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A COMMENT ON THE LOGIC OF "AGREEING TO DISAGREE" TYPE RESULTS[*]

by

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Abstract

The paper explores the logic of the "agreeing to disagree" type results. It identifies two properties of predicates over sets, being preserved under union and under difference, which are the key for the proofs of these results. We present a proposition, in which the property of being preserved by union is used, from which Aumann's, Milgrom and Stokey's and other results in this spirit follow as conclusions. We present a proposition based on both of these properties which implies Samet's generalization of Aumann's result to information structures which are not described by partitions. This result also explains why Milgrom and Stokey's result could not be extended to these information structures. The usefulness of the two generalizations is demonstrated by two additional examples of "agreeing to disagree" type results.

1. Introduction

Aumann [1976] presented a formalization of the notion of common knowledge. He used this formulation to prove that it is impossible "to agree to disagree", i.e. given that two players, 1 and 2, agree on the priors, it is impossible that it is common knowledge among the two players that player 1 assigns to some event the probability α and player 2 assigns to the same event the probability β where $\alpha \neq \beta$. Aumann's paper was the starting point for thought provoking literature.

Two interesting results concern us here:

1. Milgrom and Stokey [1982] proved a result which is often interpreted as referring to the impossibility of speculative trade. Assume that two traders agree on an ex-ante efficient allocation of goods. Then, after the traders get new information, there is no transaction with the property that it is common knowledge that both traders are willing to carry it out.

2. Bacharach (1985) and Samet (1987) explored Aumann's result for more general information structures (see also Shin(1987)). Recall that Aumann assumed that a player gets the information in form of an element in a partition of the state space. In the language of epistomologic logic, an information structure which is a partition means that the operator "to know" has the property that "not to know X" implies the knowledge of not knowing X [i.e. $-K(X) \longrightarrow K(-K(X))$]. Samet showed that Aumann's result holds even when the operator of knowledge satisfies only (a) that the knowledge of X implies that X is true, and (b) that the knowledge of X implies the knowledge of the

knowledge of X

In this note we would like to comment on those developments. We wish to clarify the logic of the "agreeing to disagree" type results, so as to better understand the relations between the works of Aumann, Milgrom and Stokey, Bacharach and Samet.

A predicate on the set of subsets of Ω is a function which assigns to every subset of Ω either the value "True" or the value "False". Consider the following two predicates:

$h_1(X)$ = True iff "the probability of the event Z conditional on X is α"

$h_2(X)$ = Ture iff "given the event X, player i prefers action a to action b".

Both h_1 and h_2 are such that if they are true for two disjoint sets X and Y, then they are true for $X \cup Y$. These facts are crucial for the proofs of Aumann's and Milgrom and Stokey's results. However, the predicates h_1 and h_2 differ in a key issue: if X contains Y and both $h_1(X)$ and $h_1(Y)$ are true, then $h_1(X-Y)$ is true, but if $h_2(X)$ and $h_2(Y)$ are true then $h_2(X-Y)$ is not necessarily true. This distinction between h_1 and h_2 appears to explain why Milgrom and Stokey's result does not hold in a world in which the information structure is not described by a partition, although Aumann's result does hold.

After presenting the basic formal concepts in Section 2, we proceed in Section 3 to present a generalization of Aumann's result which has as a special case Milgrom and Stokey's result. Following Samet we prove an analogous result for a more general information structure (section 4) and then we explain why Milgrom and Stokey's result may not hold for such information structures. The usefulness of the two generalizations is demonstrated by two additional examples of "agreeing to disagree" type results.

We would like to emphasize that the purpose of this note is merely to demonstrate the logical structure of the "agreeing to disagree" type results. As such it is only a comment on the existing literature and it is not meant to have any message about the interpretation of the notion of common knowledge (see Rubinstein(1987)). The most closely related is the work of Geanakoplos. In some sense the points that we make are dual to his results. He was interested in identifying the informational structures for which different "agreeing to disagree" type results hold, while we are interested in identifying the type of theorems which are true only in Aumann's framework and those theorems which hold in Samet's framework as well. The main results of Geanakoplos will appear in a forthcoming paper. Some of the results are mentioned in his survey on Common Knowledge, Geanakoplos' (1988).

