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TOWARD AN ENDOGENOUS CENTRAL PLACE THEORY

by

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ABSTRACT

We envisage a continuum of spatially interdependent agents, evenly distributed over a linear, homogeneous space. We let the density of agents gradually increase at a slightly uneven rate, slow enough to permit the restoration of a locally stable, perfectly dispersed equilibrium. At the same time, we also let the degree of spatial interdependence increase. Increasing spatial interdependence, in the context of our analysis, could imply an increased frequency of trips over a longer range. Then the dispersed equilibrium may eventually exhibit local instability. If there is a single type of agent, the spatial characteristics of the emerging agglomerated state cannot be predicted within our framework. If though there are two types of agent, the departure from the dispersed equilibrium may take the form of initial growth on a regularly spaced pattern of locations: specialisation here appears to be necessary for the emergence of a regular settlement pattern. The timing of emergence depends both on the preferences and on the characteristics of spatial interaction among agents. The spacing of settlements, on the other hand, depends on the characteristics of spatial interaction alone.

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1. INTRODUCTION

In this paper we study the emergence of central places over a linear, unbounded landscape occupied by a continuum of agents. Typically, a central place is understood as the focus of trade and service activity. Here it becomes a line segment where the density of agents is relatively high. Thus a system of central places resembles a multimodal density function over the real line. Our objective is to characterise the "succession of form", as Thom (1972) has named it, from a spatially uniform to a multimodal distribution of agents — especially how the timing of the succession and the emerging spatial pattern of central places are affected by the behavioural characteristics of agents. In essence, we seek to understand *how cities are born*. This has never been at issue in the classic tradition of central place theory as it evolved after Christaller (1933) and Lösch (1944).[1] There, central places simply *exist*, and the problem is to account for their perfectly even arrangement on a Euclidean plane, over which the exogenous spatial distribution of demand is itself perfectly even. In the case of a single good, where a firm and a central place can be synonymous, traditional wisdom dictates that free entry would absorb all profit and lead to the densest possible packing of central places at equilibrium. This, in turn, implies that central places will be located on a triangular lattice, giving rise to hexagonal market areas (Bollobás and Stern (1972)). In the case of many goods, on the same lattice, a hierarchy of central places arises with higher–order central places providing a larger variety of goods; and the size distribution of central places at equilibrium can be obtained using a system of multipliers, which relate the demand at different levels of the hierarchy (Beckmann (1958)). The richness and the complexity of a hierarchical system thus derived hinges, as we have already claimed, on the very existence of central places — those substantial entities somehow ready to be used by the theorist. In this respect, classical central place theory is *exogenous*. We, on the other hand, strive for an *endogenous* central place theory, one in which central places will appear as the spatial economy gradually evolves. Our aim imposes severe constraints on the scope of the central place system we can handle. Not only is our space linear, but the hierarchy of central places is reduced to a single level, and we can trace the emergence of central places inasmuch as we can point out to the reader a regular pattern of candidate locations on which Thom's "succession of form" can only happen. Nevertheless, we believe that our simple model still provides considerable intuition about the processes underlying the birth of cities.

Starrett (1978) has proven that, in a closed economy without relocation cost, where utilities and technologies are independent of location, and where complete markets exist for all goods at all locations, there is no competitive equilibrium with a positive aggregate transportation cost. In such an economy, identical agents would be arranged evenly over the land at equilibrium: Starrett's result specifies sufficient conditions under which an endogenous central place theory is impossible. Thus, for a closed economy without relocation cost, an endogenous central place theory would necessitate either utilities and/or technologies dependent on location, or incomplete markets, or both. Under complete markets, the dependence of utilities and/or technologies on location happens either because there are exogenous locational advantages (such as an unequal distribution of natural resources), or because there are population effects (such as congestion) which are determined by the spatial distribution of agents relative to the location considered, or both. Under incomplete markets, on the other hand, agents must be sensitive to the presence of others because of their need to interact. Therefore, in a closed economy with neither relocation cost nor exogenous locational advantages, *spatial interdependence* among agents is necessary for the existence of agglomerations in equilibrium.

Spatial interdependence gives rise to the concept of locational *centrality*. To fix ideas, consider an example drawn from Papageorgiou and Thisse (1985). There is a continuous distribution of identical agents over a line segment. Since spatial interaction in this economy is a good, central locations have a comparative advantage because they offer a higher potential to interact for a given level of transportation expenditure. Competition for land over the line segment would eliminate this advantage through central agglomeration, which ascertains that locations in equilibrium are characterised by a tradeoff between abundance of land and accessibility. This centripetal tendency would vanish in the absence of spatial interdependence. Now bend the line segment into a circle. Since boundaries disappear, the space loses its differentiated structure and locations become indistiguishable to each other in terms of their potential to interact. Competition for land would now disperse agents evenly over the line segment. We conclude that spatial interdependence, in this example, appears to produce agglomeration only when the space is bounded — hence inhomogeneous. An homogeneous space would seem to produce dispersion even under spatial interdependence. Conversely, spatial autarchy would produce dispersion even when our space is inhomogeneous.

Locational centrality and spatial interdependence dominate the modern treatments of agglomeration dealing with a continuum of agents (see, for example, Solow and Vickrey (1971), Amson (1972, 1973), Yellin (1974), Beckmann (1976), Odland (1976), Smith (1976), Boruchov and Hochman (1977), Fujita and Ogawa

(1982), Imai (1982), ten Raa (1984), Papageorgiou and Thisse (1985), and Tabushi (1986)). Similarly to the example of the previous paragraph, centrality in all those works arises from the bounded nature of space itself, so that spatial interdependence *and* a bounded space become sufficient for agglomeration. This contrasts with central place theory, where centrality arises in conjunction with a settlement pattern over the boundless, perfectly homogeneous landscape: whereas *centrality determines agglomeration* in the former case, *agglomeration defines centrality* in the latter. Hence central place theory requires that spatial boundaries are not necessary for agglomeration. In the context of the example, spatial interdependence alone (of agents distributed over an unbounded space) should be both necessary (Starrett (1978)) *and* sufficient for an agglomerated equilibrium. Since our arguments for a dispersed equilibrium over an homogeneous space still hold, central place theory requires the existence of agglomerated equilibria further to the dispersed equilibrium obtained in the example of the previous paragraph.

Of particular interest to us is the transition from the dispersed to an agglomerated equilibrium, as it signals the emergence of a settlement pattern. A natural way of thinking about this transition is through when and how does a stable, dispersed equilibrium become locally unstable. Whenever this occurs, perturbations around the dispersed equilibrium must lead to spatial differentiation. This approach has been adopted by Papageorgiou and Smith (1983) [PS]. Spatial interdependence in PS was represented as an externality which entered the utility function of agents and was determined over entire spatial population distributions. The externality was designed to cover a wide range of agglomerative and deglomerative factors. Local congestion, on the other hand, provided a tendency of agents to disperse. It was established that the critical moment of instability occurs when the marginal effect of increased local congestion on the utility of agents, which has been caused by an increased population density of the dispersed equilibrium, is exactly balanced by the corresponding positive externality effect. Thus the moment of critical instability was associated with a simple, intuitively satisfying notion of balance between agglomerative and deglomerative factors: as long as the latter dominate, the dispersed equilibrium must persist. Agglomerations will begin only when the positive externality effect becomes strong enough, i.e. only when the marginal utility of closedness (presumably generated by agent specialisation) overcomes the marginal disutility of local congestion.

