hausdorff as L is (see 4.9.8, pp. 164 of [1]), and will be locally compact iff L is locally compact (see 4.9.12, pp. 164 of [1]). This leaves only the continuity of the operations \oplus and ψ of $K[L]$ to show. The continuity of vector addition $+: L \times L \rightarrow L$ implies the continuity of the map $\hat{+}: [L \times L] \rightarrow [L]$ defined by $\hat{+}(P) = \{a + b \mid a, b \in P\}$ ($P \in [L \times L]$) (see 5.9.1, pp. 169 of [1]). Thus, the restriction of $\hat{+}$ to the space $B = \{C \times D \mid C, D \in K[L]\} \subset K[L \times L]$ of compact boxes is also continuous. Furthermore, the Cartesian product $\pi(C, D) = C \times D$ is continuous on $K[L] \times K[L] \rightarrow B$ (see Theorem 3 of [3]). Now \oplus is simply the composition $\oplus = \hat{+} \circ \pi: K[L] \times K[L] \rightarrow K[L]$, and so is continuous. Similarly, the continuity of scalar multiplication $R_+ \times L \rightarrow L$ implies that of scalar multiplication $\psi: R_+ \times K[L] \rightarrow K[L]$.

(ad (2)): From (1) it follows that the space $KQ[L] \subset K[L]$ is a topological semivector space. Now the map ϕ is a homeomorphism (2, pp. 155 of [1]) and is easily checked to be a homomorphism. \diamond

2.2 PROPOSITION: $KQ[L]$ is cancellative (i.e., $A \oplus B = A \oplus C \Rightarrow B = C$) and $A \oplus B \subset A \oplus C \Rightarrow B \subset C$ ($A, B, C \in KQ[L]$).

Proof: From 2.1(2) and above, $KQ[L]$ is a pointwise convex (Hausdorff) topological semivector space with singleton origin, hence, by Theorem 2.11 of [2], cancellative. Let $A, B, C \in KQ[L]$ and $A \oplus B \subset A \oplus C$. Supposing $b \in B \setminus C$, we have $A \oplus (\{b\} \cup C) = A \oplus C$ and cancelling A gives $\{b\} \cup C = C$, a contradiction. Hence, $B \setminus C = \emptyset$, implying $B \subset C$. \diamond

2.3 THEOREM: If $X \subset L$ is nonempty compact and convex,
then $KQ[X] \subset KQ[L]$ is (nonempty) compact and convex,
or

$$X \in KQ[L] \Rightarrow KQ[X] \in KQ[KQ[L]].$$

Proof: Let $X \subset L$ be nonempty compact and convex. The uniform topology which the (uniform space) X determines for $K[X]$ yields $K[X]$ compact Hausdorff, since X is compact Hausdorff (see 3.3, pp. 160, and 4.9.12, pp. 164, of [1]). Furthermore, $K[X]$ inherits the same topology as a subspace of $K[L]$ as it receives from X (see 5.2.3 and 5.2.3', pp. 167 of [1]), so that $K[X] \subset K[L]$ is compact Hausdorff.

Now $KQ[X] \subset K[X]$ is clearly nonempty and convex, since X is so. This leaves only to show that $KQ[X] \subset K[X]$ is closed. To that end, let \underline{F} be a converging filterbase in $KQ[X]$. Since $K[X]$ is compact Hausdorff, the limit point,

say Q , is unique and $Q \in K[X]$. We show that Q is also convex.