2. The Model

Let Ω be a finite state space. Let $i=1,2$ be two players. Player i's **information structure** is a function, P_i which assigns to all $\omega\in\Omega$ a non-empty subset of Ω. The interpretation of the statement $P_i(\omega)\subset S$ is that at ω agent i knows the set S. We shall say that i has a **partitional information structure** if there is a partition of Ω such that for all ω, the set $P_i(\omega)$ is the element in the partition which includes ω. Let $K_i(S)=\{\omega: P_i(\omega)\subset S\}$, that is $K_i(S)$ is the set of states in which i knows S. It is easy to check that an information structure is partitional iff it satisfies the following three conditions (compare with Bacharach (1985) and Samet (1987)):

(K-1) $K_i(S) \subset S$

(K-2) $K_i(K_i(S)) \supset K_i(S)$

(K-3) $-K_i(S) \subset K_i(-K_i(S))$.

A **predicate** on the set of subsets of Ω is a function which assigns to every subset of Ω either the value T for "True" or F for "False". Given predicate f, the interpretation of "$f(P_i(\omega))=T$" is that i knows that f is true at ω. The **meet** of two information structures P_1 and P_2 is the collection of all subsets of Ω such that for each i and for each S in the collection, S is a union of subsets of the type $P_i(\omega)$. It is said that the set S is **common knowledge at ω** if there is an element M in the meet of P_1 and P_2 such that $\omega \in M$ and $M \subset S$.

We shall say that f is **preserved under union** if for all disjoint sets R and S, if both $f(R)=T$ and $f(S)=T$ then $f(R \cup S)=T$. We say that f is **preserved under difference** if for all R and S, $R \supset S$, if both $f(R)=T$ and $f(S)=T$ then $f(R-S)=T$.

Example 1: Let "pr" be a probability measure on Ω and let X be a subset of Ω. Denote by $pr(X|Y)$ the probability of X conditional on Y. Define the predicate $f_{\alpha,X}$ by $f_{\alpha,X}(Y)=T$ if $pr(X|Y)=\alpha$. The predicate $f_{\alpha,X}$ is preserved under both union and difference.

Example 2: let C be a set of consequences. Define a **contingent contract** to be a function from Ω into C. Let A be the set of all contingent contracts. Each of the players have a von-Neumann Morgenstern (VNM) utility function u_i, defined on $C \times \Omega$. Thus, $u_i(c,\omega)$ specifies player i's utility of the consequence c at state ω. Let $a,b \epsilon A$. Define the predicate $f_{a,b,u}$ by $f_{a,b,u}(X)=T$ if the expectation of $u(a,\omega)$ conditional on X is greater then the expectations of $u(b,\omega)$ conditional on X. This predicate is preserved under union: if a is preferred to b given any of the two disjoint sets X and Y, then a is also preferred to b given $X \cup Y$. However, it is not preserved under difference: if a is preferred on b given X and given Y where $X \subset Y$, then it is not necessary that a is preferred on b given $Y-X$.

Example 3: Let ψ be a random variable on Ω and let α be a number. Define the predicate $f_{\alpha,\psi}$ by $f_{\alpha,\psi}(X)=T$ if the expected utility of ψ given X is α. The predicate $f_{\alpha,X}$ is preserved under both union and difference.

Example 4: Let ψ be a random variable on Ω. Define the predicate $f_{\alpha,\psi}$ by $f_{\alpha,\psi}(X)=T$ if the expected utility of ψ given X is strictly above (or alternatively below) α. The predicate $f_{\alpha,X}$ is preserved under union but is not preserved under difference.