In the light of our previous discussion, PS appears to provide a minimalist approach toward an endogenous central place theory. Namely, spatial interdependence *alone* among *identical* agents over a *perfectly homogeneous* landscape produces a local instability of the dispersed equilibrium. However, once the dispersed equilibrium has become locally unstable, *what are the characteristics of the emerging settlement pattern*? The method used in PS did not allow for an answer to this fundamental question. Here we propose a different method designed to resolve it. Unlike PS, who used a discrete physical space, we require a spatial continuum in order to determine the equilibrium distance between settlements. More precisely, beyond the conditions for the critical instability of the dispersed equilibrium, we are able to characterise a regular spacing of points which, at the critical instability, become the only candidates for agglomeration. Section two reexamines PS in the context of our method. We obtain the same necessary and sufficient condition for local instability. However, at the critical instability, the spacing of points on which the uniform population distribution first becomes unstable is degenerate. This seems to persist over reasonable specifications of spatial interdependence, and it points out that capturing agent specialisation in terms of spatial interdependence alone may not be sufficient to characterise a regular settlement pattern. Consequently, in section three, we introduce a second type of agent. We are now able to obtain the regular spacing of first instabilities sought, and to establish the determinants of that spacing. Section four places our results, and associated limitations, into perspective.

2. ONE TYPE OF AGENT

2.1 THE MODEL

We restate PS in continuous terms. Our landscape is the real line $\mathscr{X} \equiv]-\infty, \infty[$.[2] A continuum of agents is distributed over that space. These, for example, could represent individuals who interact with others in order to trade. For x and $x' \in \mathscr{X}$, any agent at x receives an externality from those at x' equal to $g[x, x']n[x', t]$, where $g[x, x']$ is a *distance-response function* which subsumes the spatial diffusion process of the externality (hence the particulars of spatial interdependence), and $n[x', t]$ is the density of agents in location x' at time t. The distance–response function is assumed to depend only on the distance between x and x', and to be spatially invariant, i.e.

$$(1) \qquad g[x, x'] = g[x', x] \quad \textit{and} \quad \int_{\mathscr{X}} g[x, x']\mathrm{d}x' = G \quad \textit{for } x \textit{ and } x' \in \mathscr{X}.$$

(For further details see PS.) Agents at x are sensitive to current local densities in the sense that, ceteris paribus, they prefer more land, and to the *spatial externality* $E[x, t]$, which is defined as

$$(2) \qquad E[x, t] \equiv \int_{\mathscr{X}} g[x, x'] n[x', t] \mathrm{d}x',$$

a composite of externalities, and which is a manifestation of global interdependence among agents. Hence the utility of someone located on x at time t is given by $v[n[x, t], E[x, t]]$ with $\partial v / \partial n < 0$.[3]

Agents adjust toward highest *perceived* utility. Since information is imperfect, actual and perceived distributions of utility may differ. Thus, from the viewpoint of an observer, location decisions can be determined only up to a probability distribution with density

$$(3) \qquad p[x, t] = \frac{n[x, t] v[n[x, t], E[x, t]]}{\int_{\mathscr{X}} n[x'', t] v[n[x'', t], E[x'', t]] \mathrm{d}x''}.$$

That is, the chance of finding someone at a particular location will, ceteris paribus, increase as the population density and/or the utility at this location increases. Since utilities depend on the spatial distribution of agents only, so do migration decisions. In consequence, the evolution of this system at any moment depends exclusively on its current state $\{n[x, t] \mid x \in \mathscr{X}\}$: the Markov assumption holds. Moreover, the total size of the population remains fixed, that is

$$(4) \qquad \frac{\int_{\mathscr{X}} n[x, t] \mathrm{d}x}{\int_{\mathscr{X}} \mathrm{d}x} = n^0 \quad with\ 0 < n^0 < \infty .$$

Under these circumstances, we may represent the evolution of this system by

$$(5) \qquad \frac{\partial}{\partial t} n[x, t] = \int_{\mathscr{X}} p[x, t] n[x', t] \mathrm{d}x' - \int_{\mathscr{X}} p[x', t] n[x, t] \mathrm{d}x' =$$

$$\left[\int_{\mathscr{X}} n[x', t] \mathrm{d}x' \frac{v[n[x, t], E[x, t]]}{\int_{\mathscr{X}} n[x'', t] v[n[x'', t], E[x'', t]] \mathrm{d}x''} - 1 \right] n[x, t].$$ [4,5]

2.2 STABILITY ANALYSIS

The dispersed population distribution $n\,[x\,,\,t] = n^0$ for $x \in \mathscr{X}$ is an equilibrium solution of (5). At equilibrium, everyone enjoys the same utility level $v^0 = v\,[n^0, E^0] > 0$ with $E^0 = n^0\,G$. We now perturb the dispersed equilibrium over space, holding total population constant, and perform a linear stability analysis. The corresponding solutions of (5) in the neighbourhood of the equilibrium are given by

$$(6) \qquad n\,[x\,,\,t] = n^0 + e^{\lambda t}\mathrm{d}n\,[x],$$

where λ is a parameter which determines the local stability of the equilibrium: if $\lambda < 0$, the equilibrium is asymptotically stable; and if $\lambda > 0$, it is unstable.

From now on, for simplicity, we shall avoid the time–notation. Replace (6) in (5) to obtain

$$(7) \qquad \lambda e^{\lambda t}\mathrm{d}n\,[x] = \left[\int_{\mathscr{X}} n\,[x']\mathrm{d}x' \frac{v[n^0+e^{\lambda t}\mathrm{d}n[x],E^0+\mathrm{d}E^0[x]]}{\int_{\mathscr{X}}(n^0+e^{\lambda t}\mathrm{d}n[x''])\,v[n^0+e^{\lambda t}\mathrm{d}n[x''],E^0+\mathrm{d}E^0[x'']]\mathrm{d}x''} - 1\right]\cdot$$
$$\cdot(n^0 + \lambda e^{\lambda t}\mathrm{d}n\,[x]),$$

where

$$(8) \qquad \mathrm{d}E^0\,[x] \equiv \int_{\mathscr{X}} g\,[x\,,\,x']e^{\lambda t}\mathrm{d}n\,[x']\mathrm{d}x'.$$

Expand $v\,[\cdot]$ about (n^0, E^0) and retain only linear terms:

$$(9) \qquad v\,[n^0 + e^{\lambda t}\mathrm{d}n\,[x]\,,\,E^0 + \mathrm{d}E^0\,[x]] = v^0 + \frac{\partial v}{\partial n^0}\,e^{\lambda t}\mathrm{d}n\,[x] + \frac{\partial v}{\partial E^0}\,\mathrm{d}E^0\,[x].$$

Using (9), the denominator on the RHS of (7) equals $n^0 v^0 \int_{\mathscr{X}}\mathrm{d}x.$[6] Upon replacement of this in (7), and using (9) once again, we arrive at the linearised equation

$$(10) \qquad \lambda e^{\lambda t}\mathrm{d}n\,[x] = \left[1 + \frac{1}{v^0}\left[\frac{\partial v}{\partial n^0}\,e^{\lambda t}\mathrm{d}n\,[x] + \frac{\partial v}{\partial E^0}\,\mathrm{d}E^0\,[x]\right] - 1\right](n^0 + \lambda e^{\lambda t}\mathrm{d}n\,[x]) =$$
$$\frac{n^0}{v^0}\left[\frac{\partial v}{\partial n^0}\,e^{\lambda t}\mathrm{d}n\,[x] + \frac{\partial v}{\partial E^0}\,\mathrm{d}E^0\,[x]\right].$$

By the definition (8) of $\mathrm{d}E^0\,[x]$, (10) can be written as

$$(11)\qquad \left[\lambda - \frac{n^0}{v^0}\frac{\partial v}{\partial n^0}\right]\mathrm{d}n\,[x] = \frac{n^0}{v^0}\frac{\partial v}{\partial E^0}\int_{\mathscr{X}} g\,[x\,,x']\mathrm{d}n\,[x']dx',$$

which, upon taking Fourier transforms and using the convolution property of it, simplifies into the *dispersion equation*

$$(12)\qquad \lambda = \frac{n^0}{v^0}\left[\frac{\partial v}{\partial n^0} + \frac{\partial v}{\partial E^0}\,\mathfrak{F}_s\,[g\,[x\,,x']]\right],$$

where $\mathfrak{F}_s\,[g\,[x\,,x']]$ denotes the Fourier transform of $g\,[x\,,x']$.

