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RELATIONSHIP ADMITTING FAMILIES OF CANDIDATES

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Abstract

A central theme in social choice is to determine when must there be a relationship among a group's sincere election rankings of several different subsets of candidates. This issue is completely resolved here for the positional voting methods. Namely, necessary and sufficient conditions are derived for a family of subsets of candidates to determine when there is a choice of positional voting methods so that there are relationships among the election rankings. The same issue is resolved for a related family of social choice mappings. Then, in part, these necessary and sufficient conditions are used i) to analyze sequential voting procedures, ii) to show how to create new classes of axiomatic representations for social choice mappings that uniquely characterize the Borda Count, and iii) to determine the limits of indeterminacy for positional voting election outcomes.

Theorem 1. For $N \geq$ [illegible], a family F, and a system voting vector w_p, $D(W_p)$ is a subset of [illegible]. Then [illegible], for $w \in D(W_p)$, there is a profile so that for any choice of w_p, the listing of election outcomes over the sets in F is w.

Definition [illegible]

Proposition [illegible] ■

[illegible]

Theorem 3.[illegible] If no subfamily of F satisfies the RIP, then F is not relation admitting. Namely, for any choice of w^n_F, $H(w^n_F) \neq U^n_F$.

Example [illegible]. Assume that $n \geq 3$. Let F be a family of nested sets $\{S_1, S_2, \ldots, S_n\}$ where S_i is a proper subset of S_{i+1} for $i = 1, \ldots, n-1$. The family F is not relation admitting.

[illegible]

[illegible]

Theorem 4. The family $F = \{S_1, \ldots, S_n\}$ is reflection admitting iff [illegible] $(S) \geq \dim(F) = N(N-1)/2$.

In particular, if inequality [illegible] is satisfied, then [illegible] is a proper subset of [illegible]. The number of independent relationships among the N^2 entries of [illegible] (S) [illegible] $\dim(F) = N(N-1)/2$.

[illegible]

Theorem [illegible] ... we given social choice mechanisms, f_j, [illegible] ... family, f_j, for e_j where e_1 and e_2 are disjoint. [illegible] ... the union of e_1 and e_2 is not rotation admitting. Then there exists [illegible] ... such that [illegible]

[illegible]

Theorem [illegible]. Let F = {S_1, ..., S_k} be a family that is not [illegible] relation [illegible] each S_j select [illegible] voting vectors that, along with [illegible] independent. Select [illegible] rankings of the candidates in S_j [illegible] create a profile of voters so that when the results for S_j are tallied with the [illegible] voting vector, the outcome is the kth selected ranking, [illegible]

[illegible]

[illegible]

[illegible] ranking is the [illegible] of outcomes [illegible] is the [illegible] reversed ranking is the [illegible] outcomes [illegible] is the [illegible] outcome [illegible] among these rankings.

[illegible]

[illegible]

Theorem 7. Let F be any family of subsets. For almost all choices of [illegible] is not relation admitting. In particular, if the plurality vote is used to rank each subset of candidates, then there does not exist a relation admitting family of subsets.

[illegible] be a collection of positional voting methods used to tally the [illegible] consists of all subsets of candidates. If G includes the Borda count, then the relation admitting families for G is characterized by Theorem 4. If G does not include the Borda count, then the relation admitting families for G form a proper subset of the relation admitting families for the BC.

[illegible]

Theorem 8. For a given family F, consider the set of all mappings G_i that consists of monotone, neutral social choice functions satisfying IIA that are somewhat [illegible] and consistent over each of the sets in R.

[illegible] The family R is relation admitting for this reason [illegible]

necessary [illegible] satisfies the conditions of Theorem 4. [illegible]

[illegible] where each set S_i [illegible] and the [illegible]

[illegible].

[illegible] The family R [illegible] the relationship among the [illegible] selecting the [illegible] candidates from the [illegible].

[illegible] satisfies the [illegible] there exist [illegible] among the rankings and one selection outcomes that are satisfying [illegible] if $R_a \neq R_b$, then [illegible] is a proper subset [illegible]

[illegible]

[illegible]

Example [illegible]

[illegible]

Corollary [illegible] the [illegible] subsets of [illegible] conditions [illegible] social choice function satisfying [illegible] constant over each of the sets in [illegible] impossible for the four selected [illegible] $S_3(p)$ [illegible] and [illegible] then [illegible] conditions [illegible]

[illegible]

[illegible]

Proofs

[illegible]