For each $\lambda \in [0, 1]$, denote $\lambda' = (1-\lambda)$ and define the map Ω_λ on $K[X]$ through $\Omega_\lambda(P) = \lambda P \oplus \lambda' P$ ($P \in K[X]$). By 2.1, Ω_λ for each $\lambda \in [0, 1]$ is continuous, so that $\Omega_\lambda(K[X]) \subset K[L]$; as X is convex, we actually have $\Omega_\lambda(K[X]) \subset K[X]$. Furthermore, for each $\lambda \in [0, 1]$, the restriction of Ω_λ to $KQ[X]$ is nothing but the identity map of $KQ[X]$. Also, given a $P \in K[X]$, if $\Omega_\lambda(P) \subset P$ for each $\lambda \in [0, 1]$, then $P \in KQ[X]$. Take any $\lambda \in [0, 1]$. We show that $\Omega_\lambda(Q) = Q$. Let $V \subset K[X]$ be any nbd of $\Omega_\lambda(Q) \in K[X]$. As Ω_λ is continuous, there is a nbd $U \subset K[X]$ of $Q \in K[X]$ such that $\Omega_\lambda(U) \subset V$. As \underline{F} converges to Q , there is some $W \in \underline{F}$ with $W \subset U$. But $W \subset KQ[X]$, so that $W = \Omega_\lambda(W) \subset \Omega_\lambda(U) \subset V$. This shows that \underline{F} converges to $\Omega_\lambda(Q)$; and, the limit point being unique, $\Omega_\lambda(Q) = Q$. Then, $Q \in KQ[X]$, showing that $KQ[X]$ is closed and completing the proof. \diamond

3. EMBEDDING $KQ[L]$ IN A TOPOLOGICAL VECTOR SPACE

A subset of the semivector space $[L]$, to be embeddable in a vector space, must clearly be pointwise convex and cancellative. Now the largest pointwise convex set in $[L]$ is $Q[L]$, but clearly $Q[L]$ fails to be cancellative and is, therefore, not embeddable in a vector space. On the other hand, we have just extended the operations of L to $KQ[L]$ (see 2.1), and this is a topological semivector space which is both pointwise convex and cancellative (2.2). In standard fashion (see also 2.9 of [2]) we embed it in

The Real Vector Space L : Denoting $S = KQ[L] \times KQ[L]$, equip S with coordinatewise addition $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$ and define the equivalence relation $G \subset S$ through $(A, B) G (C, D) \iff A \oplus D = B \oplus C$, so that G is a semigroup congruence and the quotient $L = S/G$ is a group. Denote the equivalence class of (A, B) by $[A, B]$, and define scalar multiplication $\psi: \mathbb{R} \times L \rightarrow L$ by setting $\psi(\lambda, [A, B]) = [\lambda A, \lambda B]$ if $\lambda \geq 0$ and $\psi(\lambda, [A, B]) = [|\lambda|B, |\lambda|A]$ if $\lambda \leq 0$. Now L is a real vector space and the map g which sends each $A \in KQ[L]$ to the equivalence class $[2A, A] \in L$ is an algebraic isomorphism embedding $KQ[L]$ into L . Evidently, L is, up to an isomorphism, the

smallest vector space in which $KQ[L]$ may be algebraically embedded. N.B.: Clearly, $[A, A] = [B, B]$ for all $A, B \in KQ[L]$, and this equivalence class is the identity element of L .

We now take a fundamental system $U = \{U_\alpha \mid \alpha \in A\}$ of symmetric open nbds of the identity e in L , and for L we define

The Topology \mathcal{T} : For each $\alpha \in A$, declare $W_\alpha = \{[A, B] \in L \mid B \subset A \oplus U_\alpha, A \subset B \oplus U_\alpha\}$ to be an open nbd of the identity element $[A, A]$ of L ; and, for each $[P, Q] \in L$, declare $[P, Q] \oplus W_\alpha$ to be an open nbd of $[P, Q]$. (We check that, if $[A, B] \in W_\alpha$ and $(C, D) \in [A, B]$, then $D \subset C \oplus U_\alpha$ and $C \subset D \oplus U_\alpha$: As $(C, D) \in [A, B]$, we have $A \oplus D = B \oplus C$, while $A \oplus D \subset B \oplus D \oplus U_\alpha$, so that $B \oplus C \subset B \oplus D \oplus U_\alpha$, from which 2.2 implies $C \subset D \oplus U_\alpha$; similarly, $D \subset C \oplus U_\alpha$.)

3.1 THEOREM: (1) L equipped with the topology \mathcal{T} is a topological vector space, and (2) g embeds $KQ[L]$ isomorphically in L .

