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SERIAL CORRELATION OF SUNSPOT EQUILIBRIA (RATIONAL BUBBLES) IN TWO POPULAR MODELS OF MONETARY ECONOMIES

by

Kiminori Matsuyama*

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Abstract

This paper first shows a simple way of determining serial correlation of stationary sunspot equilibria (i.e., rational bubbles). This result is then applied to two models of money: an overlapping generations model and a model of money-in-the-utility-function.

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*Department of Economics, Northwestern University, 2003 Sheridan Road, Evanston, IL 60208, U.S.A. I thank W. Brock for his constructive letter in response to Matsuyama (1988), which has stimulated me to write this paper.

1. Introduction

Sunspot equilibria as an explanation of economic fluctuations have attracted much attention in recent years: see Shell (1987) and Woodford (1987). "Sunspots" represent purely extrinsic uncertainty. The randomness unrelated to the economic fundamentals can cause fluctuations simply because expectations are self-fulfilling. It has been suggested that the theory of sunspots may be a useful alternative to the real business cycle theory, in which fluctuations are driven by exogenous shocks to fundamentals. However, most sunspot equilibria existing in the literature, such as Azariadis (1981) and Azariadis and Guesnerie (1986), have negative serial correlations.[1] This is somewhat unsatisfactory since most economic time series of interest are strongly positively serially correlated.[2]

This paper first shows that the monotone (oscillatory) convergence to a steady state is sufficient for the existence of stationary sunspot equilibria with positive (negative) serial correlation. This is done in section 2 within a framework of dynamics essentially devoid of economic structure. Then, section 3 applies this result in order to demonstrate the existence of positively serially correlated sunspot equilibria in two models of monetary economies. The first is an overlapping generations economy with the government financing a fixed expenditure by printing fiat money. This economy possesses two monetary steady states, one of which has positively serially correlated sunspot equilibria in its vicinity. The second is a model of money-in-the-utility-function with the infinitely lived representative agent. It provides an example in which the unique monetary steady state has

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[1] The notable exception is Woodford's (1986) example in his model with capital accumulation.

[2] See, for example, the tables in Nelson and Plosser (1982).

positively serially correlated sunspot equilibria in its vicinity. In this model, an increase in the rate of monetary growth reduces the one period lagged autocorrelation of real balances and the nominal interest rate, and as it continues to increase, local sunspot equilibria eventually disappear. It is also shown that sunspot equilibria are more likely to exist, when the agent's discount rate is small. Section 4 considers the existence of global sunspot equilibria in the money-in-the-utility-function model. It demonstrates that the determinacy of the steady state does not rule out stationary sunspot equilibria and that the oscillatory (monotone) convergence does not rule out positive (negative) serial correlation. These exercises illustrate some crucial differences between the two popular models of money. The main result is summarized in section 5.

The possible effects of purely extrinsic uncertainty have been studied in the money-in-the-utility-function approach by Obstfeld and Rogoff (1986) and Diba and Grossman (1988) under the name of "rational bubbles".[3] They are concerned with divergent bubbles (i.e., nonstationary sunspot equilibria). The present analysis suggests that their "simplifying" assumption rules out stationary sunspot equilibria. The possibility of sunspots cannot be ignored. Hopefully, this example would convince the critic of sunspot equilibria that they are not "flukes" obtained only in an overlapping generations model or its reinterpretation as a finance constrained economy.[4]

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[3]Some writers use the word "bubbles" in a different sense. In Tirole (1985) and Weil (1987), it means the difference between the price of an asset and the (expected) present discounted value of its dividends.

[4]The money-in-the-utility-function model presented below can also produce deterministic cycles and chaos: see Matsuyama (1988, 1989).

2. Serial Correlation of Sunspot Equilibria

Consider the dynamics given by the following first-order stochastic difference equation:

$$(1)\qquad E_tA(x_{t+1}) = B(x_t) ,$$

where E_t is the expectations operator, A and B are smooth functions on an interval F, the feasibility set. Time extends from zero to infinity. The variable x_t is a jump variable, thus a sequence of $\{x_0, x_1, \ldots\}$ is an "equilibrium" if it satisfies (1) and stay in F for all t. Assume that there exists x* in the interior of F that solves $A(x^*) = B(x^*)$. Clearly, $\{x^*, x^*, x^*, \ldots\}$ solves (1), providing a steady state equilibrium.

