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A NECESSARY BUT INSUFFICIENT CONDITION FOR THE STOCHASTIC BINARY CHOICE PROBLEM

by

Itzhak Gilboa*

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*Department of Managerial Economics and Decision Sciences, J.L. Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208.

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A NECESSARY BUT INSUFFICIENT CONDITION FOR THE STOCHASTIC BINARY CHOICE PROBLEM

Abstract

The "stochastic binary choice problem" is the following: let there be given n alternatives, to be denoted by $N = \{1,...,n\}$. For each of the n! possible linear orderings $\{>^m\}_{m=1}^{n}$ of the alternatives, define a matrix $Y^{(m)}_{n\times n} (1 \le m \le n!)$ as follows:

$$Y^{(m)}_{ab} = \begin{cases} 1 & a >^m b \\ 0 & \text{otherwise.} \end{cases}$$

Given a real matrix $Q_{n\times n}$, when is Q in the convex hull of $\{Y^{(m)}\}_m$?

In this paper some necessary conditions on Q--the "diagonal inequality"--are formulated and they are proved to generalize the Cohen-Falmagne conditions. A counterexample shows that the diagonal inequality (hence, perforce, the Cohen-Falmagne conditions) is insufficient. The same example is used to show that Fishburn's conditions are also insufficient.

1. Background and Motivation

The problem described above arises in the context of inconsistent decision: suppose we observe an individual choosing between pairs of alternatives under seemingly unchanged circumstances, who fails to stick to a single alternative out of each pair. We may disregard this individual, dubbing him "irrational," but the unfortunate prevalence of the phenomenon calls for a second thought: it may well be the case that the decision maker under discussion is completely rational, but some of the relevant variables which affect his/her decisions are not known to us, and consequently the circumstances which seem to be the same are in fact quite different.

Since we may assume that the probabilities Q_{ab} (of preferring alternative a to b) are observable, the question is: What are the conditions on these probabilities to justify the above explanation for consistency? Or, equivalently, when can say for sure that, no matter what relevant aspects of the decision we have failed to observe, the individual whose behavior is represented by the matrix Q is irrational?

Another interpretation of this problem is the following: let there by given a population, distributed among the n! possible preference orders according to some probability vector $p = (p_1, \ldots, p_{n!})$. The matrix $Q = \Sigma_{m=1}^{n!} p_m Y^{(m)}$ is the pairwise majority vote of this population. The question is, therefore: What are the matrices Q that may be the majority vote of some population?

Another problem, closely related to the one discussed here, is the following: for each $>^m$ define a vector $Z^{(m)} = (Z_{a,A}^{(m)})_{a \in A \subseteq N}$ (for every subset

A of N and every element a of A there is an entry in the vector $Z^{(m)}$), by:

$$Z^{(m)}_{a,A} = \begin{cases} 1 & (\forall\, b \in A,\ b \neq a)(a >^{m} b) \\ 0 & \text{otherwise.} \end{cases}$$

Given a real vector $R = (R_{a,A})_{a \in A \subseteq N}$, when is it a convex combination of $\{Z^{(m)}\}_{m=1}^{n!}$?)

The interpretation of this problem is, of course, very similar, only that we assume that the probabilities $R_{a,A}$ are given for every $A \subseteq N$, while the previous problem assumed these data to be given only for $|A| = 2$.

Necessary conditions on the vector R (to be in the convex hull of $\{Z_{(m)}\}_m$) were formulated by Block and Marschak (1960), and their sufficiency was provided by Falmagne (1978). Block and Marschak have also formulated necessary conditions for the stochastic order problem discussed in this paper, but they have not proved them to be sufficient. McFadden and Richter (1970) provided a counterexample which showed that the sufficiency conjecture was false. (This example was also found independently by Cohen and Falmagne (1978), Dridi (1980), Souza (1983), and Fishburn (1987).) Cohen and Falmagne (1978) and Fishburn (1988) also suggested new sets of necessary conditions, without solving the question of their sufficiency which will be solved in the sequel. Other works on this problem are McLennan (?), Barbera and Pattanaik (1986), and Barbera (1985). A survey which also contains additional references is given by Fishburn and Falmagne (1988).

In the following subsection we cite both Block and Marschak's necessary conditions (called "the triangle inequality") and the proof of their insufficiency. Section 2 will formulate and prove the necessity of stricter conditions, to be named "the diagonal inequality." Section 3 is devoted to the proof of the insufficiency of the diagonal inequality, while Section 4 includes some remarks concerning this paper's results' relations to the literature. More specifically, it proves that the Cohen-Falmagne condition is a special case of the diagonal inequality (hence also insufficient) and that Fishburn's conditions are insufficient (even in conjunction with the diagonal inequality).

1.1 The Triangle Inequality

We begin with some trivial conditions that any matrix $Q \in \text{conv}\{Y^{(m)}\}_m$ must satisfy:

(i) $Q_{ab} \geq 0 \quad \forall\, a,b \in N$

(ii) $Q_{aa} = 0 \quad \forall\, a \in N$

(iii) $Q_{ab} + Q_{ba} = 1 \quad \forall\, a \neq b \in N.$

Next we turn to the triangle inequality. It is easily seen that, since $>^m$ is transitive for all $m \leq n!$, each $Y^{(m)}$ has to satisfy

$$Y^{(m)}_{ab} + Y^{(m)}_{bc} \leq 1 + Y^{(m)}_{ac},\ \forall\, a,b,c \in N.$$

Hence, a convex combination of $\{Y^{(m)}\}_m$ will also satisfy this condition. Using condition (iii) one may conclude that, for all $Q \in \text{conv}\{Y^{(m)}\}_m$,

$$Q_{ab} + Q_{bc} \geq Q_{ac},\ \forall\, a,b,c \in N.$$

This condition is the famous triangle inequality.