Proof: (ad (1)): To see that the family $\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$ is a local base for a Hausdorff vector topology on L , we note

that each W_α is symmetric, and check that:

(i) For each pair $\alpha, \beta \in A$, there is a $\gamma \in A$ such that

$W_\gamma \subset W_\alpha \cap W_\beta$: Choose $\gamma \in A$ such that $U_\gamma \subset U_\alpha \cap U_\beta$;

(ii) For each $\alpha \in A$, there is a $\beta \in A$ such that

$W_\beta \oplus W_\beta \subset W_\alpha$: Choose $\beta \in A$ such that $U_\beta \oplus U_\beta \subset U_\alpha$;

(iii) For each $\alpha \in A$, there is a $\beta \in A$ such that $\lambda W_\beta \subset W_\alpha$

for each scalar $\lambda \in R$ with $|\lambda| \leq 1$: Choose $\beta \in A$ such that $\lambda U_\beta \subset U_\alpha$ for each $\lambda \in R$ with $|\lambda| \leq 1$;

(iv) Given any $[A, B] \in L$ and $\alpha \in A$, there is a $\lambda \in R$

such that $[A, B] \in \lambda W_\alpha$: Taking any $b \in B$, for each $a \in A$

find $\lambda_a \in R$ such that $a \in \lambda_a U_\alpha \oplus \{b\}$. Then, for each $a \in A$,

$a \in \lambda_a U_\alpha \oplus B$, and so $\{\lambda_a U_\alpha \oplus B \mid a \in A\}$ is an open cover of

A and, since $A \subset L$ is compact, there is a finite subcover

$\{\lambda_{a(i)} U_\alpha \oplus B \mid i = 1, \dots, m\}$. Defining $\lambda_A = \text{Max}\{\lambda_{a(1)}, \dots,$

$\lambda_{a(m)}\}$, now $A \subset \lambda_A U_\alpha \oplus B$. Finding λ_B in similar fashion

and setting $\lambda = \text{Max}\{\lambda_A, \lambda_B\}$ we see that $[A, B] \in \lambda W_\alpha$.

(v) $\bigcap_A W_\alpha = \{[A, A]\}$ (where $[A, A]$ is the identity element

of L): $[A, A] \in \bigcap_A W_\alpha$, since $[A, A] \in W_\alpha$ for each $\alpha \in A$.

On the other hand, if $B, C \in KQ[L]$ are distinct, then there is an $\beta \in A$ such that $B \notin C \oplus U_\beta$ or $C \notin B \oplus U_\beta$, so that

$[B, C] \notin W_\beta$ and $[B, C] \notin \bigcap_A W_\alpha$.

(ad (2)): Having already seen that g is an algebraic isomorphism, all we need to check here is that g is continuous and open. A basic open nbd of an element $P \in KQ[L]$ is

of the form $U_\alpha(P) = \{Q \in KQ[L] \mid P \subset Q \oplus U_\alpha, Q \subset P \oplus U_\alpha\}$
 $(\alpha \in A)$. A basic open nbd of $g(P) = [2P, P] \in L$ according
to the subspace topology of $g(KQ[L])$ determined by T is
of the form $W'_\alpha(P) = ([2P, P] \oplus W'_\alpha) \cap g(KQ[L])$ ($\alpha \in A$).
What we actually show now is the formula $g(U_\alpha(P)) = W'_\alpha(P)$.