To consider the dynamics in the vicinity of x*, assume that $A'(x^*) \neq 0$. The Implicit Function Theorem implies that there exists an open interval U containing x* in which the following dynamics can be defined:

$$(2)\qquad x_{t+1} = C(x_t) \equiv A^{-1}(B(x_t)) .$$

Any solution of (2) that stays in U for all t provides a local nonstochastic equilibrium path of (1).[5] From the well-known theorems there exists a continuum of nonstochastic equilibrium paths of (1) converging to the steady state (i.e., the steady state x* is indeterminate) if $|\rho| < 1$, where,

$$\rho \equiv C'(x^*) = B'(x^*)/A'(x^*) ,$$

and they converge monotonically to x* if $0 < \rho < 1$. On the other hand, they

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[5]Paths that leave U after finite periods may violate the feasibility condition or the transversality condition, thus unqualified for equilibria. Or, they may be legitimate equilibria, as the examples in Matsuyama (1988, 1989) and Woodford (1984, Example 6) show.

converge oscillatorily if $-1 < \rho < 0$.

There may also exist stochastic solutions of (1). Let Π be a two state Markov transition matrix:

$$\Pi = \begin{bmatrix} q_a & 1-q_a \\ 1-q_b & q_b \end{bmatrix} .$$

A two state stationary sunspot equilibrium is a quadruple (q_a, q_b, x_a, x_b) such that q_a and q_b lie in the open interval (0,1); $x_a \neq x_b$; and,

(3) $$q_a A(x_a) + (1-q_a)A(x_b) = B(x_a) ,$$

(4) $$(1-q_b)A(x_a) + q_b A(x_b) = B(x_b) .$$

Equations (3) and (4) jointly state that the Markov process with state space (x_a, x_b) and transition matrix Π is a solution of (1). Woodford (1984) shows that there exist two state stationary sunspot equilibria satisfying $x_a < x^* < x_b$ in every neighborhood of x^* if $|\rho| < 1$; i.e., if the steady state is indeterminate.

Appendix A shows that the k-period lagged autocorrelation of any time series generated by a two state Markov process with the transition matrix Π is equal to $(q_a+q_b-1)^k$. Thus, the serial correlation depends on the sign of q_a+q_b-1. Choose x_a and x_b such that $A(x_a) \neq A(x_b)$. Then, from (3) and (4),

(5) $$q_a + q_b - 1 = \frac{B(x_a) - A(x_b)}{A(x_a) - A(x_b)} + \frac{A(x_a) - B(x_b)}{A(x_a) - A(x_b)} - 1$$
$$= \frac{B(x_a) - B(x_b)}{A(x_a) - A(x_b)} .$$

The assumption $A'(x^*) \neq 0$ guarantees $A(x_a) \neq A(x_b)$ if $x_a, x_b \in (x^*-\delta, x^*+\delta)$

for a small $\delta > 0$. As δ approaches zero,

$$(6) \qquad q_a + q_b - 1 \rightarrow B'(x^*)/A'(x^*) = \rho \ .$$

from the Mean Value Theorem. Thus, the one period lagged autocorrelation of local sunspot equilibria is equal to ρ. In particular, monotone (oscillatory) convergence to the steady state implies the existence of sunspot equilibria with positive (negative) serial correlations.

Before proceeding it is worth pointing out that the above result on the local sunspot equilibria can be easily extended to more general Markov processes. Appendix B shows the case of finite state erdogic Markov processes. It may also be shown that, under mild conditions, any local stationary solution of (1) can be approximated by a AR(1) process: $x_{t+1} = (1-\rho)x^* + \rho x_t + \xi_{t+1}$, whenever $|\rho| < 1$. Two state Markov processes, however, are useful partly because they are simpler and partly because equation (5) is applicable for the global analysis.

3. Two Models of Money: Local Dynamics

The sunspot equilibria constructed by Azariadis (1981) and Azariadis and Guesnerie (1986) require $q_a + q_b < 1$, so that they are negatively serially correlated. This section discusses two models of economies with a steady state equilibrium with monotone convergence, and therefore, positively serially correlated sunspot equilibria.

Model 1. This model is due to Grandmont (1986, pp.70-71); see also Woodford (1984, Example 5) and Sargent (1987, Ch. 7). Consider an overlapping generations economy with one perishable good and no population growth. All agents live for two periods and have identical preferences $u(c_y) + v(c_0)$ and identical endowments, e_y and e_0, where c_y and e_y (c_0 and e_0) are their consumption and endowment when young (old) and u and v are both increasing, concave and satisfy the Inada conditions. Intergenerational exchange of the consumption good is possible through fiat money. In each period, the government finances its constant expenditure g by printing new money. Then, it can be shown that the equilibrium path of real balances m_t follows,

