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COMPLICATED TOPOLOGICAL STRUCTURE OF THE SET OF EQUILIBRIUM PRICES

by

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Abstract

The set of price levels consistent with perfect foresight equilibrium conditions in Brock's (1974, 1975) "Simple Perfect Foresight Monetary Model" may have a very complicated topological structure. This paper shows, for certain parameter values, that the set of equilibrium prices is uncountable, that it contains no nontrivial interval, or no isolated point, that its Lebesgue measure is zero and that it is a fractal (i.e., it is self-similar under magnification). It also characterizes the dynamic behavior of the price level using symbolic dynamics.

Keywords: Brock model, Chaos, Iterations of Maps, Fractal, Symbolic Dynamics

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1. Introduction

The possibility of multiple equilibria due to self-fulfilling expectations has been recently demonstrated in a variety of dynamic models. See Kehoe and Levine (1985), Woodford (1984) for overlapping generations models, and Gray (1984) and Obstfeld (1984) for Brock's (1974, 1975) monetary model. These studies show that the steady state may fail to have the knife-edge stability and that there exist multiple equilibria converging to the steady state. In other words, there may exist an open interval around the steady state such that any price in this interval is consistent with perfect foresight conditions. Recently, Peck (1987) provides examples of overlapping generations economies, in which the set of equilibrium prices is a union of disjoint intervals.

This paper shows that the set of equilibrium prices in the Brock model may have a complicated topological structure. For certain parameter values, the set of equilibrium prices is uncountable, it contains no nontrivial interval, or no isolated point, its Lebesgue measure is zero and it is a fractal (i.e., it is self-similar under magnification).

The paper also describes the dynamic behavior of the price level jumping around in this set, using symbolic dynamics. The price movement is chaotic in that i) there are periodic equilibria of every integer period, ii) there are uncountably many aperioidic equilibria and iii) there are equilibria which wind densely about the set of equilibrium prices.

There already exist numerous applications of the chaotic dynamics to economics: see, for example, Grandmont (1985) and the special issue (October 1986) of the Journal of Economic Theory. The models discussed in this literature are capable of generating complicated movements of endogenous variables, but the spaces on which these variables can move have simple topological structures (i.e. they are connected.) In a companion paper (Matsuyama [1988]) I discussed the model presented below, but the emphasis was made on the possibility of endogenous price fluctuations and the bifurcation as the parameters, such as the rate of money supply growth, change. (In notations introduced below, Matsuyama (1988) mainly discussed the case of $2\eta < \delta \leq \Delta(\eta)$, while the case of $\delta > \Delta(\eta)$ is the central concern here.)

The rest of paper is in three parts. Section 2 expounds a parameterized version of the Brock model. Section 3 characterizes the topological structure of the set of equilibrium prices. Section 4 describes the dynamic behavior of the price level.

2. The Brock Model

The economy is inhabited by a fixed large number of identical infinitely lived, utility maximizing agents with perfect foresight. Each agent maximizes the present discounted value of his utility stream,

$$W = \sum_{t=0}^{\infty} \beta^t \, U(c_t, m_t^d), \qquad 0 < \beta < 1 ,$$

subject to the flow budget constraint,

$$M_t^d = P_t(y-c_t) + M_{t-1}^d, \qquad \text{with } M_{-1}^d = M_{-1} \text{ given,}$$

where β is the discount factor, y is his constant endowment of the perishable consumption good, c_t denotes his consumption, and m_t^d is real balances demanded, defined by the ratio of M_t^d, nominal money holdings, and $P_t > 0$, the price level. The markets are competitive and each agent considers $\{P_t\}$ to be independent of his own money holdings. The first order condition for the agent's problem, or the arbitrage condition, is given by,

$$U_c(c_t, m_t^d) = U_m(c_t, m_t^d) + \beta U_c(c_{t+1}, m_{t+1}^d) P_t/P_{t+1}.$$

The total supply of the good in the economy is fixed and given by y. There is no government consumption and the money supply M is constant.[1] The markets clear when $M_t^d = M$ and $c_t = y$ for all t. This means that, along an equilibrium path, we have,

$$\text{(1)} \qquad \beta U_c(y, M/P_{t+1})/P_{t+1} = \{U_c(y, M/P_t) - U_m(y, M/P_t)\}/P_t .$$