Let $[2Q, Q] \in g(U_\alpha(P))$, so that $P \subset Q \oplus U_\alpha$ and
 $Q \subset P \oplus U_\alpha$. Let $[A, B] = [2Q, Q] \oplus [P, 2P] = [2Q \oplus P, Q \oplus 2P]$,
so that $A \oplus Q \oplus 2P = B \oplus 2Q \oplus P$, i.e., $A \oplus P = B \oplus Q$. As
 $A \oplus P \subset A \oplus Q \oplus U_\alpha$, we have $B \oplus Q \subset A \oplus Q \oplus U_\alpha$, and 2.2 then
yields $B \subset A \oplus U_\alpha$. Similarly, $A \subset B \oplus U_\alpha$, so that $[A, B] \in W'_\alpha$
and $[2Q, Q] = [2P, P] \oplus [A, B] \in W'_\alpha(P)$, i.e., $g(U_\alpha(P)) \subset W'_\alpha(P)$.
Now let $[2P, P] \oplus [2A, A] = [2(P \oplus A), P \oplus A] \in W'_\alpha(P)$, so that
 $2A \subset A \oplus U_\alpha$ and $A \subset 2A \oplus U_\alpha$. Then $P \oplus 2A \subset P \oplus A \oplus U_\alpha$
and $P \oplus A \subset P \oplus 2A \oplus U_\alpha$, so that 2.2 gives $P \oplus A \subset P \oplus U_\alpha$
and $P \subset P \oplus A \oplus U_\alpha$, i.e., $P \oplus A \in U_\alpha(P)$ and $[2(P \oplus A), P \oplus A]$
 $\in g(U_\alpha(P))$. Thus, $g(U_\alpha(P)) \subset W'_\alpha(P)$, and we conclude that
 $g(U_\alpha(P)) = W'_\alpha(P)$, completing the proof. \diamond

3.2 THEOREM: L with the topology T is locally convex iff
 L is locally convex.

Proof: "Only if" follows from the conjunction of 2.1(2) and
3.1(2). To see "if," assume L to be locally convex. W.l.g.,
we may assume that, for each $\alpha \in A$, U_α is convex, circled,
and radial at e and that, for each nonzero $\lambda \in R$, $\lambda U_\alpha \in U$.

Let $\alpha \in A$. It is straightforward to check that (i) W_α is circled and (ii) for each nonzero $\lambda \in R$, $\lambda W_\alpha \in W$. To check that (iii) W_α is convex, let $[A, B], [C, D] \in W_\alpha$ and $\lambda = (1-\lambda') \in [0, 1]$. Now $\lambda[A, B] \oplus \lambda'[C, D] = [\lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D]$; and, since U_α is convex, we have $\lambda U_\alpha \oplus \lambda' U_\alpha = U_\alpha$. Now $[A, B], [C, D] \in W_\alpha$ says $A \subset B \oplus U_\alpha$ and $C \subset D \oplus U_\alpha$, so that $\lambda A \oplus \lambda' C \subset \lambda B \oplus \lambda' D \oplus \lambda U_\alpha \oplus \lambda' U_\alpha$. Similarly, $\lambda B \oplus \lambda' D \subset \lambda A \oplus \lambda' C \oplus U_\alpha$. Thus, $[\lambda A \oplus \lambda' C, \lambda B \oplus \lambda' D] \in W_\alpha$, showing that W_α is convex. This in conjunction with (iv) in the proof of 3.1 (1) implies that (iv) W_α is radial at the identity element $[A, A]$ of L . Thus, W is a local base for a (unique) locally convex topology in L . \diamond

3.3 THEOREM (Rådström [4]): (1) L with the topology T is normable iff L is normable, and (2) if L is normed, L admits a norm for which f and g are isometries.

Proof: (ad (1)): "Only if" is obvious from the conjunction of 2.1(2) and 3.1(2). To see "if," assume that L is normed by a norm ρ , so that $V = \{x \in L \mid \rho(x) < 1\} = U_\alpha$ for some $\alpha \in A$. Thus, $W_\alpha = \{[A, B] \in L \mid A \subset B \oplus V, B \subset A \oplus V\} \in W$. Since V is radial at the origin, circled, convex and bounded, one easily checks (see also the proof of 3.1(1) that W_α has these properties too, so that (the Hausdorff space) L is normable, proving (1).

(ad (2)): In fact, the Minkowski functional ρ^* of W_α is a norm for L and, computing that $\rho^*[2P, P] = \sup_P \rho(p)$ for each $P \in KQ[L]$, one easily sees f and g to be isometries. \diamond

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