$$(7) \qquad E_t\ [(m_{t+1}-g)v'(e_0+m_{t+1}-g)] = m_t u'(e_y-m_t)\ ,$$

with $m_t \in [0,e_y)$. Assume that $u'(e_y) < v'(e_0)$, the "Samuelson case". Then, (7) has two steady states, m^* and m^{**} $(0 < m^* < m^{**})$, if $g > 0$ is sufficiently small. See Figure 1. There are two different inflationary tax rates consistent with the predetermined level of expenditure; the Laffer curve effect. There exist nonstochastic monotone converging paths to m^* (the hyperinflationrary one), and therefore, positively serially correlated stationary sunspot equilibria in its vicinity. The other steady state m^{**} may be indeterminate, but nonstochastic equilibrium paths converging to it cannot be monotone. This is because the nonstochastic version of (7), $(m_{t+1}-g)v'(e_0+m_{t+1}-g) = m_t u'(e_y-m_t)$, defines the unique backward dynamics $m_t = f(m_{t+1})$.[6]

As in Azariadis [1981], this model can be considered as a model of production instead of pure exchange. In this interpretation, e_y is endowment of leisure when young and each member of the young generation consumes c_y units of his own leisure and use a constant-returns-to-scale technology to transform $e_y - c_y$ units into the consumption good in order to purchase the fiat money and finance in old age consumption in excess of e_0. Then, both

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[6]The normality of old consumption, not the separability, is crucial for this result.