Brock (1974, 1975) provide the thorough analysis of the case of a separable utility function, $U(c,m) = u(c) + v(m)$. With reasonable assumptions on u and v, the first order difference equation, (1), possesses the unique[2] steady state equilibrium, $P_t = \overline{P}$, where $\overline{P}$ is given by $(1-\beta)u'(y) = v'(M/\overline{P})$, and it can be shown to be unstable. Any sequence satisfying (1), if it starts with $P_0 > \overline{P}$, is explosive (hyperinflation) and, if $P_0 < \overline{P}$, is implosive (hyperdeflation). Brock examined the conditions on u and v under which these divergent paths can be ruled out as an equilibrium.

This paper drops the separability assumption and instead considers the following specification of the utility function,

$$\text{(2)} \qquad U(c,m) = \begin{cases} (c^{\alpha} m^{1-\alpha})^{1-\gamma}/(1-\gamma) , & \text{if } \gamma \neq 1,\ \gamma > 0 , \\ \alpha \log c + (1-\alpha)\log m , & \text{if } \gamma = 1 , \end{cases}$$

for $0 < \alpha < 1$. This functional form satisfies all the standard properties of neoclassical utility functions. Namely, U_c, U_m, $U_{cc}U_{mm} - U_{cm}U_{mc} > 0$; U_{cc}, U_{mm}

[1]Matsuyama (1988) considers a money supply increase.

[2]Strictly speaking, $P_t = \infty$ is another candidate of the steady state equilibrium. This is the situation where paper money has no value. One can show that, for the class of utility functions assumed below, $P_t = \infty$ can be ruled out as an equilibrium.

< 0, as well as the normality conditions and the Inada conditions. It has been frequently used in the literature; see, for example, Fischer (1979) and Obstfeld (1985). By using (2) and normalizing the price level as $p_t = \{(1-\alpha)y/\alpha M\}P_t$, equation (1) can be written as,

$$(3) \qquad (p_{t+1})^{\eta} = (1+\delta)(p_t)^{\eta}(1-p_t) \ ,$$

where

$$\eta \equiv -(1-\alpha)(1-\gamma)-1 = (1-\alpha)\gamma + (\alpha-2) > \alpha-2 \ ,$$

$$\delta \equiv 1/\beta - 1 > 0 \ .$$

There exists the unique steady state equilibrium of (3), $\bar{p} \equiv (1-\beta) = \delta/(1+\delta)$. If $\eta = 0$, (3) simply becomes $p_t = \bar{p}$ for all t. Thus, the steady state is the unique equilibrium path. If $\alpha-2 < \eta < 0$, the qualitative properties of price dynamics (3) is similar to the case of a separable utility function; see Matsuyama (1988, section 2). In what follows, it is assumed that $\eta > 0$. Then, the price dynamics can be rewritten to,

$$(4) \qquad p_{t+1} = F(p_t) \equiv (1+\delta)^{1/\eta} p_t (1-p_t)^{1/\eta},$$

where F is defined on $I \equiv [0,1]$. Note that, if $p_t \geq 1$, (3) cannot hold with any $p_{t+1} > 0$, violating the feasibility condition. (The price level must be positive, or real balances cannot be defined.) Therefore, an initial price level consistent with the perfect foresight condition, p_0, must belong to a set Π defined by,

$$\Pi \equiv \{ p \ ; \ F^t(p) \in U \text{ for all } t \geq 0 \},$$

where $U \equiv (0,1)$ and F^t is given inductively by $F^0(p) = p$, $F^i(p) = F(F^{i-1}(p))$

for i=1,2,3,,,. By construction, $\bar{p} \in F(\Pi) \subseteq \Pi \subseteq U$, where $\subseteq$ denotes nonstrict inclusion ($\subset$ denotes strict inclusion). Thus, the price dynamics are confined to a nonempty set Π. On the other hand, it is easy to check that, if $p_0 \in \Pi$, the transversality condition for the agent's maximization problem is satisfied along the sequence $\{F^t(p_0)\}$. Thus, Π is the set of equilibrium prices at period zero. Note that Π is not closed in general. Its closure Π^* is given by,

$$\Pi^* \equiv \{ p ; F^t(p) \in I \text{ for all } t \geq 0 \}.$$

Again, by construction, $\bar{p} \in F(\Pi^*) \subseteq \Pi^* \subseteq I$. The rest of the paper examines the topological properties of Π (section 3) and the dynamics given by $F: \Pi \to \Pi$ (section 4).