It should be noted that a necessary and sufficient condition on Q to belong to $\mathrm{conv}\{Y^{(m)}\}_m$ must be representable in the form of finitely many linear inequalities. Hence it was natural to suspect that the triangle inequality was sufficient. However, the counterexample, which keeps being rediscoverd, is the following: consider the matrix Q shown in Figure 1.

Insert Figure 1 about here

Q satisfies the triangle inequality. Assume that Q is indeed in the convex hull of $\{Y^{(m)}\}_m$. Now, if $p_m > 0$, $Y^{(m)}$ must be zero where Q is zero. The preference orders $\{\succ^m\}_m$, the corresponding matrices of which satisfy this requirement, must satisfy

$$(*) \qquad \begin{array}{lll} 1 \succ^m 4 & 2 \succ^m 4 & 3 \succ^m 5 \\ 1 \succ^m 5 & 2 \succ^m 6 & 3 \succ^m 6. \end{array}$$

Now consider the set of indices A, such that for every $m \in A$, (*) holds and $6 \succ^m 1$. A contains the four indices of the preference relations satisfying

$$2,3 \succ^m 6 \succ^m 1 \succ^m 4,5.$$

Similarly, let B be those indices m, the preference relations of which satisfy (*) and $5 \succ^m 2$. They are the four relations for which

$$1,3 \succ^m 5 \succ^m 2 \succ^m 4,6.$$

And finally denote by C the indices for which both (*) and $4 >^m 3$ hold. These preference relations satisfy

$$1,2 >^m 4 >^m 3 >^m 5,6.$$

It is easily seen that these three quadruples are pairwise disjoint. However, as $Q_{61} = Q_{52} = Q_{43} = 1/2$,

$$\Sigma_{m \in A}\, p_m = \Sigma_{m \in B}\, p_m = \Sigma_{m \in C}\, p_m = 1/2$$

has to hold, which is an obvious contradiction.

2. <u>The Diagonal Inequality</u>

Let there be given two sets of indices, $A, B \subseteq N$ such that $|A| = |B| = k$ $(1 \le k \le n)$. Consider the submatrix of dimension $k \times k$, corresponding to $A \times B \subseteq N \times N$. (See Figure 2).

Insert Figure 2 about here

(A and B are not necessarily disjoint.) Enumerate the elements A and B in an arbitrary way: $A = \{a_i\}_{i=1}^k$, $B = (b_j)_{j=1}^k$ and consider the diagonal $\{(a_i, b_i)\}_{i=1}^k$.

Now choose any matrix $Y^{(m)}$, corresponding to $>^m$, and consider its submatrix defined by A and B. Suppose that for $1 \le i \ne j \le k$ it is true that $Y^{(m)}_{a_i b_i} = Y^{(m)}_{a_j b_j} = 1$.

This implies $a_i >^m b_i$ and $a_j >^m b_j$. Surely, either $a_i >^m b_j$ or $a_j >^m b_i$ (or both) must hold. (Otherwise, $a_j >^m b_j \succsim^m a_i >^m b_i \succsim^m a_j$.)

Hence $Y^{(m)}_{a_i b_i} = 1$ or $Y^{(m)}_{a_j b_j} = 1$ (or both), that is: for every pair of 1's on the diagonal there must be at least one 1 off the diagonal.

As each $Y^{(m)}$ consists solely of zeros and ones, the number of 1's on the diagonal is

$$D = \Sigma_{i=1}^{k} Y^{(m)}_{a_i b_i}$$

and the number of 1's off the diagonal is

$$S = \Sigma_{1 \le i \ne j \le k} Y^{(m)}_{a_i b_j}.$$

Hence, each $Y^{(m)}$ satisfies

$$S \ge \binom{D}{2} = (1/2)D(D - 1).$$

Let us now consider the plane DS, and translate the quadratic inequality into linear inequalities: for every r, $1 \le r \le k - 1$ we draw the string connecting the two adjacent integer points on the parabola:

$$(r, \binom{r}{2}) \text{ and } (r + 1, \binom{r+1}{2}).$$

Insert Figure 3 about here

As for each $Y^{(m)}$ D and S may assume only integer values, S must be above each of these strings. (For the integer points, this condition is equivalent to the quadratic one.)

The equation of the line connecting $(r, \binom{r}{2})$ and $(r + 1, \binom{r+1}{2})$ is $S = r \cdot D - \binom{r+1}{2}$. This proves:

<u>Theorem</u>: A necessary condition for a given matrix Q to belong to the convex hull of $\{Y^{(m)}\}_m$ is:

for every $k \leq n$, every $\{a_i\}_{i=1}^k \subseteq N$, every $\{b_j\}_{j=1}^k \subseteq N$ and every $1 \leq r \leq k - 1$,

$$\Sigma_{1 \leq i \neq j \leq k}\ Q_{a_i b_j} \geq \Sigma_{i=1}^k\ Q_{a_i b_i} - (1/2)r(r + 1).$$

<u>Remark 1</u>: Choosing $k = 2$, $A = \{a,b\}$, $B = \{b,c\}$ and $r = 1$, one gets the following necessary condition:

$$Q_{ac} + Q_{bb} \geq Q_{ab} + Q_{bc} - 1$$

or

$$Q_{ab} + Q_{bc} \leq 1 + Q_{ac}.$$

So the triangle inequality is a special case of the diagonal inequality.

<u>Remark 2</u>: The matrix Q of the famous example quoted above does not satisfy the diagonal inequality:

Let $k = 3$, $A = \{4,5,6\}$, $B = \{3,2,1\}$, for which $S = 0$, $D = 3/2$, and the condition

$$S \geq r \cdot D - (1/2)r(r + 1)$$

does not hold for $r = 1$.