A standard form of distance–response in the literature of spatial interaction is given by

$$(13)\qquad g\,[x\,,x'] = \exp(-\,a\,|x - x'\,|)\ \textit{for } x \textit{ and } x' \in \mathscr{X}$$

(see Smith (1978) for a justification). How fast interaction declines with distance is determined by the *spatial impedance parameter* $a > 0$. Its inverse represents the average distance at which spatial interaction occurs: larger values of a imply that distance is a stronger impediment to interaction. Under (13), $\mathfrak{F}_s\,[g\,[\cdot]] = 2a/(a^2 + s^2)$, where s is the variable of the Fourier transform. This variable is important for our purposes because it determines the spacing of points d^* in $\mathscr{X}$ at which the dispersed equilibrium will first become locally unstable exactly when λ changes from a negative to a positive value:

$$(14)\qquad d^* = 2\pi/s\,|_{\lambda=0} \equiv 2\pi/s^*.$$

Such points are the only candidates for initial agglomeration. Replacing $\mathfrak{F}_s\,[g\,[\cdot]]$ in (12), and taking into account that $E^0 = 2n^0/a$, we obtain

$$(15)\qquad \lambda = \frac{n^0}{v^0}\left[\frac{\partial v}{\partial n^0} + \frac{\partial v}{\partial E^0}\frac{\mathrm{d}E}{\mathrm{d}n^0}\frac{a^2}{a^2+s^2}\right].$$

The dispersed equilibrium is locally stable iff $\lambda < 0$ for $s \geq 0$. Therefore,

$$(16) \qquad \frac{\partial v}{\partial n^0} + \frac{\partial v}{\partial E^0}\frac{\mathrm{d}E}{\mathrm{d}n^0} < 0$$

is both necessary and sufficient for the local stability of the dispersed equilibrium. This is precisely the condition derived in PS. The first term in (16) denotes the marginal cost of increased congestion on the utility of agents caused by an increased, uniform level of population density; and the second term denotes the corresponding marginal effect of the spatial externality. The latter may be either a cost or a benefit, depending upon whether the spatial externality is negative or positive. In the former case, the dispersed equilibrium is locally stable. The presumption is that interaction in primitive societies is valued less. As technology evolves, the advantages of interaction become gradually stronger through an increased specialisation of economic activity, at least during the period over which the formation of the first settlement patterns has occured. Therefore, holding the marginal cost of increased congestion fixed, one would expect that the LHS of (16) gradually increases to reach zero. Precisely then, the first instabilities occur following a regular pattern determined by s^* such that the RHS of (15) equals zero. This happens at $s^* = 0$ with $\partial v/\partial E^0 > 0$. However, since the spacing of first instabilities is given by (14), we conclude that the adjustment process (5) cannot give rise to a nontrivial spatial structure emerging in the vicinity of the dispersed equilibrium. We have found that similar types of degeneracy apply for other specifications of distance–response. We conclude that spatial interdependence alone, which echoes specialisation of economic activity and which was the key to the minimalist approach of PS, is not enough to produce a nondegenerate endogenous central place theory. We are thus led to abandon a single type of agent, and to theorise that introducing specialisation of agents explicitly in our model may be necessary for the emergence of settlement patterns over an homogeneous landscape.

3. TWO TYPES OF AGENT

3.1 THE MODEL

There are two types of agent continuously distributed over $\mathscr{X}$ with densities $n_i[x, t]$, $i = 1, 2$. These, for example, could represent firms and households respectively, as in Papageorgiou and Thisse (1985) or Fujita (1988). Spatial interaction within and between the two groups gives rise to the distance–response functions $g_{ij}[x, x']$, which determine how does the impact of j-agents at x' on i-agents at x vary with distance between x and x'. As before, we assume that the distance–response functions depend only on the distance between x and x', and that they are spatially invariant, that is,

$$(17) \qquad g_{ij}[x, x'] = g_{ij}[x', x] \;\; and \;\; \int_{\mathscr{X}} g_{ij}[x, x']dx' = G_{ij} \;\; for\ i\ and\ j = 1, 2.$$

The corresponding spatial externalities (see (2)) are now given by

$$(18) \qquad E_{ij}[x, t] = \int_{\mathscr{X}} g_{ij}[x, x']n_j[x', t]\mathrm{d}x' \;\; for\ i\ and\ j = 1, 2.$$

For analytical simplicity, we assume that agents are sensitive only to spatial externality effects. Hence utilities are written $v_i[E_{ii}[x, t], E_{ij}[x, t]]$ for i and $j = 1, 2$, and $i \neq j$.[7] As before, the total size of each population remains fixed, that is

$$(19) \qquad \frac{\int_{\mathscr{X}} n_i[x,t]\mathrm{d}x}{\int_{\mathscr{X}} \mathrm{d}x} = \overset{0}{n}_i \;\; for\ i = 1, 2,$$

and the evolution of the system is determined (similarly to (5)) by the system of equations

$$(20) \qquad \frac{\partial}{\partial t} n_i[x, t] = \left[\int_{\mathscr{X}} n_i[x', t]\mathrm{d}x' \frac{v_i[E_{ii}[x, t], E_{ij}[x, t]]}{\int_{\mathscr{X}} n_i[x'', t] v_i[E_{ii}[x'', t], E_{ij}[x'', t]]\mathrm{d}x''} - 1\right] n_i[x, t]$$

for i and j = 1 , 2, *and* $i \neq j$.

3.2 STABILITY ANALYSIS

3.2.1 GENERAL FORMULATION

The dispersed population distributions $n_i[x, t] = \overset{0}{n}_i$ for $i = 1, 2$ and $x \in \mathscr{X}$ represent an equilibrium solution of the system (20) associated with utility levels $\overset{0}{v}_i$, $i = 1, 2$. As in the case of a single type, we perform a linearised stability analysis. The solutions of (20) in the neighbourhood of the equilibrium are

$$(21) \qquad n_i[x, t] = \overset{0}{n}_i + e^{\lambda t}\mathrm{d}n_i[x] \;\; for\ i = 1, 2.$$

If we follow the procedure of section 2.2, we arrive at the system

(22) $(\lambda - Q_{11})\, \mathfrak{F}_s\,[\mathrm{d}n_1\,[x']] = Q_{12}\, \mathfrak{F}_s\,[\mathrm{d}n_2\,[x']]$

(23) $(\lambda - Q_{22})\, \mathfrak{F}_s\,[\mathrm{d}n_2\,[x']) = Q_{21}\, \mathfrak{F}_s\,[\mathrm{d}n_1\,[x'])$

where

(24) $Q_{ij} \equiv C_{ij}\, \mathfrak{F}_s\,[g_{ij}\,[x\,,\,x']]$ *with* $C_{ij} \equiv \frac{n_i^0}{v_i^0}\frac{\partial v_i}{\partial E_{ij}^0}$ *for* i *and* $j = 1\,,\,2.$

Multiplying (22) by (23) gives the dispersion equation

(25) $(\lambda - Q_{11})(\lambda - Q_{22}) = Q_{12}\, Q_{21}$

which has roots denoted by $\lambda^{\pm}[s]$. A time–independent spatial structure, characterised by a regular spacing between agglomerations $d^* = 2\pi/s^*$, will emerge iff these roots are always real and negative except at s^*, $0 < s^* < \infty$, where either $\lambda^+[s^*] = 0$ and $\lambda^-[s^*] < 0$ or $\lambda^+[s^*] = \lambda^-[s^*] = 0.$[8] These requirements are equivalent to (a) $\lambda^+ + \lambda^- \leq 0$ *for all* $s \geq 0$; (b) $\lambda^+\lambda^- = 0$ *at* $s = s^*$; and (c) $\lambda^+\lambda^- > 0$ *at* $s \neq s^*$.[9] Given (25), these conditions imply the following

> PROPOSITION: *A time-independent spatial structure, characterised by a regular spacing between agglomerations* $d^* = 2\pi/s^*$, *will emerge iff there exists a unique* s^*, $0 < s^* < \infty$, *such that*
>
> P(1): $Q_{11} + Q_{22} \leq 0$ *for all* $s \geq 0$
>
> P(2): $Q_{12}\, Q_{21} = Q_{11}\, Q_{22}$ *for* $s = s^*$
>
> P(3): $Q_{12}\, Q_{21} < Q_{11}\, Q_{22}$ *for* $s \geq 0$ *and* $s \neq s^*$.

