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Given a topological space X, we denote the set of nonempty subsets of X by $[X]$, the set of open nonempty subsets of X by $\mathcal{O}[X]$, and the set of compact nonempty subsets of X by $\mathcal{K}[X]$. Given a family $\{X_\alpha \mid \alpha \in A\} = \{X_\alpha\}_A$ of topological spaces, we sometimes abbreviate $\prod_B X_\alpha$ to X_B $(B \subset A)$; we denote

$$\mathcal{B}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \in [X_\alpha] \text{ for each } \alpha \in A\},$$
$$\mathcal{B}\mathcal{O}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \in \mathcal{O}[X_\alpha] \text{ for each } \alpha \in A\},$$
$$\mathcal{B}\mathcal{K}[\{X_\alpha\}_A] = \{\prod_A P_\alpha \mid P_\alpha \in \mathcal{K}[X_\alpha] \text{ for each } \alpha \in A\}.$$

The maps whose continuity we study are the Cartesian <u>product</u> map

$$\pi: \prod_A [X_\alpha] \to \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$$

defined by $\pi(\{P_\alpha\}_A) = \prod_A P_\alpha = P_A$ $(\{P_\alpha\}_A \in \prod_A [X_\alpha])$ and <u>factorisation</u>, i.e., π^{-1}. (Clearly, π and π^{-1} are bijections.) In doing this, we always equip hyperspaces (i.e., spaces of subsets) with the finite topology [1]. Given a topological space Y, by the finite topology on $[Y]$ is meant the topology generated by taking as a basis for open collections in $[Y]$ all collections of the form $\langle U^i \mid i \in M\rangle = \{P \in [Y] \mid P \subset \bigcup_M U^i$ and, for each $i \in M$, $P \cap U^i \neq \emptyset\}$ with M a finite set and $U^i \subset Y$ open for each $i \in M$. Given any hyperspace $\mathcal{H}[Y] \subset [Y]$, the finite topology on $\mathcal{H}[Y]$ is then the subspace topology on $\mathcal{H}[Y]$ determined by the finite topology on $[Y]$.

Let $\{X_\alpha\}_A$ be a family of topological spaces.

1. PROPOSITION: *Factorisation π^{-1}: $\mathcal{B}[\{X_\alpha\}_A] \to \prod_A [X_\alpha]$ is continuous (i.e. π is an open map).*

Proof: Let $\mathcal{W} \subset \prod_A [X_\alpha]$ be a sub-basic open set, i.e., a set of the form $\mathcal{W} = \mathcal{W}_\beta \times \prod_{A\setminus\{\beta\}} [X_\alpha]$ with $\beta \in A$ and $\mathcal{W}_\beta \subset [X_\beta]$ open, and assume furthermore that $\mathcal{W}_\beta$ is basic, i.e., of the form $\mathcal{W}_\beta = \langle W^i \mid i \in M\rangle$ with M finite and $W^i \subset X_\beta$ open for each $i \in M$. It suffices to show that $(\pi^{-1})^{-1}(\mathcal{W}) = \pi(\mathcal{W}) \subset \mathcal{B}[\{X_\alpha\}_A] \subset [X_A]$ is open. Defining $V^i = W^i \times \prod_{A\setminus\{\beta\}} X_\alpha \subset X_A$ for each $i \in M$, we see that the collection $\mathcal{V} = \langle V^i \mid i \in M\rangle \subset [X_A]$ is open and that $\mathcal{V} \cap \mathcal{B}[\{X_\alpha\}_A] = \pi(\mathcal{W})$. ◊

2. LEMMA: Let $P_A \in \mathcal{B}K[\{X_\alpha\}_A]$ and let $W \subset X_A$ be any open set with $P_A \subset W$. Then there is an open "tube" $T_A = \prod_A T_\alpha \subset W$ (with $T_\alpha = X_\alpha$ for all but a finite set $N \subset A$ of indices) such that $P_A \subset T_A$.

Proof: For each $x \in P_A$ find an open tube nbd $T(x)$ of x with $T(x) \subset W$, so that, for each $x \in P_A$, $T_\alpha(x) = X_\alpha$ for all but a finite set $N(x) \subset A$ of indices.
$\{T(x) \mid x \in P_A\}$, being an open cover of the compact P_A, admits a finite subcover $\{T(x_i) \mid i \in M\}$ of P_A. Define $N = \bigcup_M N(x_i)$ and $V = \bigcup_M T_N(x_i)$. Now $V \subset X_N$ is open with $P_N \subset V$ and, since P_N is compact and N finite, there is an open box $T_N \subset X_N$ with $P_N \subset T_N \subset V$. Writing $T_A = T_N \times X_{A \setminus N}$, T_A is thus an open tube of the desired sort. ◊

3. THEOREM: Cartesian product π is continuous on $\prod_A K[X_\alpha]$ and so this space is homeomorphic to $\mathcal{B}K[\{X_\alpha\}_A]$.