3. The Insufficiency of the Diagonal Inequality

This section is organized as follows: first we define the term "graph decomposition"; then we prove the existence of a graph that is not 2/3-decomposable, and only afterwards we prove the insufficiency of the diagonal inequality using the graph which is not 2/3-decomposable.

3.1 Definition of graph decomposition

First we define the half complete graphs over N: a directed graph G(N,E) is called half-complete (over N) iff for any $a \neq b \in N$ either $(a,b) \in E$ or $(b,a) \in E$ (but not both), and for every $a \in N$, $(a,a) \notin E$.

The set of half-complete graphs will be denoted by $\mathcal{E}$. (This notation as well as the rest of the discussion presupposes a given N. As long as no confusion may result, we will suppress unnecessary subscripts.)

Denote by $\mathcal{E}^T$ the transitive half-complete graphs over N: $(G \in \mathcal{E})$ is transitive iff

$$(a,b),\ (b,c) \in E \Rightarrow (a,c) \in E.)$$

$$\mathcal{E} \supset \mathcal{E}^T = \{G_m^T(N, E_m^T)\}_{m=1}^{n!}$$

(The set of edges of G_m^T will henceforth be denoted by E_m^T.)

It is obvious that there is a one-to-one correspondence between $\{G^T_m\}_m$ and $\{Y^{(m)}\}_m$, since every transitive half-complete graph defines a linear preference relation over N, and vice versa.

We are interested in the distributions over $\mathcal{E}^T$. Let G^T_R be a random variable assuming values in $\mathcal{E}^T$ according to the probability vector $p = (p_1,\ldots,p_{n!})$. For $(a,b) \in N \times N$ define an event $Pref_{ab}$ (a is preferred to b) as follows:

$$Pref_{ab} = \cup_{m\in M_{ab}} (G^T_R = G^T_m)$$

where $M_{ab} = \{1 \le m \le n! \mid (a,b) \in E^T_m\}$. (For $a = b$ $Pref_{ab} = \emptyset$.) By this definition, $\text{Prob } (Pref_{ab}) = \Sigma_{m\in M_{ab}} p_m$.

<u>Definition</u>: A graph $G(N,E) \in \mathcal{E}$ is μ-decomposable for ($\mu \in [0,1]$) iff there exists a probability vector $p = (p_1,\ldots,p_{n!})$ such that for all $(a,b) \in E$,

$$\text{Prob } (Pref_{ab}) \ge \mu.$$

For instance, every $G \in \mathcal{E}$ is 1/2-decomposable, since $p_m = 1/n!$ defines $\text{Prob } (Pref_{ab}) = 1/2$ for all $a \ne b$. We would like to know whether every $G \in \mathcal{E}$ is 2/3-decomposable for all n.

The negative answer is given in the next subsection.

3.2 <u>The existence of a graph which is not 2/3-decomposable</u>

We will need:

<u>Lemma</u>: Let $G \in \mathcal{E}$ be 2/3-decomposable, and suppose that $(a,b),(b,c),(c,a) \in E$. Then, if p is a probability vector of a G-decomposition, and Prob denotes the probability measure defined by p,

(i) $\quad$ $\text{Prob } (\text{Pref}_{ba} \cap \text{Pref}_{cb}) = 0$

(ii) $\quad$ $\text{Prob } (\text{Pref}_{ab}) = 2/3.$

<u>Proof</u>:

(i) $\quad$ If $\text{Prob } (\text{Pref}_{ba} \cap \text{Pref}_{cb}) = \varepsilon > 0$, then

$$\text{Prob } (\text{Pref}_{ba} \cup \text{Pref}_{cb}) = \text{Prob } (\text{Pref}_{ba}) + \text{Prob } (\text{Pref}_{cb}) - \text{Prob } (\text{Pref}_{ba} \cap \text{Pref}_{cb}) \leq 2/3 - \varepsilon,$$

whence $\text{Prob } (\text{Pref}_{ab} \cap \text{Pref}_{bc}) \geq 1/3 + \varepsilon$.

But, since for all $G^T \in \mathcal{E}^T$ in which (a,b), $(b,c) \in E^T$, it is true that $(a,c) \in E^T$,

$$\text{Prob } (\text{Pref}_{ac}) \geq 1/3 + \varepsilon$$

and $\text{Prob } (\text{Pref}_{ca}) < 2/3$, in contradiction to the 2/3-decomposability of G.

(ii) $\quad$ By definition of decomposability,

$$\text{Prob } (\text{Pref}_{ab}) \geq 2/3.$$

If the inequality is strict, $\text{Prob } (\text{Pref}_{ba}) < 1/3$.

This implies

$$\text{Prob } (\text{Pref}_{ba} \cup \text{Pref}_{cb} \cup \text{Pref}_{ac}) \leq \text{Prob } (\text{Pref}_{ba}) + \text{Prob } (\text{Pref}_{cb}) + \text{Prob } (\text{Pref}_{ac}) < 1$$

whence $\text{Prof } (\text{Pref}_{ab} \cap \text{Pref}_{bc} \cap \text{Pref}_{ca}) > 0$

which is impossible since all the graphs in $\mathcal{E}^T$ are transitive. //

The definition of the graph

We need 54 vertices:

- 6 vertices will be called a, b, c, d, e, f.
- 48 vertices will be called (i,j,k) for specific values of i,j,k:
  - i will assume the values {1,2,3}.
  - For each of these i-values, j will assume the values {1,...,8}.
  - The possible values of k depend upon the value of j, according to the following table:

j-value	1	2	3	4	5	6	7	8
k values	1	1,2	1,2	1	1,2,3	1,2,3	1,2,3	1

(Thus, typical vertices are (1,1,1), (3,6,3), etc.) The edges between the vertices are the following:

$$(a,b),\ (b,c),\ (c,a) \in E;\ (d,e),\ (e,f)\ (f,d) \in E$$

$$\{(x,y) \mid x \in \{a,b,c\},\ y \in \{d,e,f\}\} \subseteq E$$

Insert Figure 4 about here

(abc is a circle and so is def, where all the edges between them are directed from abc to def; see also Figure 4.) To define the edges of the vertices (i,j,k) we will need a few abbreviations.