3.2.2 NEGATIVE EXPONENTIAL DISTANCE–RESPONSE

Our subsequent analysis will be based on the standard specification

(26) $\quad g_{ij}[x, x'] = \exp(-a_{ij}|x - x'|)$ *for i and j* $= 1, 2$

where, as in section 1.2, $a_{ij} > 0$ represent impedance parameters. Under (26), $\mathfrak{F}_s[g_{ij}[\cdot]] = 2a_{ij}/(a_{ij}^2 + s^2)$ for i and $j = 1, 2$; and $E_{ij} = 2n_j^0/a_{ij}$. In consequence,

(27) $$Q_{ij} = C_{ij}\frac{a_{ij}^2}{a_{ij}^2 + s^2}.$$

When $a_{11} = a_{22}$ and $a_{12} = a_{21}$, it can be shown that $\lambda = 0$ is a solution of the dispersion equation (25) for all $s \in [0, \infty[$.[10] Therefore, when the two types of agent have the same interaction behaviour, no spatial structure can emerge. From now on, we shall assume that $a_{11} \neq a_{22}$ and $a_{12} \neq a_{21}$.

We begin the study of conditions P(1) – P(3). Substituting (27) into P(1) yields

(28) $$(a_{11}^2 C_{11} + a_{22}^2 C_{22})s^2 + a_{11}^2 a_{22}^2 (C_{11} + C_{22}) \leq 0 \quad \textit{for all } s \geq 0.$$

In (28), $s \to 0$ implies $C_{11} + C_{22} \leq 0$; and $s \to \infty$ implies $a_{11}^2 C_{11} + a_{22}^2 C_{22} \leq 0$. It can be shown easily that those facts imply, in turn,

COROLLARY 1: *Under the specification* (26), P(1) *is satisfied iff*

C1(1): $\quad C_{11} + C_{22} \leq 0$

C1(2): $\quad a_{11}^2 C_{11} + a_{22}^2 C_{22} \leq 0.$

Substituting now (27) into P(2) and P(3) yields

(29) $$AC(a_{11}^2 + s^2)(a_{22}^2 + s^2) \leq (a_{12}^2 + s^2)(a_{21}^2 + s^2) \quad \textit{for all } s \geq 0$$

with

(30) $$A \equiv \frac{a_{12}^2 a_{21}^2}{a_{11}^2 a_{22}^2} \quad \textit{and} \quad C \equiv \frac{C_{12}C_{21}}{C_{11}C_{22}},$$

where C_{ij}, i and $j = 1, 2$, are defined by (24). Equality in (29) occurs iff $s = s^*$.

Upon rearrangement, this equation becomes

$$(31) \qquad f\,[s^2] \equiv (D-1)s^4 + (D\beta - \alpha)s^2 + Da_{11}^2\, a_{22}^2 - a_{12}^2\, a_{21}^2 \le 0$$

with

$$(32) \qquad D \equiv AC,\ \alpha \equiv a_{12}^2 + a_{21}^2 \ \ \textit{and}\ \ \beta \equiv a_{11}^2 + a_{22}^2\,.$$

A unique solution to $f\,[\cdot] = 0$ (corresponding to $s = s^*$) implies $(D\beta - \alpha)^2 - 4(D-1)(Da_{11}^2\, a_{22}^2 - a_{12}^2\, a_{21}^2) = 0$. Moreover, in (31), $s \to 0$ and $0 < s^* < \infty$ imply $D < A$; and $s \to \infty$ implies $D - 1 < 0$. Finally, when (31) admits a unique solution, it is given by

$$(33) \qquad s^* = \sqrt{\frac{D\beta - \alpha}{2\,(1-D)}}\,.$$

Using $D - 1 < 0$, the condition $0 < s^* < \infty$ requires $D\beta - \alpha > 0$ and, therefore, $0 < \alpha/\beta < 1$. In summary,

COROLLARY 2: *Under the specification* (26), P(2) *and* P(3) *are satisfied iff*

C2(1): $\qquad h\,[D] \equiv (D\beta - \alpha)^2 - 4(D-1)(Da_{11}^2\, a_{22}^2 - a_{12}^2\, a_{21}^2) = 0$

has a real solution D^* *such that*

C2(2): $\qquad 0 < \alpha/\beta < D^* < 1.$

When $AC = D^* \in\ ]\alpha/\beta\,,\, 1[$, $f\,[s^2] < 0$ for $s \neq s^*$ and $f\,[s^2] = 0$ for $s = s^*$; and when $AC < D^*$, $f\,[s^2] < 0$ for all $s \ge 0$. Therefore, using C as a *bifurcation parameter*, we can claim

COROLLARY 3: *If the conditions of corollary* 1 *and* 2 *are satisfied, the dispersed equilibrium will be locally stable iff* $C < D^*/A \equiv C^*$; *and a time-indepndent spatial structure, characterised by a regular spacing between agglomerations* $d^* = 2\pi/s^*$, *will emerge at*

C3: $\qquad C = C^*.$[11]

3.2.3 AGGLOMERATED SPATIAL STRUCTURE

The conditions under which C2(2) is satisfied are summarised by the following two lemmata:

LEMMA 1: *$h[D] = 0$ admits two real solutions $D^{\pm}$ with $0 < D^{\pm} \leq 1$ iff*

L1: $\min\{a_{11}, a_{22}\} < \min\{a_{12}, a_{21}\} < \max\{a_{12}, a_{21}\} < \max\{a_{11}, a_{22}\}$.

LEMMA 2: *If* L1 *is satisfied then* (a) $D^{-} < \alpha/\beta$; *and* (b) $D^{+} > \alpha/\beta$ *iff*

L2: $$\frac{a_{12}^2 + a_{21}^2}{a_{11}^2 + a_{22}^2} < \min\left\{1, \frac{a_{12}^2 a_{21}^2}{a_{11}^2 a_{22}^2}\right\}.$$

Thus L1 and L2 are necessary for the emergence of an agglomerated, time–independent spatial structure. The corresponding characteristics of this structure can be determined through

LEMMA 3: *If $h[D] = 0$ admits a real solution $D^{*} \in \,]\alpha/\beta\, , 1[$,*

L3: $$s^{*} = \sqrt{\theta - a_{22}^2 + \sqrt{\theta(a_{11}^2 - a_{22}^2 + \theta)}} \quad \textit{with} \quad \theta \equiv \frac{(a_{12}^2 - a_{22}^2)(a_{21}^2 - a_{22}^2)}{a_{11}^2 + a_{22}^2 - (a_{12}^2 + a_{21}^2)}.$$

The proof of all three lemmata is in the appendix.[12] Using these, we can now state our results in final form. Let $d_{ij} \equiv 1/a_{ij}$, i and $j = 1, 2$, representing the average distance over which the corresponding spatial interaction occurs. This notation will allow us to express our conditions in a more intuitive way.