3. Partitional Information Structure

The next proposition is a unification of Aumann's and Milgrom and Stokey's results for the partitional information structure. It is related to a result due to Cave (1983):

Proposition 1: Assume that the information structures of the two players are partitional, i.e. they satisfy (K-1,2,3). If f and g are two predicates such that

(1) for no S both $f(S)=T$ and $g(S)=T$

(2) f and g are preserved under union

then there is no ω^* for which the set $\{\omega: f(P_1(\omega))=T \text{ and } g(P_2(\omega))=T\}$ is common knowledge at ω^*.

Proof: By the definition of common knowledge, there is a set Y in the meet of P_1 and P_2 such that $Y\subset\{\omega: f(P_1(\omega))=T\}$ and $Y\subset\{\omega: g(P_2(\omega))=T\}$. The set Y is a union of disjoint sets at which f and g get the value true and since f and g are preserved under union, $f(Y)=g(Y)=T$, a contradiction to (1). ■

The proposition implies immediately Aumann's "agreeing to disagree" result that there is no ω at which it is common knowledge that player 1 believes that the posterior of X given his information is α and player 2 disagrees with him and believes that the posterior of X given what he knows is $\beta\neq\alpha$.

Conclusion 1 (Aumann): There is no ω^*, α and X for which the set $\{\omega: f_{\alpha,X}(P_1(\omega))=T \text{ and } g_{\alpha,X}(P_2(\omega))=T\}$ is common knowledge where $\alpha\neq\beta$.

Proof: Follows from proposition 1 since $f_{\alpha,X}$ and $f_{\beta,X}$ cannot be true at the same set and they are preserved under union. ■

To address Milgrom and Stokey's result define a contingent contract b to be **ex-ante efficient** if there is no contract a satisfying that, for both i, $Ex\ u_i(a(\omega),\omega) > Ex\ u_i(b(\omega),\omega)$. Denote by $Ex\ [u_i(a(\omega),\omega) \mid X]$ the expected utility of u_i of the contract a conditional on the set X.

Conclusion 2 (Milgrom and Stokey): If b is ex-ante efficient, then there is no ω^* at which the set $\{\omega: \text{Ex}\ [u_i(a(\omega),\omega)\ |\ P_i(\omega)] > \text{Ex}\ [u_i(b(\omega))\ |\ P_i(\omega)]\}$ is common knowledge at ω^*.

Proof: Consider f_{a,b,u_1} and f_{a,b,u_2}. These predicates are preserved under union. The ex-ante efficiency of b implies condition (1) of the proposition: if there is a set X such that for both i,
$\text{Ex}\ [u_i(a(\omega),\omega)\ |\ X] > \text{Ex}\ [u_i(b(\omega),\omega))\ |\ X]$, then the contract which is identical with the contract a on X and with b on Ω–X is better for the two players than the contract b. The rest follows from proposition 1. ∎

Remark: Milgrom and Stokey's result was extended in Dow, Madrigal and Werlang (1988), where instead of expected utility the decision makers are assumed to follow Schmeidler's theory of expected utility with non-additive probability measures. Actually, it is clear from Conclusion 2 that Milgrom and Stokey's result is valid for any theory of choice under uncertainty as long as it satisfies the condition that if the contract a is preferred to the contract b given two disjoint sets X and Y, then a is also preferred to b given $X \cup Y$.

The above two observations provide a scheme for producing more "agreeing to disagree" type results. For example:

Conclusion 3: Let ψ be a random variable on Ω and let α and β be two distinct numbers. There is no ω at which it is common knowledge that, conditional on his knowledge, 1 believes that the expectation of ψ is α and, conditional on his knowledge, 2 believes that the expectation is β.

Conclusion 4: Let ψ be a random variable on Ω and α a number. There is no ω at which it is common knowledge that, conditional on his knowledge, 1 believes that the expectation of ψ is strictly above α and, conditional on his knowledge, 2 believes that the expectation is strictly below α.