First, we define only those edges touching the vertices $\{(i,j,k)\}_{j,k}$, where the edges touching $\{(2,j,k)\}_{j,k}$ and $\{(3,j,k)\}_{j,k}$ will be defined according to a cyclic symmetry: the edges of $\{(2,j,k)\}_{j,k}$ are like those of $\{(1,j,k)\}_{j,k}$, where d is replaced by e, e by f, and f by d. The edges of $\{(3,j,k)\}_{j,k}$ are again like those of $\{(1,j,k)\}_{j,k}$, where d is replaced by f, e by d, and f by e.

Next, we define some abbreviations:

(1) Five vertices (x_1,x_2,x_3,x_4,x_5) are in an <u>A-structure</u> if (x_1,x_5), (x_5,x_2), (x_3,x_5), $(x_5,x_4) \in E$

Insert Figure 5 about here

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(2) Six vertices $(x_1,x_2,x_3,x_4,x_5,x_6)$ are in an <u>upper B-structure</u> if:

(i) (x_2,x_5), (x_5,x_3), $(x_4,x_5) \in E$

(ii) (x_1,x_2,x_5,x_4,x_6) are in an A-structure.

Insert Figure 6 about here

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(3) Six vertices $(x_1,x_2,x_3,x_4,x_5,x_6)$ are in a <u>lower B-structure</u> if:

(i) (x_5,x_1), (x_2,x_5), $(x_5,x_3) \in E$

(ii) (x_1,x_5,x_3,x_4,x_6) are in an A-structure.

Insert Figure 7 about here

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(4) Eight vertices $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ are in an <u>upper C-structure</u> if:

(i) (x_6,x_1), (x_2,x_6), (x_6,x_3), (x_4,x_6) $\in E$

(ii) $(x_1,x_6,x_4,x_5,x_7,x_8)$ are in an upper B-structure.

Insert Figure 8 about here

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(5) Eight vertices $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ are in a <u>lower C-structure</u> if:

(i) (x_6,x_2), (x_3,x_6), (x_6,x_4), (x_5,x_6) $\in E$

(ii) $(x_1,x_2,x_6,x_5,x_7,x_8)$ are in a lower B-structure.

Insert Figure 9 about here

With these abbreviations we finally specify the direction of edges touching the vertices $\{(1,j,k)\}_{j,k}$:

<u>For j = 1</u> (a,c,d,f,(1,1,1)) are in an A-structure.

<u>For j = 2</u> (a,d,f,c,(1,2,1),(1,2,2)) are in a lower B-structure.

<u>For j = 3</u> (d,a,c,f,(1,3,1),(1,3,2)) are in an upper B-structure.

<u>For j = 4</u> (d,a,f,c,(1,4,1)) are in an A-structure.

<u>For j = 5</u> (a,b,d,c,f,(1,5,1),(1,5,2),(1,5,3)) are in a lower C-structure.

<u>For j = 6</u> (a,d,c,e,f,(1,6,1),(1,6,2),(1,6,3)) are in an upper C-structure.

<u>For $j = 7$</u> $(a,d,e,b,c,(1,7,1),(1,7,2),(1,7,3))$ are in a lower C-structure.

<u>For $j = 8$</u> $(d,b,e,c,(1,8,1))$ are in an A-structure.

Since all the structures defined for $\{(1,j,k)\}_{j,k}$ do not involve edges touching the vertices $\{(i,j,k)\}_{j,k}$ for $i \neq 1$, the symmetric structures defined for $\{(2,j,k)\}_{j,k}$ and $\{(3,j,k)\}_{j,k}$ will have the same property, and hence these definitions do not contradict each other. The rest of the edges in G (that must belong to E for $G \in \mathcal{E}$ to hold) may be directed in an arbitrary way.

<u>The Main Claim</u>: The Graph G defined above is not 2/3-decomposable.

<u>Proof</u>: Suppose G were 2/3 decomposable, and let $p = (p_1,\dots,p_{n!})$ be a decomposition probability vector. By the lemma proved above, $\text{Prob}\ (\text{Pref}_{ac}) = 1/3$, whence there exists at least one index m for which $p_m > 0$ and $(a,c) \in E^T_m$. The lemma also implies, as $(c,a) \in E$ has an opposite direction in G^T_m (i.e., $(a,c) \in E^T_m$ and not $(c,a) \in E^T_m$), that the two other edges in the same circle are directed in G^T_m as in G, that is, $(a,b),(b,c) \in E^T_m$, or $a >^m b >^m c$.

Similarly, the vertices d,e,f may appear in G^T_m in only one of the following three permutations:

$$d >^m e >^m f, \quad e >^m f >^m d, \quad f >^m d >^m e.$$

(The other three permutations are possible only if there are two edges in G, the direction of which is reversed in G_m^T, which is impossible by the lemma.)

Claim A: The permutation of d,e,f in G_m^T cannot be $d >^m e >^m f$.

Proof: For the proof we have to define some new abbreviations. These definitions are dependent upon both the original graph G and the new graph G_m^T discussed above:

(1) Four vertices (x_1,x_2,x_3,x_4) are in position 1 iff:

(i) $(x_2,x_1),(x_4,x_3) \in E$

(ii) $(x_1,x_2),(x_3,x_4) \in E_m^T$

(iii) $(x_2,x_3) \in E_m^T$ or

$(x_4,x_1) \in E_m^T$

Insert Figure 10 about here

(In the figure, the straight line indicates the direction of the edges in G_m^T, where the arcs are original edges of G, which are reversed in G_m^T.)