THEOREM: *Let $g_{ij}[x, x'] = \exp(-a_{ij}|x - x'|)$ for i and $j = 1, 2$, and $x, x' \geq 0$.*

(a) *A time-independent spatial structure will emerge iff the average distances of spatial interaction satisfy*

T(1): $\min\{d_{11}, d_{22}\} < \min\{d_{12}, d_{21}\} < \max\{d_{12}, d_{21}\} < \max\{d_{11}, d_{22}\}$

T(2): *if $d_{11}^2 d_{22}^2 < d_{12}^2 d_{21}^2$ then $d_{12}^2 + d_{21}^2 < d_{11}^2 + d_{22}^2$ must hold*

T(3): *if* $d_{11}^2 d_{22}^2 > d_{12}^2 d_{21}^2$ *then* $\frac{1}{d_{12}^2} + \frac{1}{d_{21}^2} < \frac{1}{d_{11}^2} + \frac{1}{d_{22}^2}$ *must hold*

and the marginal utilities satisfy

$$\text{T(4):} \quad \frac{n_1^0}{v_1^0}\frac{\partial v_1}{\partial E_{11}^0} + \frac{n_2^0}{v_2^0}\frac{\partial v_2}{\partial E_{22}^0} \le 0$$

$$\text{T(5):} \quad \frac{1}{d_{11}^2}\frac{n_1^0}{v_1^0}\frac{\partial v_1}{\partial E_{11}^0} + \frac{1}{d_{22}^2}\frac{n_2^0}{v_2^0}\frac{\partial v_2}{\partial E_{22}^0} \le 0.$$

(b) *The spatial structure will emerge at*

$$\text{T(6):} \quad C \equiv \left[\frac{\partial v_1}{\partial E_{12}^0}\frac{\partial v_2}{\partial E_{21}^0}\right] \div \left[\frac{\partial v_1}{\partial E_{11}^0}\frac{\partial v_2}{\partial E_{22}^0}\right] = \frac{d_{12}^2 d_{21}^2}{d_{11}^2 d_{22}^2} D^* \equiv C^* > 0$$

where D^* *is the larger solution to*

$$\left[D\left[\frac{1}{d_{11}^2} + \frac{1}{d_{22}^2}\right] - \left[\frac{1}{d_{12}^2} + \frac{1}{d_{21}^2}\right]\right]^2 - 4(D-1)\left[\frac{D}{d_{11}^2 d_{22}^2} - \frac{1}{d_{12}^2 d_{21}^2}\right] = 0.$$

(c) T*he distance between potential agglomerations will be given by*

$$\text{T(7):} \quad d^* = 2\pi \div \sqrt{\theta - \frac{1}{d_{22}^2} + \sqrt{\theta\left[\frac{1}{d_{11}^2} - \frac{1}{d_{22}^2} + \theta\right]}} \quad \textit{with}$$

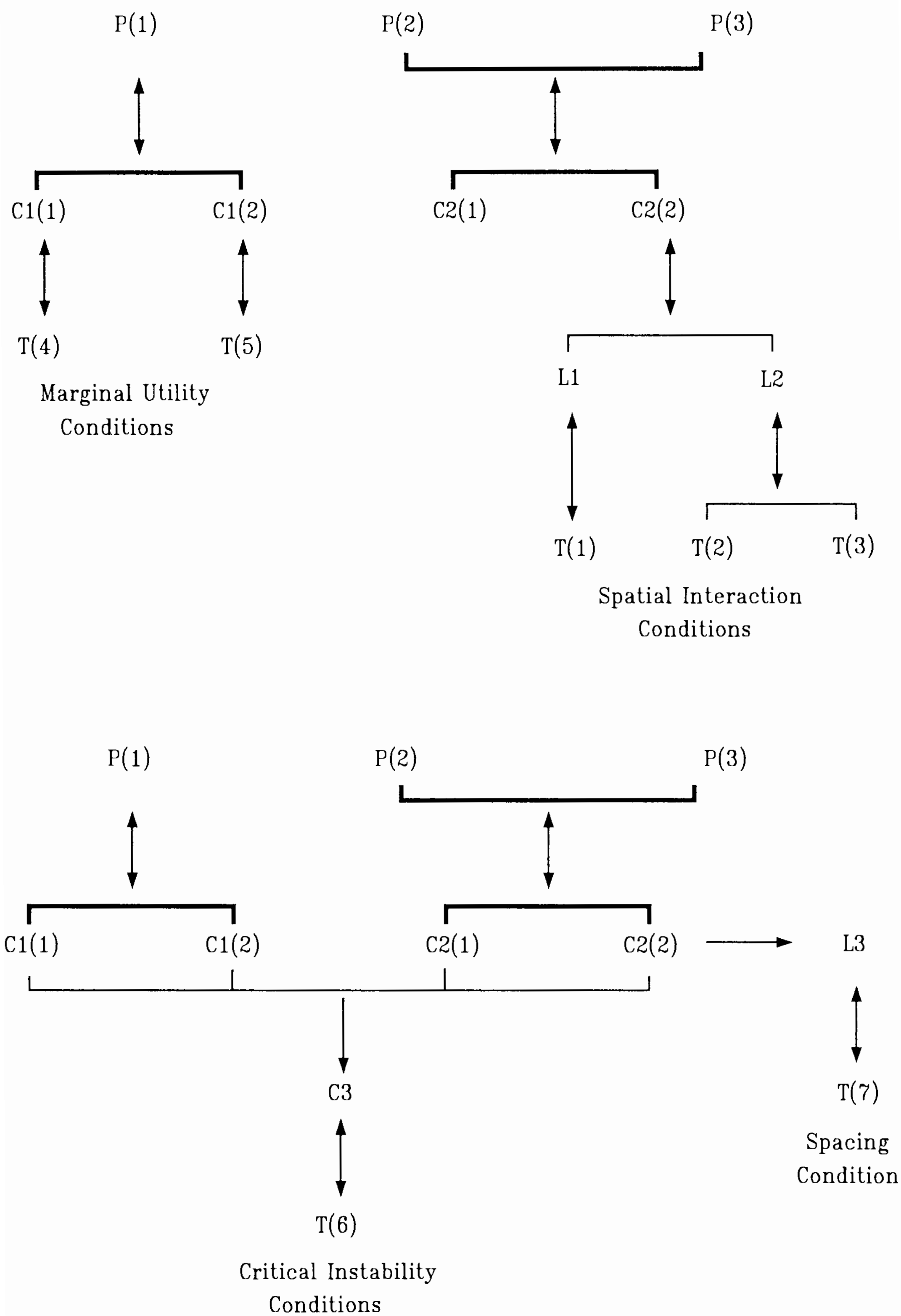
$$\theta \equiv \left[\frac{1}{d_{12}^2} - \frac{1}{d_{22}^2}\right]\left[\frac{1}{d_{21}^2} - \frac{1}{d_{22}^2}\right] \div \left[\left[\frac{1}{d_{11}^2} + \frac{1}{d_{22}^2}\right] - \left[\frac{1}{d_{12}^2} + \frac{1}{d_{21}^2}\right]\right].$$

The deductive structure which led to the statement of the theorem appears in figure one, where double arrows denote equivalence and single arrows denote sufficiency.

3.2.4 INTERPRETATION

As we have argued in section 3.2.2, when the two types of agent are characterised by

FIGURE 1: Logical structure of the argument.



the same average distance of interaction ($d_{11} = d_{22}$ and $d_{12} = d_{21}$), no spatial structure can emerge. Let $d_{11} > d_{22}$. Condition T(1) implies that, for agents of type one, the average distance of interaction with agents of the same type is longer than that corresponding to agents of the other type. The converse holds for agents of type two. Such an asymmetry makes both types of agent less sensitive to the proximity of type one than type two. It also ensures a relatively strong spatial interdependence between the two types through the implication that both average distances for cross–type interaction must be longer than one of the two average distances for own–type interaction. Finally, condition T(1) is compatible with either one of the alternative conditions T(2) and T(3) for a wide variety of situations which can be interpreted in a straightforward manner.[13]

If every agent experiences a positive externality effect from agents of the same type, the system is unstable. If, on the other hand, every agent dislikes the presence of his own kind, conditions T(4) and T(5) of the theorem are satisfied with strict inequality. However, it is not necessary that both types be subject to negative own–type externality effects if a spatial structure is to emerge. Conditions T(4) and T(5) can still be satisfied when the positive own–type externality effect of one type is balanced by a negative own–type externality effect of the other type which is strong enough. The positivity requirement in T(6) imposes further constraints upon the various patterns of spatial interdependence which are consistent with the transition from a dispersed to an agglomerated equilibrium. Namely, that if the own–type externality effects have the same sign, so must the cross–type externality effects; and if the own–type externality effects have opposing signs, so must the cross–type externality effects.