Proof: π being a bijection, and noting that $\pi(\prod_A K[X_\alpha]) = \mathcal{B}K[\{X_\alpha\}_A]$, the above proposition leaves only the continuity of π on $\prod_A K[X_\alpha]$ to show. Take any $\{P_\alpha\}_A \in \prod_A K[X_\alpha]$ and let $\mathcal{W} = \langle W^1 \mid i \in M \rangle \cap \mathcal{B}K[\{X_\alpha\}_A]$ be a basic open nbd of $P_A = \pi(\{P_\alpha\}_A)$ with $M = \{1, \ldots, m\}$.

By the lemma above, there is an open tube $T_A = T_N \times X_{A\setminus N}$ such that $P_A \subset T_A \subset W = \coprod_M W^i$, where $N \subset A$ is finite and $T_N = \prod_N T_\alpha \subset X_N$ is an open box. For each $i \in M$, let $p^i \in W^i \cap P_A$, and find an open tube nbd $T_A{}^i$ of p^i contained in $W^i \cap T_A$. Now defining $\mathcal{U}_\alpha = \langle T_\alpha, T_\alpha^1, \ldots, T_\alpha^m \rangle \cap K[X_\alpha]$ for each $\alpha \in N$, and writing $\mathcal{U} = (\prod_N \mathcal{U}_\alpha) \times \prod_{A\setminus N} K[X_\alpha]$, $\mathcal{U} \subset \prod_A K[X_\alpha]$, is an open nbd of $\{P_\alpha\}_A$ and $\pi(\mathcal{U}) \subset \mathcal{W}$, so we conclude that π is continuous. ◊

4. <u>PROPOSITION</u>: <u>If A is a finite set, then Cartesian product π is continuous on $\prod_A \mathcal{O}[X_\alpha]$ and so this space is homeomorphic to $\mathcal{BO}[\{X_\alpha\}_A]$.</u>

<u>Proof</u>: Imitate the last proof. ◊

5. <u>APPLICATIONS</u>: Let X, Y be topological spaces and consider an <u>application</u> of X to Y, i.e., a continuous map $\mathfrak{f}: X \times Y \to Y$. Then the map $\mathfrak{f}^*: [X \times Y] \to [Y]$, defined by $\mathfrak{f}^*(S) = \{\mathfrak{f}(s) \mid s \in S\}$ $(S \in [X \times Y])$, is continuous (see Theorem 5.10.1, pp. 170, of [1]), so that the restriction of $\mathfrak{f}^*$ to $\mathcal{B}[X, Y] \subset [X \times Y]$ is also continuous. Let $\mathcal{H}_X \subset [X]$ and $\mathcal{H}_Y \subset [Y]$, and define $F: \mathcal{H}_X \times \mathcal{H}_Y \to [Y]$ through $F(P, Q) = \mathfrak{f}^*(P \times Q)$ $(P \in \mathcal{H}_X, Q \in \mathcal{H}_Y)$. Now F is continuous if π is continuous on $\mathcal{H}_X \times \mathcal{H}_Y$, since F is the composition $\mathfrak{f}^* \circ \pi$; and F is actually an application if furthermore $F(\mathcal{H}_X \times \mathcal{H}_Y) \subset \mathcal{H}_Y$.

<u>Examples</u>: (1) If X is a topological semigroup, then so is $K[X]$. (2) If X is a topological semigroup whose multiplication is an open map, then $O[X]$ is also a topological semigroup. (3) If X is a topological vector space, then the space of convex compact nonempty subsets of X forms a topological semivector space (see 2.1 of [2]), and this allows us to embed it in a topological vector space (see Theorem 3.1 of [2]).

REFERENCES

[1] Michael, Ernest, "Topologies on Spaces of Subsets," <u>Transactions of the American Mathematical Society</u>, Vol. 71 (1951), pp. 152-182.

[2] Prakash, Prem and Murat R. Sertel, "Hyperspaces of Topological Vector Spaces: Their Embedding in Topological Vector Spaces," Preprint Series No. I/74-17, Easter 1974, International Institute of Management, West Berlin, or Discussion Paper No. 83 , Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois, (1974).