(2) Four vertices (x_1,x_2,x_3,x_4) are in an upper position 2 iff:

(i) $(x_2,x_1),(x_3,x_2) \in E$

(ii) $(x_1,x_2),(x_2,x_3),(x_3,x_4) \in E_m^T$

Insert Figure 11 about here

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(3) Four vertices (x_1,x_2,x_3,x_4) are in a lower position 2 iff:

(i) $(x_3,x_2),(x_4,x_3) \in E$

(ii) $(x_1,x_2),(x_2,x_3),(x_3,x_4) \in E_m^T$

Insert Figure 12 about here

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(4) Five vertices (x_1,x_2,x_3,x_4,x_5) are in an upper position 3 iff:

(i) $(x_3,x_2) \in E$

(ii) $(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5) \in E_m^T$

Insert Figure 13 about here

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(5) Five vertices (x_1,x_2,x_3,x_4,x_5) are in a lower position 3 iff:

(i) $(x_4,x_3) \in E$

(ii) $(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5) \in E_m^T$

Insert Figure 14 about here

We have to prove a few auxiliary claims:

Claim A.1: If there is a vertex x_5 such that (x_1,x_2,x_3,x_4,x_5) are in an A-structure in G, it is false that (x_1,x_2,x_3,x_4) are in position 1. (Recall that the definitions of the structures refer to a single graph, which is always G in our discussion, whereas the definitions of the positions refer to both G and G_m^T.)

Proof: By the definition of an A-structure,

$$(x_1, x_5), (x_5, x_2) \in E.$$

If (x_1, x_2, x_3, x_4) were in position 1, $(x_2, x_1) \in E$. But $(x_1, x_2) \in E_m^T$ (this edge is reversed in G_m^T), whence $(x_1, x_5), (x_5, x_2) \in E_m^T$. (The other two edges in the circle x_1, x_2, x_5 must have in G_m^T the same direction as in G.) Similarly, $(x_3, x_5), (x_5, x_4) \in E_m^T$.

By the definition of position 1, either $(x_2, x_3) \in E_m^T$ or $(x_4, x_1) \in E_m^T$. In the first case $(x_5, x_2), (x_2, x_3), (x_3, x_5) \in E_m^T$, and in the second $(x_5, x_4), (x_4, x_1), (x_1, x_5) \in E_m^T$. Both possibilities contradict the transitivity of G_m^T, whence (x_1, x_2, x_3, x_4) are not in position 1. //

Claim A.2: If there are vertices, x_5, x_6 for which $(x_1, x_2, x_3, x_4, x_5, x_6)$ are in an upper (lower) B-structure in G, it cannot happen that (x_1, x_2, x_3, x_4) are in an upper (lower) position 2.

Proof: We will prove the claim only for the upper structure and position, since the proof for the other case is symmetric.

By the definition of the B-structure:

$$(x_2, x_5), (x_5, x_3) \in E$$

By that of position 2,

$(x_3, x_2) \in E$; $(x_2, x_3) \in E_m^T$.

This implies $(x_2, x_5), (x_5, x_3) \in E_m^T$. (Since only one of $\{(x_3, x_2), (x_2, x_5), (x_5, x_3)\} \subseteq E$ may be reversed in G_m^T and (x_3, x_2) is indeed reversed.) Consequently, $(x_5, x_4) \in E_m^T$ while $(x_4, x_5) \in E$ and $(x_2, x_1) \in E$, but $(x_1, x_2) \in E_m^T$.

Insert Figure 15 about here

Therefore (x_1, x_2, x_5, x_4) are in position 1. By the definition of the B-structure, $(x_1, x_2, x_5, x_4, x_6)$ are in an A-structure, which contradicts claim A.1. //

Claim A.3: If there are vertices x_6, x_7, x_8 such that $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ are in an upper (lower) C-structure, then $(x_1, x_2, x_3, x_4, x_5)$ cannot be in an upper (lower) position 3.

Proof: Again we give the proof only for the upper structure and upper position since the lower structure and position are dealt with symmetrically. As in the preceding proofs, here we have $(x_2, x_6), (x_6, x_3) \in E_m^T$, whence $(x_1, x_6), (x_6, x_4), (x_4, x_5) \in E_m^T$, while $(x_6, x_1), (x_4, x_6) \in E$. That is, (x_1, x_6, x_4, x_5) are in an upper position 2. But, by the definition of the C-structure, $(x_1, x_6, x_4, x_5, x_7, x_8)$ are in an upper B-structure, in contradiction to Claim A.2. //

Insert Figure 16 about here

We now proceed to prove Claim A, that is, that d $>^m$ e $>^m$ f is impossible. Assume the contrary, i.e.: (a,b),(b,c),(a,c),(d,e),(e,f),(d,f) $\in E_m^T$. The two triangles abc and def, when "spanned" in the linear ordering $>^m$, may be in one of the following eight positions:

(1) <u>One of the triangles is "above" the other one</u>, i.e., (c,d) $\in E_m^T$ or (f,a) $\in E_m^T$. In this case, (a,c,d,f) are in position 1, but, by the definition of G, (a,c,d,f,(1,1,1)) are in an A-structure, a contradiction.

Insert Figure 17 about here

---

(2) <u>The triangle def is "covered" by abc</u>, that is: (a,d),(f,c) $\in E_m^T$. Note that (d,f),(f,c) $\in E_m^T$ while (f,d),(c,f) $\in$ E, whence (a,d,f,c) are in a lower position 2. However, (a,d,f,c,(1,2,1),(1,2,2)) are in a lower B-structure, and hence this possibility has to be excluded.