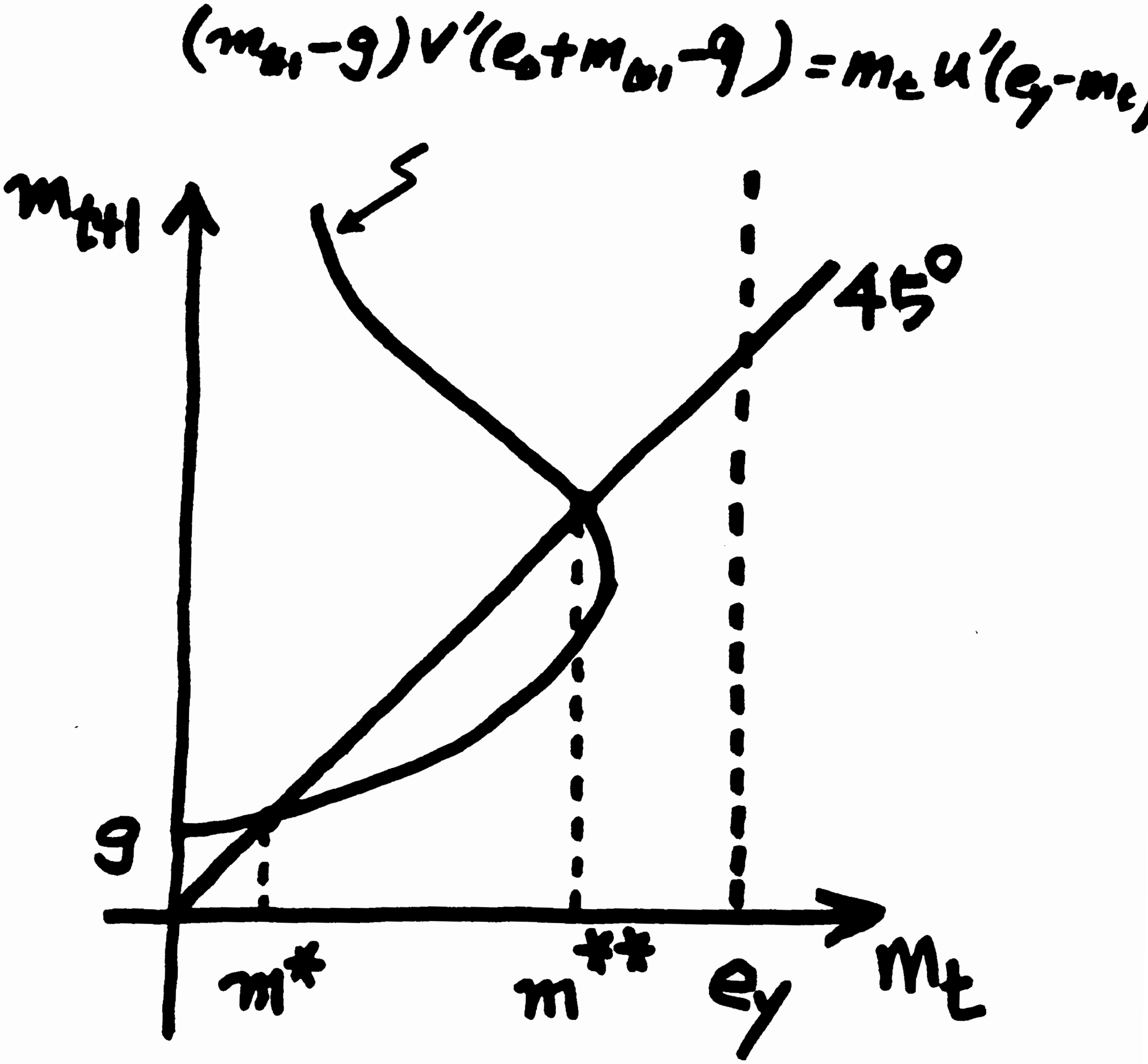

Fig. 1

equilibrium employment and output are equal to m_t, so that there are positive serial correlations in output and employment in sunspot equilibria in the vicinity of m*, the "slippery side of the Laffer curve", while they are negatively serially correlated in sunspot equilibria in the vicinity of m**, if they exist.

<u>Remark 1.</u> In this class of overlapping generations models, the one examined by Azariadis and Guesnerie (1986) in detail, local sunspot equilibria with positive serial correlations cannot exist under the "laissez-faire" ($g = 0$). This is because, $g = 0$ implies $m^* = 0$. Although this nonmonetary steady state is still indeterminate, all converging paths must approach it from above. Feasibility requires $m_t \geq 0$. The existence of stationary sunspot equilibria requires the steady state is in the interior of the feasibility set.[7] Furthermore, once the gross substitutability of consumption when young and old is assumed, the monetary steady state is determinate and no sunspot equilibria can exist.[8]

The next model provides an example in which the local dynamics around the unique monetary steady state can be monotone convergent. It also demonstrates that sunspots can occur in a model of infinitely lived agents.

<u>Model 2.</u> Consider the Brock (1975) model of money-in-the-utility-function. The representative agent maximizes $E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, m_t)$, $(0 < \beta < 1)$, subject to the flow budget constraint, $M_t = P_t(y - c_t) + H_t + M_{t-1}$, with M_{-1} given,

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[7] However, the existence of nonstationary sunspot equilibria does not, as demonstrated in Peck (1988).

[8] This has been proved in more general models; see Kehoe, Levine, Mas-Colell and Woodford (1988).

where β is the discount factor, y is his constant endowment of the perishable consumption good, c_t denotes his consumption, and m_t is real balances demanded, defined by the ratio of M_t, nominal money holdings, and $P_t > 0$, the price level. At the beginning of period t, the agent receives H_t units of paper money from the government through a "helicopter drop", thus considered to be independent of his money holdings. There is no government consumption and the money supply grows at the rate $\mu > \beta$, which implies $H_t = (\mu-1)M_{t-1}$. The markets clear when $M_t = \mu^t M_0$ and $c_t = y$ for all t. Then, one can show that the equilibrium dynamics follows,[9]

$$(8) \qquad E_t[\beta U_c(y,m_{t+1})m_{t+1}] = \mu\{U_c(y,m_t) - U_m(y,m_t)\}m_t \text{ , with } m_t \geq 0 \text{ .}$$

The steady state is given by $[(\mu-\beta)U_c(y,m^*) - \mu U_m(y,m^*)]m^* = 0$. The monetary steady state $m^* \neq 0$ satisfies $(\mu-\beta)U_c(y,m^*) = \mu U_m(y,m^*)$ and is unique if real balances are normal. Standard assumptions of preferences do not place any stronger restriction. For example, let the one period utility function be $U(c,m) = (c^\alpha m^{1-\alpha})^{1-\gamma}/(1-\gamma)$ with $\gamma \neq 1$, $\gamma > 0$ and $0 < \alpha < 1$.[10] Then, real balances are normal and one can show, after some algebra,

$$(9) \qquad \rho = \frac{\beta(\gamma-1)(1-\alpha) - \mu}{\beta(\gamma-1)(1-\alpha) - \beta} \text{ .}$$

Therefore, if $\beta(\gamma-1)(1-\alpha) > \mu$, there are monotone convergent paths to the unique steady state and positively serially correlated stationary sunspot

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[9]Along any convergent path and stationary sunspot equilibrium, the transversality condition for the agent's maximization problem is satisfied automatically.

[10]One can easily show that this functional form satisfies all standard properties of utility functions, as well as the normality and Inada conditions. It has been used in Fischer (1979) and Obstfeld (1985).

equilibria. This requires $(\gamma-1)(1-\alpha) > 1$. In this case, the serial correlation of local sunspot equilibria is decreasing in the growth rate of money supply. As one increases μ from β to infinity, it changes from one to zero [$\beta < \mu < \beta(\gamma-1)(1-\alpha)$] and from zero to minus one [$\beta(\gamma-1)(1-\alpha) < \mu < 2\beta(\gamma-1)(1-\alpha)-\beta$] and finally, local sunspot equilibria disappear [$\mu > 2\beta(\gamma-1)(1-\alpha)-\beta$].[11] It is also worth pointing out that, when $(\gamma-1)(1-\alpha) > 1$, sunspots equilibria are more likely to exist when β is <u>larger</u>. The persistence of endogenous fluctuations cannot be attributed to the short-sightedness of the agent.