As long as $C < C^*$, the dispersed equilibrium is locally stable. We may now imagine that the cross–type externality effects become gradually stronger relative to the own–type externality effects. This, for example, could be accounted for through an increasing degree of specialisation with development, which would imply an increasing frequency of trips toward agents of the other type and which would affect the corresponding marginal utilities. Under these circumstances, C will gradually increase. At the same time, the average distances of interaction must also increase because development facilitates spatial interaction. In consequence, C^* could be affected either way. If however the average distances for own–type interaction increase enough relative to the corresponding distances for cross–type interaction, C may approach C^*; and the moment the two become equal, a regular pattern of locations on which first instabilities occur is created. In our context, this marks the potential emergence of a settlement pattern, and it is characterised by a delicate balance between agglomerative and deglomerative factors. Emergence itself occurs only when the former just become more important than the latter. For example, consider the case in which all agents are subject to negative own–type and positive cross–type externality effects. In this case, where conditions T(4) – T(6) are satisfied, the two types of agent may represent households (type one) and firms (type two). "Households are attracted to places where the density of firms is high because opportunities there are more numerous; and they are repulsed by places where the density of households is high because they dislike congestion. Firms are attracted to places where the density of consumers is high because there the expected volume of business is large; and they are repulsed by places where the density of sellers is high because of the stronger competition prevailing there." [14] For $C = C^*$, congestion effects within each type just balance the need to interact with the other type. If $d_{11} > d_{12} > d_{21} > d_{22}$, a regular pattern of central places is possible, where household agglomerations coincide with smaller firm agglomerations — smaller because firms in this example are more sensitive than households to negative own–type externality effects.

Condition T(7) of the theorem implies that the spacing of settlements depends only upon the average distances of interaction. Neither the population densities and the utility levels of the dispersed equilibrium, nor the preferences of agents for interaction, appear to affect spacing. The way average distances affect spacing is determined directly through condition T(7) as

$$(34) \qquad \frac{\partial d^*}{\partial d_{ij}} > 0 \;\; \textit{and} \;\; \frac{\partial d^*}{\partial d_{ii}} < 0 \;\; \textit{for } i \textit{ and } j = 1, 2, \textit{ and } i \neq j.$$

Consider once more the case in which all agents are subject to negative own–type and positive cross–type externality effects. As d_{ij} increases with development, the need to interact with agents of the other type is satisfied with larger, more widely spaced agglomerations at equilibrium. This conforms with our intuition about the rôle of better transportation networks on the evolution of a settlement pattern. As d_{ii} increases, on the other hand, the contrary should happen: smaller, less widely spaced agglomerations would arise in response to the aversion toward agents of the same type. More generally, T(7) suggests that the connection between technological development and spatial structure is quite complex, and this might be a reason why related theoretical issues (such as the impact of telecommunications on the equilibrium settlement pattern) have become so controversial.

4. CONCLUDING REMARKS

Until recently, say around 3,500 BC, human populations were evenly dispersed over homogeneous land. Shortly before that time, societal evolution was well under way, gradually fostering more specialisation of economic agents (Parsons (1977)). Specialisation, in turn, implied a fundamental need to interact — hence spatial interdependence among agents. It seems that around 3,500 BC spatial interdependence reached a critical level and the first cities were born: the spatially uniform population pattern was replaced by a spatially differentiated one. Over the next two millenia settlement patterns became established. Any one of these reflected well local geography: dense in the fertile crescents, sparse over rugged terrain, attracted by great waterways and other advantageous features of the landscape, settlements appeared almost randomly distributed. If however any such systematic distortions were removed, a regular pattern of central places would have emerged over the boundless plain. This ideal regular pattern is the object of our concern.

Our arguments support the emergence of a regular settlement pattern only in the neighbourhood of the dispersed equilibrium. Beyond, we need to employ an assumption that ascertains continuing accumulation on the regular grid of initial growth. This is a direct consequence of retaining only linear terms in the Taylor expansions of our model. Although second–order terms could provide us with further information about the character of settlement growth, technical difficulties involved discouraged their retention.

We have provided the rudiments of a naive, but trully endogenous central place theory. Nevertheless, the crucial step of translating all this onto the homogeneous plain, thus testing the fundamental conjecture of Christaller (1933), Lösch (1944) and other classical location theorists about hexagonal market areas, is yet to be taken. There are some indications that this conjecture may be sound: hexagons arise as an equilibrium solution of an adjustment process in various instances. For example, the Fokker–Planck equation can generate this type of spatial structure (Haken (1978, chapter 8)). However, spatial interdependence in the Fokker–Planck equation is limited to purely local interactions; and it seems difficult to extend these results into the kind of spatial interdependence encountered over a system of human settlements.

For a class of models which contains our own, ten Raa (1984) has demonstrated that an equilibrium over a compact region exhibits lower density further away from the centre, thus forming a single agglomeration. As the region becomes larger, the equilibrium distribution becomes flatter until at the limit, an instance of which is represented by our model, the dispersed equilibrium obtains. We are therefore led to believe that our local instability conditions could be extended to this more general framework of compact regions. There, we expect that the settlement pattern will be more closely packed over more densely populated, central regions. In consequence, the spatial structure of a settlement pattern over a compact region would emerge as the superposition of two agglomerative trends, one reflecting the regional centrality of locations, and another reflecting the spatial behaviour of agents. It may also well be that nonlinear terms in the Taylor series expansion would reveal differential growth favouring centrally located settlements. All this bears remarkable similarity to Isard's (1956, p. 272) classic diagram, about a central place system warped by agglomeration. Clearly though, between here and there, there is much to do indeed.

FOOTNOTES

1 For a review of central place theory, see Mulligan (1984).

2 Geographical homogeneity in PS was achieved by placing the economy on a circle, thereby eliminating the impact of boundaries. We, on the other hand, eliminate boundaries by placing the economy on a line of infinite length. Equivalently, we could have employed a finite system with periodic boundaries.

3 Utility here should be understood to represent the reduced form of some constrained optimisation problem. For example, it could be the indirect utility function of an individual. In the context of urban economics, let the indirect utility of an individual in location x at time t be given by

$$V\,[R\,[x\,,\,t]\,,\,Y - C\,[x\,,\,t]] \equiv \max_{z\,,q} \left\{ U\,[z\,,\,q] \,\middle|\, z + R\,[x\,,\,t]q = Y - C\,[x\,,\,t] \right\},$$

where U is the direct utility, z is the amount of a numéraire good consumed, q is the amount of land consumed (the inverse of population density), R is the rent on land, C is the cost of transportation and Y is income. Let $C\,[x\,,\,t] = c\,[E\,[x\,,\,t]]$, i.e. transportation costs for someone in location x at time t depend upon how the population is currently distributed relative to that location (see Papageorgiou and Thisse (1985), and Fugita (1988) for related justifications).

If the rent is determined by local demand only, $R[x, t] = r[n[x, t]] = r[q^{-1}[x, t]]$, then

$$V[R[x, t], Y - C[x, t]] = v[n[x, t], E[x, t]],$$

which is the formulation in PS.

4 The adjustment process for the deterministic counterpart of a continuous–time and space Markovian model is given by

$$\frac{\partial}{\partial t} n[x, t] = \int_{\mathscr{X}} q_{xx'}[t] n[x', t] \mathrm{d}x' - \int_{\mathscr{X}} q_{x'x}[t] n[x, t] \mathrm{d}x',$$

where $q_{xx'}[t]$ is the migration rate from x' to x at time t. (For a derivation in the case of a continuous–time Markovian process see de Palma and Lefèvre (1983).) Migration rates, in turn, are defined as $q_{xx'}[t] \equiv \partial p_{xx'}[t, \Delta t]/\partial \Delta t \big|_{\Delta t=0}$ where $p_{xx'}[t, \Delta t]$ is the probability density that an agent at x' will migrate to x during the time–interval $[t, t + \Delta t[$. Let the decision to migrate be a two–stage process, namely, $p_{xx'}[t, \Delta t] = p_{x|x'}[t] p_{x'\cdot}[t, \Delta t]$, where $p_{x'\cdot}[t, \Delta t]$ is the probability density that someone in x' will reassess the advantages of his location during $[t, t + \Delta t[$, and $p_{x|x'}[t]$ is the conditional probability density that he will then migrate to x. In consequence, $q_{xx'}[t] = p_{x|x'}[t] \cdot \cdot \partial p_{x'\cdot}[t, \Delta t]/\partial \Delta t \big|_{\Delta t=0}$; and if the rate at which locations are reassessed is a constant then $q_{xx'}[t] = p_{x|x'}[t]$ — provided the unit of time has been chosen in an appropriate manner. Finally, if migrations are costless, we can write $p_{x|x'}[t] = p[x, t]$ and (5) follows.