Insert Figure 18 about here

---

(3) <u>The triangle abc is "covered" be def</u>, that is: (d,a),(c,f) $\in E_m^T$. In this case, (d,a),(a,c) $\in E_m^T$, but (a,d),(c,a) $\in$ E, whence (d,a,c,f) is in an upper position 2. As (d,a,c,f,(1,3,1),(1,3,2))

are in an upper B-structure, abc cannot be "covered" by def.

Insert Figure 19 about here

---

(4) The triangles "intersect" each other, where def is "higher", or: $(d,a),(a,f),(f,c) \in E_m^T$. But $(a,d),(c,f) \in E$, so that (d,a,f,c) are in position 1, a contradiction to the fact that $(d,a,f,c,(1,4,1))$ are in an A-structure.

Insert Figure 20 about here

If none of the situations (1)-(4) occurs, the triangles are bound to "intersect" each other, with abc "higher" than def, that is: $(a,d),(d,c),(c,f) \in E_m^T$. Describing the remaining possibilities, (5)-(8), we will not repeat this fact. We are therefore left with one of:

Insert Figure 21 about here

---

(5) b is "above" d, i.e., $(a,b),(b,d) \in E_m^T$. As $(c,d) \in E$, (a,b,d,c,f) are in a lower position 3. Since $(a,b,d,c,f,(1,5,1),(1,5,2),(1,5,3))$ are in a lower C-structure, this possibility contradicts Claim A.3.

Insert Figure 22 about here

---

(6) <u>b is "below" d, e is "below" c</u>, namely, $(a,d),(d,c),(c,e),(e,f) \in E_m^T$. Here (a,d,c,e,f) are in an upper position 3, while (a,d,c,e,f,(1,6,1),(1,6,2),(1,6,3)) are in an upper C-structure, again a contradiction.

Insert Figure 23 about here

----

(7) <u>b and e are "between" d and c, and e is "above" b</u> or: $(a,d),(d,e),(e,b),(b,c) \in E_m^T$. $(b,e) \in E$, hence (a,d,e,b,c) are in a lower position 3, while (a,d,e,b,c,(1,7,1),(1,7,2),(1,7,3)) are in a lower C-structure, which is impossible.

Insert Figure 24 about here

----

(8) <u>b and e are "between" d and c, and b is "above" e</u>: $(d,b),(b,e),(e,c) \in E_m^T$. Recall that $(b,d),(c,e) \in E$, whence (d,b,e,c) are in position 1. However, since (d,b,e,c,(1,8,1)) are in an A-structure, this possibility must also be excluded.

Insert Figure 25 about here

It is easily seen that possibilities (1)-(8) exhaust all possible inter-relations between the triangles abc and def, and as they were excluded one by one, we have proved Claim A, that is: it is impossible that $d >^m e$

$>^m f$.//

We may now write:

Claim B: It is impossible that $e >^m f >^m d$.

Claim C: It is impossible that $f >^m d >^m e$.

The proofs of these claims are identical to that of Claim A, where the vertices $\{(1,j,k)\}_{j,k}$ are replaced by $\{(2,j,k)\}_{j,k}$ and $\{(3,j,k)\}_{j,k}$, respectively. Since the remaining three permutations of def were proved impossible by the lemma, G is not 2/3-decomposable. //

3.3 Proof of the insufficiency of the diagonal inequality

In view of subsection 3.2, the main point is:

Lemma: Let $G(N,E) \in \mathcal{E}$. Define

$$Q_{ab} = \begin{cases} 2/3 & a \neq b,\ (a,b) \in E \\ 1/3 & a \neq b,\ (a,b) \notin E \\ 0 & a = b \end{cases}$$

Then Q satisfies the diagonal inequality.

Proof: Let there be given two indices sets $A,B \subseteq N$:

$$A = \{a_i\}_{i=1}^k, \quad B = \{b_j\}_{j=1}^k.$$

Denote

$$D = \Sigma_{i=1}^k Q_{a_i b_i}, \quad S = \Sigma_{1 \le i \ne j \le k} Q_{a_i b_j}.$$

We wish to prove that, for all $1 \le r \le k - 1$,

$$S \ge rD - \binom{r+1}{2}.$$

Distinguish between two cases:

<u>Case (a)</u>: <u>$k = 2$</u>, whence $r = 1$. Here $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and

$$S = Q_{a_1 b_2} + Q_{a_2 b_1}, \quad D = Q_{a_1 b_1} + Q_{a_2 b_2}.$$

If $S > 0$, then $S \ge 1/3$. As $Q_{ab} \le 2/3$ for all $a, b \in N$, $D \le 4/3$. This implies

$$S - rD + \binom{r+1}{2} \ge 1/3 - 4/3 + 1 = 0.$$

If, on the other hand, $S = 0$, we necessarily have $Q_{a_1 b_2} = Q_{a_2 b_1} = 0$, whence $a_1 = b_2$, $a_2 = b_1$.

In this case, $D = Q_{a_1 b_1} + Q_{a_2 b_2} = 1$ and again $S - rD + \binom{r+1}{2} = 0 - 1 + 1 = 0$, that is, the diagonal inequality holds.

Case (b): $k > 2$. As $Q_{ab} \leq 2/3$ for all $a, b \in N$, $D \leq 2/3\, k$. To have a lower bound for S, we need:

Observation: Out of the $k(k - 1)$ elements $Q_{a_i b_j}$ $(i \neq j)$, at most k may be zero.