In the money-in-the-utility-function model, the nominal interest rate $r_t = U_m(y,m_t)/U_c(y,m_t)$ also fluctuates with real balances. If $U(c,m) = (c^{\alpha}m^{1-\alpha})^{1-\gamma}/(1-\gamma)$, $r_t = (1-\alpha)y/(\alpha m_t)$. Thus, in local two state sunspot equilibria, the one period lagged autocorrelation of the nominal interest rate is also given by (9). This cannot occur in Model 1 or any overlapping generations model of money since the nominal interest rate in these models are always equal to zero.

<u>Remark 2.</u> It should be noted that, since we are concerned with bounded stationary equilibria, our example only requires that the utility function has the form specified above only in an interval containing the steady state. Therefore, many conditions introduced in order to eliminate divergent rational bubbles, such as in Obstfeld and Rogoff (1986), including the boundedness of utility functions, cannot rule out sunspot equilibria shown above.

The reason why the (local) indeterminacy arises despite it is an

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[11]An increase in money supply growth through lump sum transfer eliminates the sunspot equilibria in overlapping generations models, too (See Grandmont [1986]). But, it cannot change the sign of serial correlation.

infinitely lived representative agent model is that the agent's utility depends positively on real balances, which are not a scarce resource. For any given sequence of money supply, one can always improve welfare by assigning prices lower than equilibrium prices. Thus, the market outcome is not optimal, making the standard proof of local determinacy untenable.

The above condition $(\gamma-1)(1-\alpha) > 1$ implies $U_{cm} < 0$, which seems necessary to have sunspot equilibria in a model of money-in-the-utility-function. Obstfeld and Rogoff (1986) and Diba and Grossman (1988) also consider the possibility of sunspot equilibria in the Brock model. They assume the additive separability $U(c,m) = U(c) + V(m)$.[12] This assumption is crucial for their nonexistence results. As observed by Feenstra (1986), we have little to say about the cross derivative of a utility function.[13] Thus, their results should be taken with some caution.

4. Global Analysis

In the previous sections we are only concerned with local dynamics. The global analysis of overlapping generations have been done thoroughly by Azariadis and Guesnerie (1986). In this section, we discuss the existence of global sunspot equilibria in Model 2.

Let $U(c,m) = (c^{\alpha}m^{1-\alpha})^{1-\gamma}/(1-\gamma)$ if $\gamma \neq 1$, $\gamma > 0$ and $U(c,m) = \alpha \log c + (1-\alpha)\log m$ if $\gamma = 1$ for $0 < \alpha < 1$. Then, by defining $\eta = (\gamma-1)(1-\alpha) - 1$, equation (8) can be rewritten as,

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[12]Impose the additive separability in equation (8) above and let $\sigma = 1$, then we obtain equation (5) of Obstfeld and Rogoff (1986) and equation (20) of Diba and Grossman (1988).

[13]In fact, Feenstra's analysis is still limited in that he assumes the homogeneous consumption good. If the resources saved by holding real balances are imperfect substitutes of the "consumption good", then one can impose less restrictions on the cross derivative of the induced utility function.

(10) $$E_t\, A(r_{t+1}) = B(r_t) \;,$$

where $A(r) = r^{\eta}$ and $B(r) = (\mu/\beta)(r)^{\eta}(1-r)$ and $r_t = (1-\alpha)y/(\alpha m_t)$ is the nominal interest rate. Equation (10) has the unique steady state $r^* = 1-(\beta/\mu) > 0$.[14] If $\eta = 0$, (10) simply becomes $r_t = r^*$: the steady state is the only equilibrium. If $\eta \neq 0$, A is invertible and the nonstochastic dynamics can be expressed as,

(11) $$r_{t+1} = C(r_t) = A^{-1}(B(r_t)) \;,$$

where $C(r) = (\mu/\beta)^{1/\eta} r(1-r)^{1/\eta}$. It is straightforward to show,

(12a) If $\eta < 0$, $C(r) \lesseqqgtr r$ as $r \lesseqqgtr r^*$,

(12b) If $\eta > 0$, $r \lesseqqgtr C(r)$ as $r \lesseqqgtr r^*$. $C(0) = C(1) = 0$. C is uni-modal; that is, C is strictly increasing on $[0, \eta/(1+\eta))$ and strictly decreasing on $(\eta/(1+\eta), 1]$. (See Figure 2).