5 Let $0 < v_{min} \leq v[n[x, t], E[x,t]] \leq v_{max} < \infty$, where v_{min} and v_{max} are constants. Then

$$\frac{1}{v_{max}} \leq \frac{\int_{\mathscr{X}} n[x', t] \mathrm{d}x'}{\int_{\mathscr{X}} n[x'', t]\, v[n[x'', t], E[x'', t]] \mathrm{d}x''} \leq \frac{1}{v_{min}},$$

which implies that (5) is well–behaved.

6 Disregarding second–order terms, we have

$$\int_{\mathscr{X}} (n^0 + e^{\lambda t}\mathrm{d}n\,[x''])(v^0 + \frac{\partial v}{\partial n^0}\, e^{\lambda t}\mathrm{d}n\,[x''] + \frac{\partial v}{\partial E^0}\,\mathrm{d}E^0\,[x''])\mathrm{d}x'' =$$
$$n^0 v^0 \int_{\mathscr{X}} \mathrm{d}x'' + n^0 \frac{\partial v}{\partial n^0}\, e^{\lambda t} \int_{\mathscr{X}} \mathrm{d}n\,[x'']\mathrm{d}x'' + n^0 \frac{\partial v}{\partial E^0} \int_{\mathscr{X}} \mathrm{d}E^0\,[x'']\mathrm{d}x'' +$$
$$v^0 e^{\lambda t} \int_{\mathscr{X}} \mathrm{d}n\,[x'']\mathrm{d}x'' =$$
$$n^0 v^0 \int_{\mathscr{X}} \mathrm{d}x'' + n^0 \frac{\partial v}{\partial E^0} \int_{\mathscr{X}}\int_{\mathscr{X}} g\,[x'', x']e^{\lambda t}\mathrm{d}n\,[x']\mathrm{d}x'\mathrm{d}x'' =$$
$$n^0 v^0 \int_{\mathscr{X}} \mathrm{d}x'' + n^0 \frac{\partial v}{\partial E^0}\, G e^{\lambda t} \int_{\mathscr{X}} \mathrm{d}n\,[x']\mathrm{d}x' = n^0 v^0 \int_{\mathscr{X}} \mathrm{d}x'',$$

since $\int_{\mathscr{X}} \mathrm{d}n\,[x]\mathrm{d}x = 0$ because of fixed total population size.

7 This is a more general formulation than the one proposed in footnote three. There, the rent was determined by local demand only which, here, could be expressed as $R_{\mathrm{i}}[x , t] = r_{\mathrm{i}}[n_{\mathrm{i}}[x , t] , n_{\mathrm{j}}[x , t]]$ for i and $j = 1 , 2$, and $i \neq j$. Under this assumption, $v_{\mathrm{i}}[\cdot]$ would depend on both local populations and spatial externalities. If, however, the rent is determined by global demand, $R_{\mathrm{i}}[x , t] = r_{\mathrm{i}}[E_{\mathrm{ii}}[x , t] , E_{\mathrm{ij}}[x, t]]$, utility can be written as a function of spatial externalities only.

8 We exclude the case of complex roots because the corresponding solution (21) would exhibit oscillations in time — a possibility which is meaningless in the context of our study.

9 At s^*, using condition (b), the constant term of the dispersion equation is zero because one of the roots is zero. Therefore condition (b) also implies that the roots must be real at $s = s^*$.

10 The dispersion equation (25) becomes

$$\left[\lambda - \frac{a}{a_{11}^2+s^2}\right]\left[\lambda - \frac{b}{a_{11}^2+s^2}\right] = \frac{c}{(a_{12}^2+s^2)^2}$$

with a , b , c constants. Taking into account that, at s^*, $\lambda = 0$ and $\partial\lambda/\partial s = 0$ for the implicit function $\lambda[s]$ defined by the dispersion equation, we can write this equation as

$$\lambda^2 - \frac{(a+b)\lambda}{a_{11}^2+s^2} = 0$$

which establishes our claim.

11 Since $A > 0$, $D > 0$ and $A > D^*$, we have $0 < C^* < 1$.

12 A more direct proof of lemma three is the following. Since $f[s^2]$ is a maximum at $s = s^*$, taking into account (29), we conclude that s^* satisfies the cubic equation

$$\frac{d}{ds}\left[\frac{a_{12}^2+s^2}{a_{11}^2+s^2}\,\frac{a_{21}^2+s^2}{a_{22}^2+s^2}\right] = 0,$$

which has L3 as a solution. We have been unable to solve this equation directly.

13 For example, let $d_{11} = c_1 + \varepsilon_{11}$, $d_{22} = c_1 - \varepsilon_{11}$, $d_{12} = c_2 \pm \varepsilon_{12}$, and $d_{21} = c_2 \mp \varepsilon_{12}$. Then, condition T(2) implies $0 < \varepsilon_{12} < \varepsilon_{11}$ which is compatible with T(1). Condition T(3) arises under similar constraints, applied to the impedance parameters rather than to the average distances: let $1/d_{11} = c_1 - \varepsilon_{11}$, $1/d_{22} = c_1 + \varepsilon_{11}$, $1/d_{12} = c_2 \mp \varepsilon_{12}$, and $1/d_{21} = c_2 \pm \varepsilon_{12}$ which, as before, is compatible with T(1).

14 Papageorgiou and Thisse (1985, p. 20).

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APPENDIX

This appendix contains proofs for the three lemmata. We first define the following reduced variables ($\tilde{\alpha}_{11} \neq 0$ since $\alpha_{11}^2 \neq \alpha_{22}^2$):

$$
\begin{aligned}
\tilde{a}_{11}^2 &\equiv a_{11}^2 - a_{22}^2 \\
\tilde{a}_{12}^2 &\equiv a_{12}^2 - a_{22}^2 \\
\tilde{a}_{21}^2 &\equiv a_{21}^2 - a_{22}^2 \\
\tilde{a}_{22}^2 &\equiv a_{22}^2 - a_{22}^2 = 0
\end{aligned}
\qquad\qquad
\begin{aligned}
\hat{a}_{11}^2 &\equiv \tilde{a}_{11}^2 / \tilde{a}_{11}^2 = 1 \\
\hat{a}_{12}^2 &\equiv \tilde{a}_{12}^2 / \tilde{a}_{11}^2 \\
\hat{a}_{21}^2 &\equiv \tilde{a}_{21}^2 / \tilde{a}_{11}^2 \\
\hat{a}_{22}^2 &\equiv \tilde{a}_{22}^2 / \tilde{a}_{11}^2 = 0\,.
\end{aligned}
$$