Proof: Assume there are at least $(k + 1)$ different pairs (i, j) for which $a_i = b_j$. Then there must be at least one index j for which there are $i_1 \neq i_2$ such that $a_{i_1} = b_j = a_{i_2}$. This contradicts the assumption that $|A| = k$. //

Since for $a \neq b$, $Q_{ab} \geq 1/3$, we have

$$S \geq k(k - 2) \bullet 1/3$$

whence

$$S - rD + 1/2\, r(r + 1) \geq 1/3\, k(k - 2) - 2/3\, kr + 1/2\, r(r + 1) =$$
$$1/3\, (k - r)(k - r - 2) + 1/6\, r(r - 1).$$

Again we distinguish between two cases:

(b.1) $r \leq k - 2$, which implies $S - rD + 1/2\, r(r + 1) \geq 0$ immediately, and

(b.2) $r = k - 1$ in which case

$$S - rD + 1/2\, r(r + 1) \geq -1/3 + 1/6\, (k - 1)(k - 2).$$

But $k > 2$ implies $(k - 1)(k - 2) \geq 2$, and hence

$$S - rD + 1/2\, r(r + 1) \geq 0.$$

So that Q satisfies the diagonal inequality. //

Our desired conclusion is:

Claim: The diagonal inequality is not sufficient for a matrix Q to belong to $\text{conv}\{Y^{(m)}\}_m$.

Proof: Let G be the graph constructed in subsection 3.2, which is not 2/3-decomposable. Define Q as in the lemma, which also assures that Q satisfies the diagonal inequality. Note that were Q in $\text{conv}\{Y^{(m)}\}_m$, G would have been 2/3-decomposable. //

4. Remarks

In this section we will show that all the conditions mentioned in Fishburn and Falmagne (1988) are insufficient.

4.1 The diagonal inequality generalizes the Cohen-Falmagne conditions

Proof: The Cohen-Falmagne conditions are the following: for every two subsets A,B with $|A| = |B| = m$ and $A \cap B = \emptyset$, and every 1-1 function f mapping A onto B,

$$\Sigma_{i\in A} \Sigma_{k\in B\setminus\{f(i)\}} Q_{ik} + \Sigma_{i\in A} Q_{f(i)i} \le m(m - 1) + 1.$$

Given A, B and f, let us enumerate the elements of A and B such that $f(a_j) = b_j$ for $i \le j \le m$. Then in our notation this condition can be rewritten as

$$\Sigma_{1\le i\ne j\le m} Q_{a_i b_j} + \Sigma_{i=1}^{m} Q_{b_i a_i} \le m(m - 1) + 1$$

or

$$\Sigma_{1\le i\ne j\le m} (1 - Q_{b_j a_i}) + \Sigma_{i=1}^{m} Q_{b_i a_i} \le m(m - 1) + 1$$

which is equivalent to

$$\Sigma_{1\le i\ne j\le m} Q_{b_j a_i} \ge \Sigma_{i=1}^{m} b_i a_i - 1.$$

Note that this is exactly the diagonal inequality for $r = 1$ ($k = m$ and A and B are, unfortunately, in reverse roles). //

4.2 <u>Fishburn's conditions are insufficient</u>

<u>Proof</u>: Fishburn's conditions are of the form:

$$\Sigma_{(i,j)\in C^+} Q_{ij} - \Sigma_{(i,j)\in C^-} Q_{ij} \le 3k - 2$$

where $|C^+| = 2|C^-| = 4k - 2$ and $k \geq 2$. Let us assume that $Q_{ij} \in [1/3, 2/3]$ for all i,j. Then it is easy to see that

$$\sum_{(i,j)\in C^+} Q_{ij} - \sum_{(i,j)\in C^-} Q_{ij} \leq 2/3\ (4k - 2) - 1/3\ (2k - 1) =$$
$$= 2k - 1 \leq 3k - 2.$$

Hence these conditions are always satisfied for Q-matrices that do not contain numbers smaller than 1/3 (equivalently, larger than 2/3). In particular, the matrix Q of Section 3.3 above satisfies these inequalities, although it is not in $\text{conv}\{Y^{(m)}\}_m$. //

(Note that we have in fact shown that the diagonal inequaltiy, the triangle inequality, Cohen-Falmagne conditions, and Fishburn's conditions taken together do not constitute a sufficient condition.)

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Figure 1

$$Q = \begin{vmatrix} 0 & 1/2 & 1/2 & 1 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 0 & 1/2 & 1 & 1 \\ 0 & 0 & 1/2 & 0 & 1/2 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1/2 & 0 \end{vmatrix}$$

Fig. 2

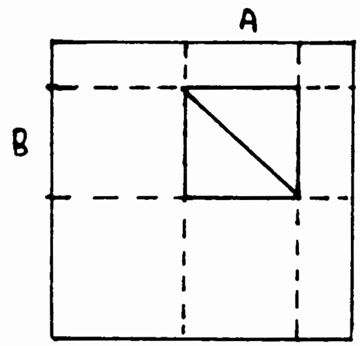


Fig. 3

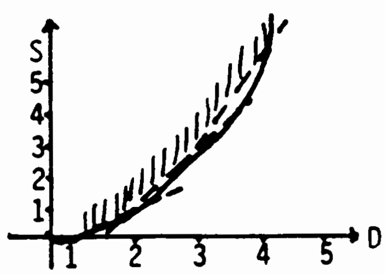


Fig. 4

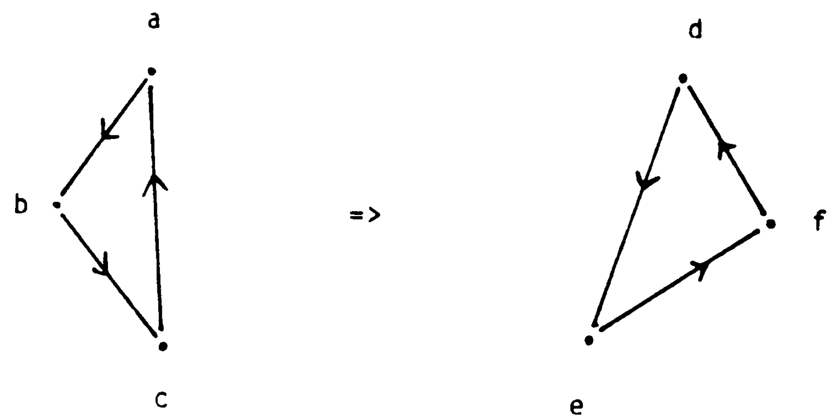
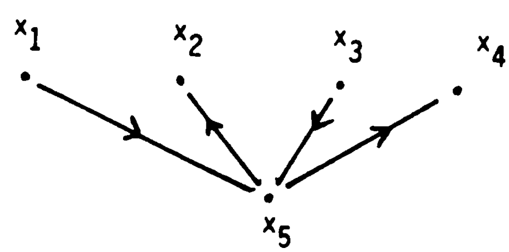