3. Topological Structure of Π

It is straightforward to verify that, with $\eta > 0$, F has the following properties.

(P.1) $F(0) = F(1) = 0$, $F'(0) = (1+\delta)^{1/\eta} > 1$.

(P.2) F has a unique, nondegenerate critical point, $p^* \equiv \eta/(1+\eta)$; $F'(p^*) = 0$, $F''(p^*) < 0$. F is strictly increasing on $[0,p^*)$, strictly decreasing on $(p^*,1]$.

(P.3) F is C^3. If $0 < \eta \leq 1$, its Schwartzian derivative, $SF \equiv F'''/F' - (3/2)[F''/F']^2$, is negative on $I\backslash\{p^*\}$.

(P.4) $F'(\bar{p}) = 1 - \delta/\eta$.

(P.5) Let $\Delta(\eta) \equiv \eta^{-\eta}(1+\eta)^{1+\eta} - 1$. If $\delta < \Delta(\eta)$, $F(p^*) < 1$ and thus $F(U) \subset$

U and $F(I) \subset I$. If $\delta > \Delta(\eta)$, $F(p^*) > 1$, $U \subset F(U)$ and $I \subset F(I)$.[3]

From (P.5), if $\delta < \Delta(\eta)$, $F(U) \subset U$ so that $F^t(U) \subset U$ for all $t \geq 0$. Hence, $\Pi = U$. The set of equilibrium prices is an open interval containing the steady state price.[4] The situation is more complicated if $\delta > \Delta(\eta)$. See Figure. Note that there exists a closed interval of initial prices, I_0, that leave U after one period. There are two open intervals of initial prices, U_0 and U_1, that remain in U after one period. Note that F maps both U_0 and U_1 monotonically onto U. This implies that there are two disjoint closed intervals of prices that leave U after two periods. The remaining four open intervals are mapped monotonically onto U by F^2. Continuing in this manner one can construct Π by successively removing closed intervals from the "middles" of a set of open intervals.

To determine the topological structure of Π when $\delta > \Delta(\eta)$, some definitions need to be introduced.

Definition 1. Let f be a function on I. A nonempty subset $X \subset I$ is called a f-invariant set if $f(X) = X$.

Definition 2. Let $X \subset I$ be a f-invariant set. A map f is expanding on X if there exists a constant $C > 0$ and a constant $K > 1$ such that $|(f^t)'(p)| \geq C \cdot K^t$ for every positive integer t and every point $p \in X$.

Definition 3. Two maps $f: X \to X$, $g: Y \to Y$ are topologically conjugate if there is a homeomorphism $h: X \to Y$ such that $h(f(x)) = g(h(x))$ for every $x \in X$.

