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ON THE CONTINUITY OF  
CARTESIAN PRODUCT AND FACTORISATION

by

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Given a topological space  $X$ , we denote the set of non-empty subsets of  $X$  by  $[X]$ , the set of open nonempty subsets of  $X$  by  $O[X]$ , and the set of compact nonempty subsets of  $X$  by  $K[X]$ . Given a family  $\{X_\alpha \mid \alpha \in A\} = \{X_\alpha\}_A$  of topological spaces, we sometimes abbreviate  $\prod_B X_\alpha$  to  $X_B$  ( $B \subset A$ ); we denote

$$\begin{aligned} B[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in [X_\alpha] \text{ for each } \alpha \in A\}, \\ BO[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in O[X_\alpha] \text{ for each } \alpha \in A\}, \\ BK[\{X_\alpha\}_A] &= \{\prod_A P_\alpha \mid P_\alpha \in K[X_\alpha] \text{ for each } \alpha \in A\}. \end{aligned}$$

The maps whose continuity we study are the Cartesian product map

$$\pi: \prod_A [X_\alpha] \rightarrow B[\{X_\alpha\}_A] \subset [X_A]$$

defined by  $\pi(\{P_\alpha\}_A) = \prod_A P_\alpha = P_A$  ( $\{P_\alpha\}_A \in \prod_A [X_\alpha]$ ) and factorisation, i.e.,  $\pi^{-1}$ . (Clearly,  $\pi$  and  $\pi^{-1}$  are bijections.) In doing this, we always equip hyperspaces

(i.e., spaces of subsets) with the finite topology [1].

Given a topological space  $Y$ , by the finite topology on  $[Y]$  is meant the topology generated by taking as a basis for open collections in  $[Y]$  all collections of the form  $\langle U^i \mid i \in M \rangle = \{P \in [Y] \mid P \subset \coprod U^i \text{ and, for each } i \in M, P \cap U^i \neq \emptyset\}$  with  $M$  a finite set and  $U^i \subset Y$  open for each  $i \in M$ . Given any hyperspace  $H[Y] \subset [Y]$ , the finite topology on  $H[Y]$  is then the subspace topology on  $H[Y]$  determined by the finite topology on  $[Y]$ .

Let  $\{X_\alpha\}_A$  be a family of topological spaces.

1. PROPOSITION: Factorisation  $\pi^{-1}: \mathcal{B}[\{X_\alpha\}_A]$

$\rightarrow \prod_A [X_\alpha]$  is continuous (i.e.  $\pi$  is an open map).

Proof: Let  $\omega \subset \prod_A [X_\alpha]$  be a sub-basic open set, i.e., a set of the form  $\omega = \omega_\beta \times \prod_{A \setminus \{\beta\}} [X_\alpha]$  with  $\beta \in A$  and  $\omega_\beta \subset [X_\beta]$  open, and assume furthermore that  $\omega_\beta$  is basic, i.e., of the form  $\omega_\beta = \langle W^i \mid i \in M \rangle$  with  $M$  finite and  $W^i \subset X_\beta$  open for each  $i \in M$ . It suffices to show that  $(\pi^{-1})^{-1}(\omega) = \pi(\omega) \subset \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$  is open. Defining  $V^i = W^i \times \prod_{A \setminus \{\beta\}} X_\alpha \subset X_A$  for each  $i \in M$ , we see that the collection  $V = \langle V^i \mid i \in M \rangle \subset [X_A]$  is open and that  $V \cap \mathcal{B}[\{X_\alpha\}_A] = \pi(\omega)$ .  $\diamond$

2. LEMMA: Let  $P_A \in \mathcal{BK}[\{X_\alpha\}_A]$  and let  $W \subset X_A$  be  
any open set with  $P_A \subset W$ . Then there is an open "tube"  
 $T_A = \prod_A T_\alpha \subset W$  (with  $T_\alpha = X_\alpha$  for all but a finite set  
 $N \subset A$  of indices) such that  $P_A \subset T_A$ .

Proof: For each  $x \in P_A$  find an open tube nbd  $T(x)$  of  $x$   
with  $T(x) \subset W$ , so that, for each  $x \in P_A$ ,  $T_\alpha(x) = X_\alpha$

for all but a finite set  $N(x) \subset A$  of indices.