A two state sunspot equilibria of (10) is given by $0 < q_a, q_b < 1$; $r_a \neq r_b$, and,

(13) $$q_a A(r_a) + (1-q_a)A(r_b) = B(r_a) \;,$$

(14) $$(1-q_b)A(r_a) + q_b A(r_b) = B(r_b) \;.$$

Let $r_a < r_b$ without loss of generality. If $\eta < 0$, A is a decreasing function so that $A(r_a) > A(r_b)$. Thus, there exists $0 < q_a < 1$ for which (13) holds, if and only if $A(r_a) > B(r_a)$, or equivalently $r_a < C(r_a)$, which implies $r_a >$

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[14]Although $r_t = 0$ also solves (10), zero is not in the feasibility set.

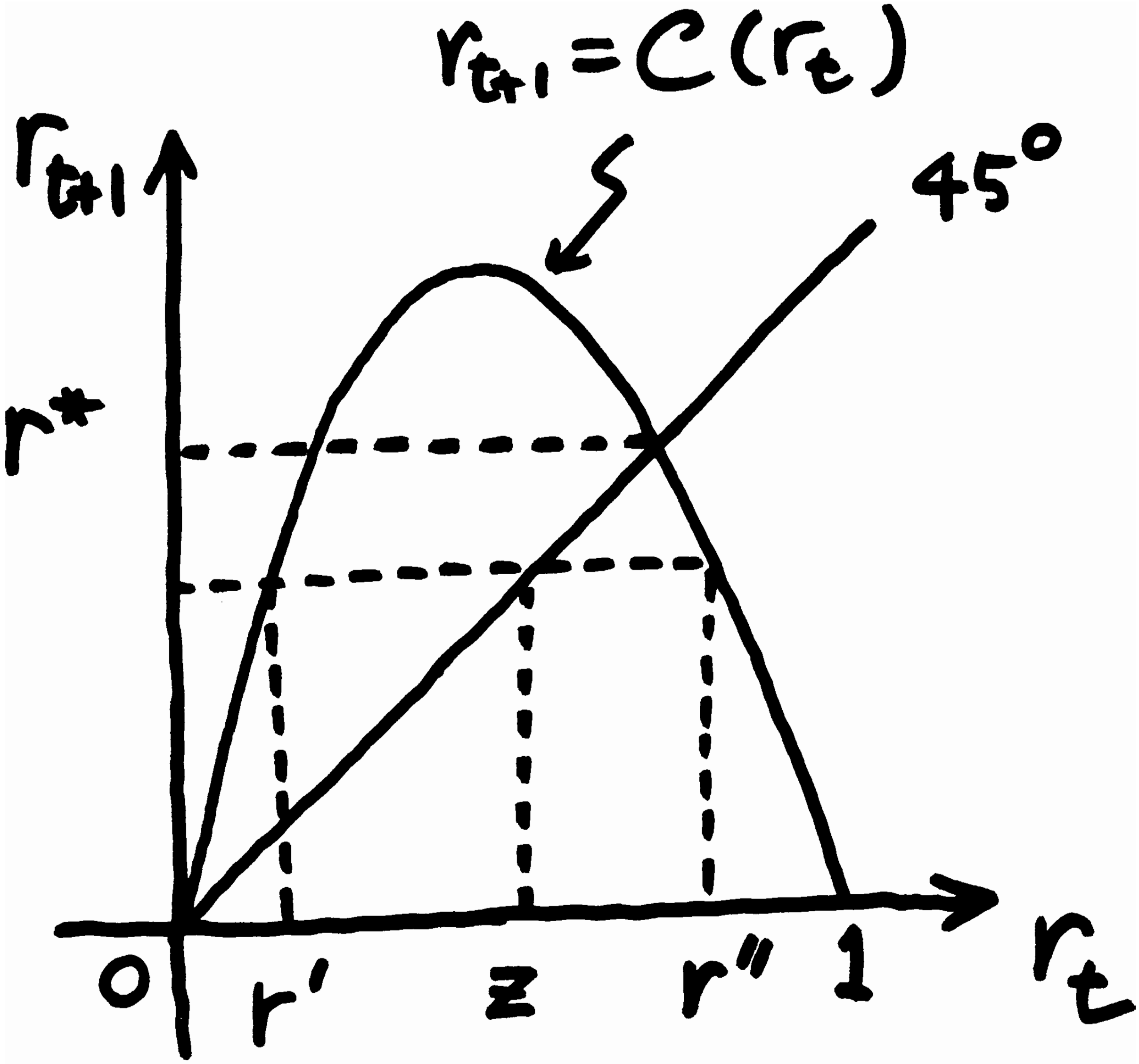

Fig. 2 ($\eta > 0$)

r^* from (12a). Similarly, the existence of $0 < q_b < 1$ that solves (14) requires $r_b < r^*$, thus $r_a > r^* > r_b$, which contradicts $r_a < r_b$. Therefore, there exist no two state sunspot equilibrium if $\eta < 0$.