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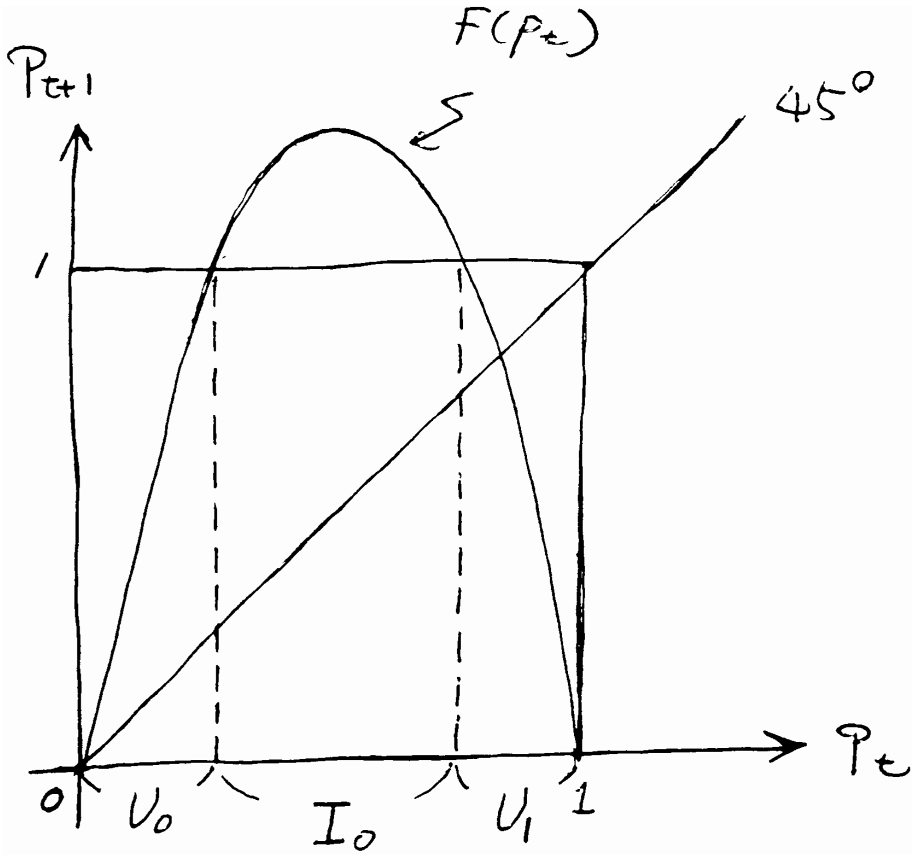
[3] Matsuyama (1988, Lemma) shows that $\Delta(\eta) = \eta^{-\eta}(1+\eta)^{1+\eta} - 1$, defined on $(0,\infty)$, is strictly increasing, $\lim_{\eta\to 0} \Delta(\eta) = 0$ and $\Delta(\eta) > 2\eta$ for all $\eta > 0$. The nongeneric case $\delta = \Delta(\eta)$ will not be discussed below; see Matsuyama (1988, Corollary) for this case.

[4] This is not to say that the price dynamics is simple in this case; see Matsuyama (1988).



Figure

The homeomorphism h is called a <u>topological conjugacy</u>.

If f and g are topologically conjugate via h, then the qualitative properties of f-dynamics can be examined by looking at g-dynamics and vice versa. Moreover, if h is differentiable, f is expanding on X if and only if g is expanding on h(X).

<u>Lemma 1.</u> Assume that $\delta > \Delta(\eta)$. Then,

(1.1) Π^* is F-invariant.

(1.2) If $0 < \eta \leq 1$, F is expanding on Π^*.

(1.3) If $\eta \geq 1$ and $(2+\delta)^{\eta}(1+\delta-\eta) > (1+\delta)^{\eta}[1+\Delta(\eta)]$, F is expanding on Π^*.

<u>Proof.</u> (1.1) Since Π^* is nonempty and $F(\Pi^*) \subseteq \Pi^*$, it suffices to show $\Pi^* \subseteq F(\Pi^*)$. If $\delta > \Delta(\eta)$, $\Pi^* \subset I \subseteq F(I)$, so that, for any $p \in \Pi^*$, there exists a p' $\in I$ with $p = F(p')$. If $p' \notin \Pi^*$, then there exists a integer $t \geq 1$ such that $F^t(p') = F^{t-1}(p) \notin I$, which contradicts with $p \in \Pi^*$. Thus, $p = F(p') \in F(\Pi^*)$.

(1.2) From (P.3), $SF < 0$ on $I\backslash\{p^*\}$. Therefore, one can construct a function f such that i) f maps an interval $I' = [a,b]$, with $a < 0$ and $b \geq F(p^*)$, into itself, ii) $f = F$ on I, iii) f has a unique stable fixed point in $[a,0)$ and every $p \in I'\backslash I$ converges to it, and iv) $Sf < 0$ on $I'\backslash\{p^*\}$. Then f satisfies the assumption of Theorem 2 in Nusse (1987) and Π^* is the set of points that do not converge to the stable fixed point in $[a,0)$. Therefore, the result follows immediately from Lemma 4.14 and Corollary 6.8 in Nusse (1987).