4. Non-partitional Information Structure

Following Bacharach and Samet we turn now to a discussion of the agreeing to disagree type results for information structures which are not described by a partition but only satisfy (K-1,2). As shown by Samet, (K-1,2) imply the following property of the functions P_i. Let $\mathbf{P}_i=\{S: \exists\ \omega \text{ such that } P_i(\omega)=S\}$. For all R and S in $\mathbf{P}_i$, $R \cap S$ is a union of elements in $\mathbf{P}_i$.

Proposition 2: Suppose that P_1 and P_2 satisfy (K-1,2) and let f and g be two predicates such that

(1) there is no S for which f(S)=g(S)=T

(2) f and g are preserved under both union and difference.

Then, there is no ω^* at which the set $\{\omega: f(P_1(\omega))=T \text{ and } g(P_2(\omega))=T\}$ is common knowledge.

Proof: Let S be an element in the meet of P_1 and P_2 which includes ω^*. We shall argue that both f(S)=g(S)=T and thus get a contradiction. Let $S=S_1 \cup S_2 \cup \ldots \cup S_J$ where S_j is in $\mathbf{P}_1$. For all S_j, $f(S_j)=T$. It will suffice to show that if Q and R are non empty unions of elements in $\mathbf{P}_i$ for which f is true, then $f(Q \cup R)=T$. The proof is by induction on the size of $Q \cup R$. If $Q \cap R=\phi$, then by the inductive hypothesis and since f is preserved under union $f(Q \cup R)=T$. Otherwise, $Q \cap R$ is a union of sets each of which is an intersection of elements in $\mathbf{P}_i$ and hence $Q \cap R$ is a union of elements in $\mathbf{P}_i$. Furthermore $Q \cap R$

is of cardinality strictly smaller then $Q \cup R$. By the inductive hypothesis $f(Q \cap R) = T$. Since f is preserved under difference, $f(Q - Q \cap R) = (R - Q \cap R) = T$ and since f is preserved under union, $f(Q \cup R) = T$. ■

Conclusion 1' (Samet): There is no ω^*, X and $\alpha \neq \beta$ such that the set $\{\omega: f_{\alpha,X}(P_1(\omega)) = T \text{ and } f_{\beta,X}(P_2(\omega)) = T\}$ is common knowledge at w^*.

Notice that the proof of proposition 2 used the finiteness of Ω, while Samet proved the result in the more complicated setting of infinite state space.

Conclusion 3': Let ψ be a random variable on Ω and let α and β be two distinct numbers. There is no ω^* at which the set {ω: 1 believes that the expectation of ψ is α and 2 believes that the expectation of ψ is β} is common knowledge.

Conclusions 2 and 4 are not necessarily true if the information structure is not partitional. Consider the space $\Omega = \{\omega_1, \omega_2, \omega_3\}$ where all states are equally likely. Assume

$P_1(\omega) = \{\omega_1, \omega_2, \omega_3\}$ and

$P_2(\omega_1) = \{\omega_1, \omega_2\}$, $P_2(\omega_2) = \{\omega_2\}$ and $P_2(\omega_3) = \{\omega_2, \omega_3\}$.

Both P_i satisfy K-1 and K-2 but P_2 does not satisfy K-3.

A counter example to conclusion 2′:

Let $B=\{a,b\}$ and let u_1 and u_2 be VNM utilities presented by the following table:

	Player 1		Player 2	
	a	b	a	b
ω_1	3	0	0	3
ω_2	0	5	5	0
ω_3	3	0	0	3

The contingent contract $x(\omega)=b$ is ex ante efficient, but for all ω it is common knowledge that $y(\omega)=a$ is preferred by both players to $x(\omega)$.

A counter example to conclusion 4′:

Let ψ be the random variable $\psi(\omega_2)=1$ and $\psi(\omega_1)=\psi(\omega_3)=0$. Pick $\alpha=0.35$. For all ω it is common knowledge that 1 believes that the expectation of ψ is 1/3 (which is less than 0.35) and that 2 believes that the expectation of ψ is 0.5 or 1 (which are strictly above 0.35).

The observation that conclusion 2 does not hold for the information structure without partitions appears first in Brown and Geanakoplos (1988).

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