If $\eta > 0$, A is an increasing function so that $A(r_a) < A(r_b)$. Thus, there exist $0 < q_a$, $q_b < 1$ for which (13) and (14) hold, if and only if $A(r_a) < B(r_a)$, $B(r_b) < A(r_b)$. Recall also that equation (5) is applicable whenever $A(r_a) \neq A(r_b)$. Thus, it is positively serially correlated if and only if $A(r_a) < B(r_a) < B(r_b) < A(r_b)$, and it is negatively serially correlated if and only if $A(r_a) < B(r_b) < B(r_a) < A(r_b)$. These conditions can be further rewritten to,

(15a) $$r_a < C(r_a) < C(r_b) < r_b \quad ,$$

and

(15b) $$r_a < C(r_b) < C(r_a) < r_b \quad .$$

One can always find r_a and r_b that satisfies (15a) and (15b). Figure 2 shows how this can be done. Note that (12b) implies that, for any z such that $0 < z < r^*$, there are two solutions of $z = C(r)$, $r'$and r'', and they satisfy $r' < \eta/(1+\eta) < r''$, and,

$$r' < C(r') = z = C(r'') < r'' \quad .$$

Since C is strictly increasing at $r = r'$, (15a) is satisfied by letting $r_a = r' - \epsilon$ and $r_b = r''$ for a sufficiently small $\epsilon > 0$. Similarly, (15b) is satisfied by letting $r_a = r' + \epsilon$ and $r_b = r''$ for a sufficiently small $\epsilon > 0$. Thus, both positively and negatively serially correlated sunspot equilibria exist whenever $\eta > 0$, or $(\gamma-1)(1-\alpha) > 1$. Note that we did not impose any restriction on $\rho = [\beta(\gamma-1)(1-\alpha)-\mu]/[\beta(\gamma-1)(1-\alpha)-\beta]$. Therefore, the local determinacy of the unique steady state does not rule out the existence of sunspot equilibria in the large. Also, the monotone (oscillatory) convergence to the unique steady state does not rule out negative (positive) serial correlation.

Remark 3. Matsuyama (1988, Proposition 1) shows that in this model there exists no deterministic cycle of period two if $\beta < \mu \leq 2\beta(\gamma-1)(1-\alpha)-\beta$. Nevertheless, sunspot equilibria exist. This suggests that the main result of Azariadis and Guesnerie (1986), that is, sunspots exist if and only if period-two cycles exist, does not carry over outside of the class of models they examined. This is due to the fact that the backward dynamics is globally well defined in their models, while it is the forward dynamics that is globally well defined in our model. In our notations, a function B is strictly monotone in their models, while a function A is strictly monotone in our model.[15] For the existence of perfect foresight cycles and chaos, it does not matter whether the dynamics is backward or forward. However, for the analysis of rational expectations paths, this distinction becomes crucial since uncertainty resolves only when time moves forward.

5. Concluding Remarks

This paper has considered two popular models of money: an overlapping generations economy and a money-in-the-utility-function. However, it is not

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[15]This is not to say that overlapping generations models cannot generate well defined forward dynamics. If each member of the young generation is allowed to borrow from the government in the "classical case" and the government maintains a zero budget deficit, then the forward dynamics of the government's net worth is globally well defined. See Benhabib and Day (1982) and Boldrin and Woodford (1988).

an attempt to make a judgment of which is better than the other. Obviously, each approach has its own pluses and minuses. Although many alternatives have been proposed to overcome their deficiencies, it seems likely that the two approaches will continue to be used for the foreseeable future because of their tractability. Therefore, it seems highly useful to understand the differences between the predictions that the two approaches make. Among our major findings are;

i) In order for an overlapping generations model of money to produce positively serially correlated sunspot equilibria, it needs to have two monetary steady states. Then, they can be found in the vicinity of the hyperinflationary steady state.

ii) A money-in-the-utility-function model can produce sunspot equilibria with either positive or negative serial correlation in real balances and the nominal interest rate even when the monetary steady state is unique, if the separability assumption is dropped. Moreover, a high discounting is not necessary. Sunspots may have effects in an economy with an infinitely lived patient representative agent.