Fig. 5



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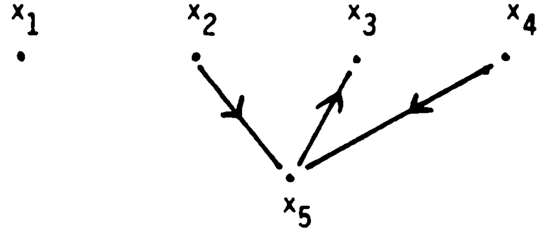

Fig. 7

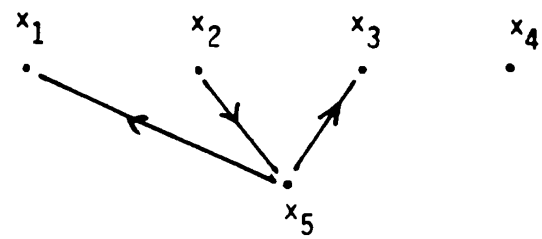

Fig. 8

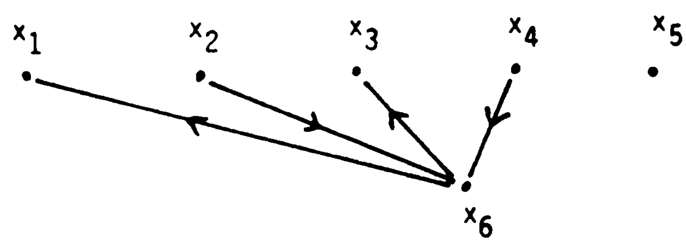

Fig. 9

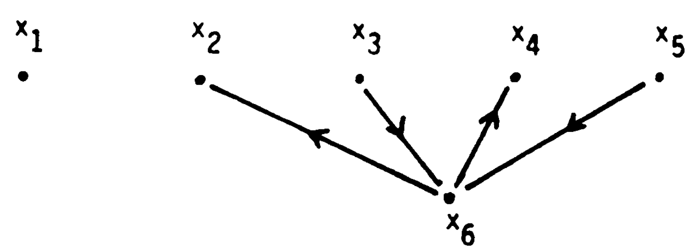

Fig. 10

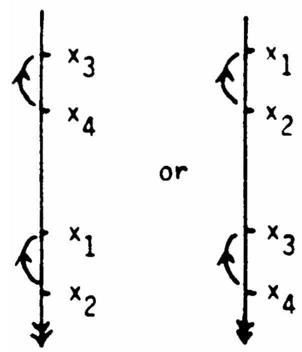

Fig. 11

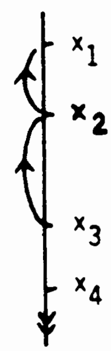

Fig. 12

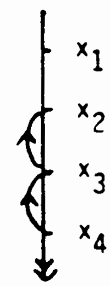

Fig. 13

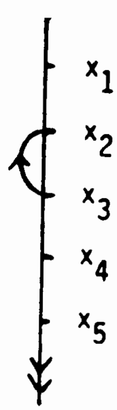

Fig. 14

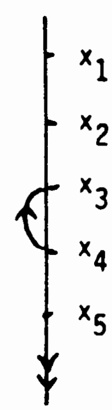

Fig. 15

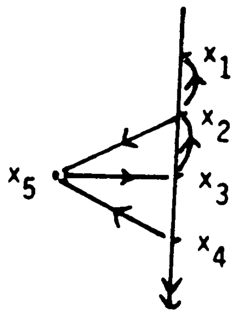

Fig. 16

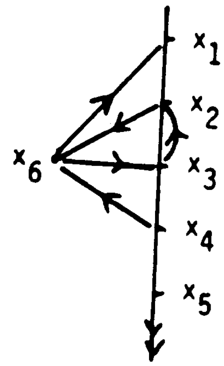

Fig. 17

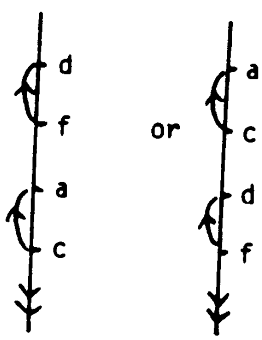

Fig. 18

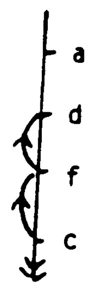

Fig. 19

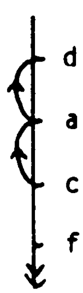

Fig. 20

Fig 21

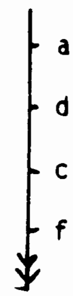

Fig. 22

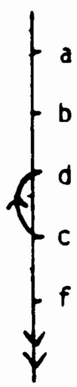

Fig. 23

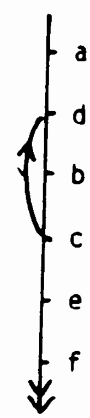

Fig. 24

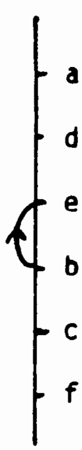

Fig. 25