(1.3) Let $G(q) = (1+\delta)q(1-q^{1/\eta})$. Then F and G are topologically conjugate via $H: I \to I$, where $q = H(p) \equiv p^{\eta}$. Thus, it suffices to show G is expanding on $H(\Pi^*) \subset I$. If there exists a $K > 1$ such that $|G'(q)| \geq K$ for all q such that $G(q) \in I$, then the chain rule implies that $|G^{t\prime}(q)| \geq K^t$ for all t and all $q \in H(\Pi^*)$ and we are done. Since G is concave, this condition is equivalent to that if $|G'(q)| = 1$ implies $G(q) > 1$, which is in turn equivalent to $(2+\delta)^{\eta}(1+\delta-\eta) > (1+\delta)^{\eta}[1+\Delta(\eta)]$ when $\eta \geq 1$. q.e.d.

Remark 1. The result of Lemma 1 also holds when Π^* is replaced by Π. In (1.2), negative Schwartzian derivative plays a crucial role. The role of $Sf < 0$ in iterations of maps on an interval is discovered by Singer (1978). He showed, among others, i) if Sf is negative, Sf^t is negative for all t, ii) an increasing map with negative Sf cannot have a stable fixed point between two unstable fixed points. From i) and ii), each stable cycles attracts a critical point of f. When $\delta > \Delta(\eta)$, p^* escapes from I and thus there exists no stable cycle in I, from which the result would follow (cf. Collet and Eckmann [1980, Part II.5] and Misiurewicz [1981]). The condition of (1.3) is not vacuous since it is satisfied whenever δ is sufficiently large ($\delta > 1 + \sqrt{5} = 3.23606...$ when $\eta = 1$; $\delta > 6.20491...$ when $\eta = 2$. It is approximately $\delta > (1.079)\Delta(\eta)$.) It does not require $\eta \leq 1$, and thus provides a sufficient condition when negative Schwartzian derivative does not hold.

The next step is to construct sets homeomorphic to Π and Π^*. First, let Σ_2^* denote the set of infinite sequences of 0's and 1's and Σ_2 denote the set of sequences of (possibly finitely many) 0's and infinitely many 1's. That is,

$$\Sigma_2^* \equiv \{\ a = (a_0 a_1 a_2 \ldots)\ ;\ a_t = 0 \text{ or } 1\ \},$$

$$\Sigma_2 \equiv \{\ a \in \Sigma_2^*\ ;\ \text{for any } T, \text{ there exists } t \geq T \text{ such that } a_t = 1\ \},$$

Clearly, $\Sigma_2 \subset \Sigma_2^*$. Note also that we can regard a point in Σ_2 as the binary expansion of a real number in (0,1]. Endow Σ_2^* (and Σ_2) with the metric, d: $\Sigma_2^* \times \Sigma_2^* \to R_+$, defined by $d(a,a') = \sum_{t=0}^{\infty} |a_t - a'_t| 2^{-t}$. It is easy to show that, with the metric d, Σ_2^* is the closure of Σ_2.

<u>Lemma 2</u>. If either $\delta > \Delta(\eta)$ and $0 < \eta \le 1$, or $(2+\delta)^\eta(1+\delta-\eta) > (1+\delta)^\eta[1+\Delta(\eta)]$ and $\eta \ge 1$, then Π^* and Σ_2^* are homeomorphic and so are Π and Σ_2.