We also define

$$
\begin{aligned}
\alpha &\equiv a_{12}^2 + a_{21}^2 \ \text{(see (32))} \\
\beta &\equiv a_{11}^2 + a_{22}^2 \ \text{(see (32))} \\
\gamma &\equiv a_{11}^2 - a_{12}^2 = \tilde{a}_{11}^2 - \tilde{a}_{12}^2 \equiv \tilde{\gamma} \\
\delta &\equiv a_{11}^2 - a_{21}^2 = \tilde{a}_{11}^2 - \tilde{a}_{21}^2 \equiv \tilde{\delta} \\
\zeta &\equiv a_{22}^2 - a_{21}^2 = \tilde{a}_{22}^2 - \tilde{a}_{21}^2 \equiv \tilde{\zeta} \\
\eta &\equiv a_{22}^2 - a_{12}^2 = \tilde{a}_{22}^2 - \tilde{a}_{12}^2 \equiv \tilde{\eta}
\end{aligned}
\qquad\qquad
\begin{aligned}
\hat{\alpha} &\equiv \hat{a}_{12}^2 + \hat{a}_{21}^2 \\
& \\
\hat{\gamma} &\equiv \tilde{\gamma}/\tilde{a}_{11}^2 = 1 - \hat{a}_{12}^2 \\
\hat{\delta} &\equiv \tilde{\delta}/\tilde{a}_{11}^2 = 1 - \hat{a}_{21}^2 \\
\hat{\zeta} &\equiv \tilde{\zeta}/\tilde{a}_{11}^2 = 1 - \hat{a}_{21}^2 \\
\hat{\eta} &\equiv \tilde{\eta}/\tilde{a}_{11}^2 = 1 - \hat{a}_{12}^2\,.
\end{aligned}
$$

PROOF OF LEMMA 1: After a long but straightforward calculation, the solutions to $h[D] = 0$ are given by

(A1) $$D^{\pm} = (-\gamma\zeta - \delta\eta \pm \sqrt{4\gamma\delta\zeta\eta}\,) \div (\eta - \gamma)^2.$$

Note that $\eta \neq \gamma$ since $\alpha_{11} \neq \alpha_{22}$. Thus $h[D] = 0$ admits two real solutions iff $\gamma\delta\zeta\eta > 0$. Moreover, these solutions are strictly positive iff $\pm\sqrt{4\gamma\delta\zeta\eta} > \gamma\zeta + \delta\eta$. Now $\gamma\zeta + \delta\eta < 0$ is sufficient for $D^+ > 0$: if $\gamma\zeta + \delta\eta \geq 0$, raising $\sqrt{4\gamma\delta\zeta\eta} > \gamma\zeta + \delta\eta$ to its square, we obtain $(\gamma\zeta - \delta\eta)^2 < 0$ if $\gamma\zeta \neq \delta\eta$ — an impossibility. Likewise, $\gamma\zeta + \delta\eta < 0$ and $\gamma\zeta \neq \delta\eta$ imply that $D^- > 0$. Since $\alpha_{11} \neq \alpha_{22}$ and $\alpha_{12} \neq \alpha_{21}$, $\gamma\zeta \neq \delta\eta$ holds. In consequence

(A2) $$D^{\pm} > 0 \ \textit{iff} \ \gamma\delta\zeta\eta > 0,\ \gamma\zeta + \delta\eta < 0 \ \textit{and} \ \gamma\zeta \neq \delta\eta.$$

Using (A1), it can be shown that $D^{\pm} < 1$ iff $\pm\sqrt{4\gamma\delta\zeta\eta} < \gamma\delta + \zeta\eta$. Now $\gamma\delta + \zeta\eta > 0$ is necessary for $D^{+} \leq 1$: raising $\sqrt{4\gamma\delta\zeta\eta} < \gamma\delta + \zeta\eta$ to its square, we obtain $(\gamma\delta - \zeta\eta)^2 \geq 0$. Also, $\gamma\delta + \zeta\eta \geq 0$ is sufficient for $D^{-} < 1$. Therefore

(A3) $\quad D^{\pm} < 1$ *iff* $\gamma\delta\zeta\eta > 0$ *and* $\gamma\delta + \zeta\eta > 0$.

TABLE 1: Alternative sign patterns.

	γ	δ	ζ	η
(a)	+	+	+	+
(b)	+	+	−	−
(c)	−	+	+	−
(d)	−	−	+	+
(e)	+	−	−	+
(f)	+	−	+	−
(g)	−	+	−	+
(h)	−	−	−	−

Consider (A2) and (A3). The inequality $\gamma\delta\zeta\eta > 0$ holds iff one of (a) – (h) in table one holds. Of these, (c), (e), (f) and (g) violate $\gamma\delta + \zeta\eta > 0$, while (a) and (h) violate $\gamma\zeta + \delta\eta < 0$. Thus only (b) and (d) are consistent with (A2) and (A3). It can be shown that these imply L1. QED

PROOF OF LEMMA 2: Using our definitions, we have

(A4) $$\frac{\alpha}{\beta} = \frac{\hat{\alpha}+2B}{1+2B} \quad \textit{with } B \equiv \frac{a_{22}^2}{\tilde{a}_{11}^2} > 0,$$

where $\tilde{\alpha}_{11}^2 \neq 0$ because $\alpha_{11}^2 \neq \alpha_{22}^2$. In consequence,

(A5) $$D^{\pm} > \frac{\alpha}{\beta} \;\; \textit{iff} \;\; 2B < \frac{D^{\pm} - \hat{\alpha}}{1 - D^{\pm}}.$$

Observe that L1 and $D^{\pm} > \alpha/\beta$ imply $\alpha/\beta < 1$. Based on L1, we conclude that $D^{\pm} > \alpha/\beta$ only if $D^{\pm} - \hat{\alpha} > 0$; and since $D^{\pm} - \hat{\alpha} = 2(-\hat{a}_{12}^2\hat{a}_{21}^2 \pm \sqrt{\hat{\gamma}\hat{\delta}\hat{\zeta}\hat{\eta}}\,)$, we have $D^{-} - \hat{\alpha} <$

0, which implies part (a) of the lemma.

We now turn to part (b). Using (A5)

$$\text{(A6)} \qquad D^+ > \frac{\alpha}{\beta} \ \mathit{iff}\ 2B < \frac{2\left[-\hat{a}_{12}^2\hat{a}_{21}^2+\sqrt{\hat{\gamma}\hat{\delta}\hat{\zeta}\hat{\eta}}\,\right]}{1-\hat{a}+2\left[\hat{a}_{12}^2\hat{a}_{21}^2-\sqrt{\hat{\gamma}\hat{\delta}\hat{\zeta}\hat{\eta}}\,\right]}.$$

Simplification and rearrangement yields

$$\text{(A7)} \qquad D^+ > \frac{\alpha}{\beta} \ \mathit{iff}\ B - \Omega < \sqrt{\Omega(1+\Omega)} \ \ \mathit{with}\ \ \Omega \equiv \frac{\hat{a}_{12}^2\hat{a}_{21}^2}{1-\hat{\alpha}}.$$

There are two cases. Firstly note that $B - \Omega$ >(<) 0 iff A <(>) 1. If $B - \Omega > 0$, (A7) implies

$$\text{(A8)} \qquad (B - \Omega)^2 < \Omega(1 + \Omega)$$

which, in turn, implies $\alpha/\beta < A$. Therefore, in this case, $D^+ > \alpha/\beta$ iff $\alpha/\beta < A < 1$. Secondly, if $B - \Omega < 0$, $D^+ > \alpha/\beta$ (see (A7)) and $\alpha/\beta < A < 1$. We conclude that $D^+ > \alpha/\beta$ iff $\alpha/\beta < \min\{1\ , A\}$. QED

PROOF OF LEMMA 3: We first note that

$$\text{(A9)} \qquad \frac{\alpha}{\tilde{a}_{11}^2} = \hat{a}_{12}^2 + \hat{a}_{21}^2 + 2B \ \ \mathit{and}\ \ \frac{\beta}{\tilde{a}_{11}^2} = 2B + 1.$$

Using these in conjunction with (33), we arrive at

$$\text{(A10)} \qquad \frac{s^{*2}}{\tilde{a}_{11}^2} = -B + \frac{D^+ - \hat{\alpha}}{2(1 - D)} = -B + \sqrt{\Omega(1+\Omega)} + \Omega.$$

Therefore $s^{*2} = -a_{22}^2 + \tilde{a}_{11}^2\left(\sqrt{\Omega(1+\Omega)} + \Omega\right)$, and L3 follows by taking into account that $\tilde{a}_{11}^2\,\Omega = \theta$. QED