$\{T(x) \mid x \in P_A\}$ , being an open cover of the compact  $P_A$ ,

admits a finite subcover  $\{T(x_i) \mid i \in M\}$  of  $P_A$ . Define

$N = \bigcup_M N(x_i)$  and  $V = \bigcup_M T_N(x_i)$ . Now  $V \subset X_N$  is open with

$P_N \subset V$  and, since  $P_N$  is compact and  $N$  finite, there is

an open box  $T_N \subset X_N$  with  $P_N \subset T_N \subset V$ . Writing

$T_A = T_N \times X_{A \setminus N}$ ,  $T_A$  is thus an open tube of the desired

sort.  $\diamond$

3. THEOREM: Cartesian product  $\pi$  is continuous on

$\prod_A K[X_\alpha]$  and so this space is homeomorphic to

$\mathcal{BK}[\{X_\alpha\}_A]$ .

Proof:  $\pi$  being a bijection, and noting that

$\pi(\prod_A K[X_\alpha]) = \mathcal{BK}[\{X_\alpha\}_A]$ , the above proposition leaves only the

continuity of  $\pi$  on  $\prod_A K[X_\alpha]$  to show. Take any

$\{P_\alpha\}_A \in \prod_A K[X_\alpha]$  and let  $W = \langle W^i \mid i \in M \rangle \cap \mathcal{BK}[\{X_\alpha\}_A]$  be a

basic open nbd of  $P_A = \pi(\{P_\alpha\}_A)$  with  $M = \{1, \dots, m\}$ .

By the lemma above, there is an open tube  $T_A = T_N \times X_{A \setminus N}$  such that  $P_A \subset T_A \subset W = \bigcup_M W^i$ , where  $N \subset A$  is finite and  $T_N = \prod_N T_\alpha \subset X_N$  is an open box. For each  $i \in M$ , let  $p^i \in W^i \cap P_A$ , and find an open tube nbd  $T_A^i$  of  $p^i$  contained in  $W^i \cap T_A$ . Now defining  $U_\alpha = \langle T_\alpha^1, T_\alpha^2, \dots, T_\alpha^m \rangle \cap K[X_\alpha]$  for each  $\alpha \in N$ , and writing  $U = (\prod_N U_\alpha) \times \prod_{A \setminus N} K[X_\alpha]$ ,  $U \subset \prod_A K[X_\alpha]$ , is an open nbd of  $\{P_\alpha\}_A$  and  $\pi(U) \subset W$ , so we conclude that  $\pi$  is continuous.  $\diamond$

4. PROPOSITION: If  $A$  is a finite set, then Cartesian product  $\pi$  is continuous on  $\prod_A O[X_\alpha]$  and so this space is homeomorphic to  $BO[\{X_\alpha\}_A]$ .

Proof: Imitate the last proof.  $\diamond$

5. APPLICATIONS: Let  $X, Y$  be topological spaces and consider an application of  $X$  to  $Y$ , i.e., a continuous map  $f: X \times Y \rightarrow Y$ . Then the map  $f^*: [X \times Y] \rightarrow [Y]$ , defined by  $f^*(S) = \{f(s) \mid s \in S\}$  ( $S \in [X \times Y]$ ), is continuous (see Theorem 5.10.1, pp. 170, of [1]), so that the restriction of  $f^*$  to  $B[X, Y] \subset [X \times Y]$  is also continuous. Let  $H_X \subset [X]$  and  $H_Y \subset [Y]$ , and define  $F: H_X \times H_Y \rightarrow [Y]$  through  $F(P, Q) = f^*(P \times Q)$  ( $P \in H_X, Q \in H_Y$ ). Now  $F$  is continuous if  $\pi$  is continuous on  $H_X \times H_Y$ , since  $F$  is the composition  $f^* \circ \pi$ ; and  $F$  is actually an application if furthermore  $F(H_X \times H_Y) \subset H_Y$ .

Examples: (1) If  $X$  is a topological semigroup, then so is  $K[X]$ . (2) If  $X$  is a topological semigroup whose multiplication is an open map, then  $O[X]$  is also a topological semigroup. (3) If  $X$  is a topological vector space, then the space of convex compact nonempty subsets of  $X$  forms a topological semivector space (see 2.1 of [2]), and this allows us to embed it in a topological vector space (see Theorem 3.1 of [2]).

#### REFERENCES

- [1] Michael, Ernest, "Topologies on Spaces of Subsets," Transactions of the American Mathematical Society, Vol. 71 (1951), pp. 152-182.
- [2] Prakash, Prem and Murat R. Sertel, "Hyperspaces of Topological Vector Spaces: Their Embedding in Topological Vector Spaces," Preprint Series No. I/74-17, Easter 1974, International Institute of Management, West Berlin, or Discussion Paper No. 83, Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, (1974).