<u>Proof</u>. For every $p \in \Pi^*$, define its <u>address in period t</u> $A_t(p)$ by $A_t(p) = 0$ if $F^t(p)$ belongs to the closure of U_0 and $A_t(p) = 1$ if $F^t(p)$ belongs to the closure of U_1. The <u>itinerary</u> of p, A(p), is the sequence of $\{A_t(p)\}$ of its successive addresses. Then, A is a map from Π^* to Σ_2^*. Devaney (1987, Theorem 7.2) shows that A is a homeomorphism between Π^* and Σ_2^* for the case of $\eta = 1$ and $\delta > 1 + \sqrt{5}$. His proof rests on the fact that F is expanding on Π^*. Therefore, from Lemma 1, one can show that Π^* and Σ_2^* are homeomorphic under the assumption in a manner similar to his. To prove Π and Σ_2 are homeomorphic, note that $\Pi^*\backslash\Pi = \{ p \in I \ ; \text{ for sufficiently large } t, F^t(p) = 0 \}$ as $F(0) = F(1) = 1$. Since $A(0) = (000\ldots)$, $\Pi^*\backslash\Pi$ and $\Sigma_2^*\backslash\Sigma_2$ (the set of sequences with finitely many 1's) are homeomorphic via A. Therefore, Π and Σ_2 are homeomorphic via A. q.e.d.

The next theorem puts forward our main result.

<u>Theorem 1</u>. If either $\delta > \Delta(\eta)$ and $0 < \eta \le 1$, or $(2+\delta)^\eta(1+\delta-\eta) > (1+\delta)^\eta[1+\Delta(\eta)]$ and $\eta \ge 1$, then Π is uncountable, it contains no nontrivial interval, or no isolated points, its Lebesgue measure is zero.

<u>Proof</u>. Suppose that Π contains a nontrivial interval so that there are p^-, p^+ $\in \Pi$ such that $p^- \neq p^+$ and $[p^-, p^+] \in \Pi$. Then, all points in the interval between $F^t(p^-)$ and $F^t(p^+)$ lie either in U_0 or U_1 for all t. In other words, the points in this interval have the same itinerary, which contradicts the fact that A is a one-to-one between Π and Σ_2. Thus Π contains no nontrivial interval. Next, fix any point $a \in \Sigma_2$ and any $\epsilon > 0$. Choose a positive

integer $s > -\log\epsilon/\log 2$ and define $a' \in \Sigma_2$ by $a'_t = a_t$ if $t \neq s$ and $a'_s \neq a_s$, then $d(a,a') = 2^{-s} < \epsilon$, which proves that Σ_2, and therefore Π, contain no isolated point. That Lebesgue measure of Π is zero is shown by constructing an extension of F as done in the proof of (1.2) of Lemma 1 and then by applying Theorem B in Nusse (1987). q.e.d.

Remark 2. When $\eta = 1$, (4) becomes the so-called logistic map: $p_{t+1} = (1+\delta)p_t(1-p_t)$. In this case, the first condition becomes $1+\delta > 4$. Henry (1973) proved that Π^* has Lebesgue measure zero under this condition. With $\eta = 1$, the second condition becomes $1+\delta > 2 + \sqrt{5}$. Devaney (1987) provides the proof that Π^* has the properties given in Theorem 1 under this condition; see also Guckenheimer and Holmes (1986, pp.228-230). Besides, I did not find any studies that explicitly discuss iterations of a map f satisfying $I \subset f(I)$. The reason for this paucity of relevant studies may be that iteration of such a map is not interesting in physical, biological systems, where initial conditions are given by history. For example, if the population dynamics of a certain species of insects were given by such a map, this species would have been extinct with probability one, thus unobserved. In an economic system such as ours, the only constraint on the initial price level is that expectations must be self-fulfilling, thus there is no reason to ignore a map with $I \subset f(I)$.

Remark 3. The set of equilibrium prices in our model is an example of a fractal. A set $X \subset R^n$ is a fractal if its Hausdorff dimension is not an integer. Intuitively, a fractal is a set which is self-similar under magnification. For example, if we magnify subsets of Π, $\Pi \cup U_0$ and $\Pi \cup U_1$, with a "microscope" F, then they look exactly like the original set: $F(\Pi \cup U_0) = F(\Pi \cup U_1) = \Pi$. Likewise, the restriction of Π by a component (i.e., a maximal connected subset) of $F^{-t}(U) = \{ p \in U ; F^{t}(p) \in U \}$ looks exactly like Π under a magnification F^t for all t.