Appendix A: Two State Markov Processes

Let x_t be a Markov process with state space (x_a, x_b) with transition matrix Π given in the text. Then, some algebra yields,

$$\Pi^k = \begin{bmatrix} q^*+(1-q^*)\lambda^k & (1-q^*)(1-\lambda^k) \\ q^*(1-\lambda^k) & (1-q^*)+q^*\lambda^k \end{bmatrix} ,$$

where $q^* = (1-q_b)/(2-q_a-q_b)$ and $\lambda = q_a+q_b-1$. Since $0 < q_a, q_b < 1$ implies $|\lambda| < 1$,

$$\lim_{k\to\infty} \Pi^k = \begin{bmatrix} q^* & 1-q^* \\ q^* & 1-q^* \end{bmatrix} .$$

Therefore, this Markov process is ergodic and $0 < q^* < 1$ is the steady state probability of $x_t = x_a$. Hence, for $k \geq 0$, $E(x_t x_{t+k}) = E(x_t E(x_{t+k}|x_t)) = q^*[\{q^*+(1-q^*)\lambda^k\}x_a^2+(1-q^*)(1-\lambda^k)x_a x_b] + (1-q^*)[q^*(1-\lambda^k)x_a x_b+\{(1-q^*)+q^*\lambda^k\}x_b^2]$, and $E(x_t) = q^*x_a+(1-q^*)x_b$, so that,

$$\begin{aligned} Cov(x_{t+k}, x_t) &= E(x_t x_{t+k}) - E(x_t)E(x_{t+k}) \\ &= q^*\{q^*+(1-q^*)\lambda^k\}x_a^2 + 2q^*(1-q^*)(1-\lambda^k)x_a x_b \\ &\qquad + (1-q^*)\{(1-q^*)+q^*\lambda^k\}x_b^2] - [q^*x_a+(1-q^*)x_b]^2 \\ &= \lambda^k q^*(1-q^*)[x_a-x_b]^2 = \lambda^k Cov(x_t, x_t) = \lambda^k Var(x_t) . \end{aligned}$$

Since $x_a \neq x_b$ implies $Var(x_t) \neq 0$, the k-period lagged autocorrelation of x_t is given by,

$$\begin{aligned} &Cov(x_{t+k}, x_t)/[Var(x_{t+k})\cdot Var(x_t)]^{1/2} \\ &\qquad = Cov(x_{t+k}, x_t)/Var(x_t) = \lambda^k = (q_a+q_b-1)^k . \end{aligned}$$

Appendix B: Finite State Local Sunspot Equilibria

Assume $|\rho| < 1$ and consider a Markov process with state space $X^T = (x_a, x_b, \ldots, x_n)$ with a transition matrix Π that solves (1), where Π is a n by n nonnegative matrix and π_{ij} denotes the probability that the state is x_j in the next period given that the current state is x_i. It satisfies $\Pi J = J$, where J is the column vector of 1's. By linearizing (1) around the steady state, which is set to zero without loss of generality, $\Pi X = \rho X$ and thus $\Pi^k X = \rho^k X$ for all $k \geq 0$.[16] Suppose that Π is irreducible and aperiodic (it suffices to assume $\pi_{ij} > 0$ for all i and j). Then, there exists a unique stable limit distribution $q^* = (q^*_a, q^*_b, \ldots, q^*_n)$, where q^*_i the steady state probability of $x_t = x_i$. Each element of q^* is positive and it satisfies $q^*\Pi = q^*$ and $q^*J = 1$. Then $E(x_t) = q^*X = q^*\Pi X = \rho q^*X = \rho E(x_t)$, or, $E(x_t) = 0$. Hence, by denoting $\Omega = \mathrm{Diag}(q^*)$,

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\begin{aligned}
\mathrm{Cov}(x_{t+k}, x_t) &= E[x_{t+k}x_t] = E[x_t E(x_{t+k}|x_t)] \\
&= X^T \Omega \Pi^k X = \rho^k X^T \Omega X = \rho^k \mathrm{Cov}(x_t, x_t) = \rho^k \mathrm{Var}(x_t) \ .
\end{aligned}
$$

Note that Ω is positive definite so that $\mathrm{Var}(x_t) = X^T \Omega X \neq 0$ whenever $X \neq 0$. Therefore, the k-period lagged autocorrelation of the local sunspot equilibria is equal to ρ^k.

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[16]Note that the existence of sunspots $X \neq 0$ implies that ρ is an eigenvalue of Π. Since the modules of the eigenvalues of the transition matrix are no greater than one [see, for example, Gantmacher (1964, Ch.13, sec.6)], one can conclude that the determinacy of the steady state implies the nonexistence of local sunspots. Laitner (1989) proves this result more generally, using a different technique.

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