4. Symbolic Dynamics

This section describes the dynamic behavior of the price level given by F: Π → Π under the assumptions of Theorem 1. Again, the topological equivalence between Π and Σ_2 turns out to be very useful.

Let us define a map of Σ_2 onto itself σ, called the shift map, by $\sigma(a_0 a_1 a_2 \ldots) = (a_1 a_2 a_3 \ldots)$. Then, by construction, $\sigma(A(p)) = A(F(p))$. Also it is easy to verify that σ is onto and continuous (cf. Devaney [1987, Proposition 6.5 and Theorem 7.3]). In other words, F and σ are topologically conjugate via the homeomorphism A. Therefore, the quantitative properties of F-dynamics are completely understood by analyzing σ-dynamics. This technique is called <u>symbolic dynamics</u>.

First, note that a point $p \in \Pi$ is periodic with period k if $F^k(p) = p$ but $F^i(p) \neq p$ for $i < k$. It is clear that p is a periodic point of period k under F: Π → Π if and only if A(p) is a periodic point of period k under $\sigma: \Sigma_2 \rightarrow \Sigma_2$. It is easy to see that there are $2^k - 1$ solutions of $\sigma^k(a) = a$ in Σ_2. (That is, there are $2^k - 1$ sequences satisfying $a_t = a_{t+k}$ for all t. Note that it is not 2^k, since (000...) does not belong to Σ_2.) Therefore, the number of initial prices that lead to cycles of period k or a divisor of k is $2^k - 1$. From this, one can easily calculate the number of periodic prices and periodic equilibria. See Table. In particular, if k is a prime number, there are $2^k - 2$ ($2^k - 1$ minus 1, because one of them $\bar{p}$ or $A(\bar{p})$ = (111...) is the steady state) prices lie in a period-k cycle and there are $(2^k-2)/k$ different period-k cycles.

Note that the above argument implies that periodic equilibria are

<u>Table</u>

k (period length)	2^k-1	#(periodic points)	#(cycles)
1	1	1	1
2	3	2	1
3	7	6	2
4	15	12	3
5	31	30	6
6	63	54	9
7	127	126	18
8	255	240	30
9	511	504	56
10	1023	990	99
11	2047	2046	186
12	4095	4020	335
13	8191	8190	630
14	16383	16254	1161
15	32767	32730	2182
16	65535	65280	4080
17	131071	131070	7710
18	262143	261576	14532
19	524287	524286	27594
20	1048575	1047540	52377

countable. It is equally easy to show that eventually periodic equilibria are countable. Since Π is uncountable, there are uncountably many aperiodic equilibria. It is also easy to show that there are equilibrium paths that wind densely about Π. To demonstrate it, it is sufficient to find a point in Σ_2 that comes arbitrarily close to every point in Σ_2 under iteration of σ. Such a point must contain all finite strings of 0's and 1's. It can be constructed by successively listing strings of length 1, then length 2, and so on, as $a^* = (0,1)(00,01,10,11)(000,001,010,\ldots.$ Then, for every $a \in \Sigma_2$ and every positive integer n, there exists a t such that $\sigma^t(a^*)$ and a have the same addresses until period n so that $d(\sigma^t(a^*),a) = \sum_{i=n+1}^{\infty} |\sigma_i^t(a^*)-a_i|2^{-i} \leq \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n}$, which can be made arbitrarily small.

Theorem 2 summarizes the results obtained above.

Theorem 2. Suppose either $\delta > \Delta(\eta)$ and $0 < \eta \leq 1$, or $(2+\delta)^{\eta}(1+\delta-\eta) > (1+\delta)^{\eta}[1+\Delta(\eta)]$ and $\eta \geq 1$. Then,

(2.1) There is a countable set of initial prices that lead to cycles. The number of initial prices lead to cycles of period k or a divisor of k is 2^k-1. In particular, there are cycles of every integer period.

(2.2) There is an uncountable set of initial prices that lead to aperiodic equilibria.

(2.3) There is an initial price that leads to an equilibrium path that winds densely about the set of equilibrium prices.

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