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ENDOGENOUS PRICE FLUCTUATIONS IN AN OPTIMIZING MODEL OF A MONETARY ECONOMY

By

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Abstract

This paper demonstrates that an optimizing model of a monetary economy can produce perfect foresight equilibria, in which the price level fluctuates forever in a stationary environment. These equilibrium paths are bounded so that neither the transversality condition nor the fractional backing of paper money proposed by Obstfeld and Rogoff (1983) can rule them out.

The chaotic dynamics is also considered. Although an economy with a high rate of money supply growth is more likely to be in the chaotic region (an increase in the growth rate of money supply leads to a period-doubling transition to chaos), the chaos can emerge even with a constant money supply and an arbitrarily small discount rate.

The paper also shows that some fluctuating equilibria give higher welfare than the steady state equilibrium.

Keywords: The Brock model, Endogenous Cycles, Chaos

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1. Introduction

It is almost an axiom among monetary economists that stable money supply ensures stable prices in a stable environment. Recently, Obstfeld and Rogoff (1983, 1986), among others, have subjected this assertion to logical scrutiny. Following Brock (1974, 1975), they studied an optimizing model of a monetary economy, where a demand for money arises through the assumption that an agent's utility depends on his real money balances as well as his consumption level. Their model can produce divergent price paths with a constant money supply. They showed, however, that some weak and economically sensible conditions are sufficient to rule out these divergent paths and to select the steady state as the unique equilibrium path.[1]

Their analysis crucially depends on the assumption that the one period utility function is separable in real money balances and the consumption good. With a nonseparable utility function, this class of models can generate multiple perfect foresight equilibria. For example, it is relatively well known that, when real money balances and the consumption good are Edgeworth substitutes, there may exist multiple convergent path to the steady state; see Gray (1984) and Obstfeld (1984).[2] Although this type of multiplicity has profound implications concerning the assumption of perfect foresight and the validity of comparative dynamics, its empirical significance seems unclear, since the price level eventually stabilizes along any equilibrium path.

What is relatively unknown is that the model can produce multiple bounded equilibrium paths that do not converge to the steady state. This paper considers constant elasticity utility functions and demonstrates that, for certain parameter values, there exist a continuum of equilibrium paths, and that the price level fluctuates forever in almost all of these equilibria. In other words, the model can produce endogenous price fluctuations. These paths are bounded so that neither the transversality condition nor the fractional backing of paper money proposed by Obstfeld and Rogoff (1983) can rule them out.[3] Moreover, some fluctuating equilibria give higher welfare than the steady state equilibrium. It will be also shown that equilibrium price fluctuations can be either periodic or arbitrarily complicated, by utilizing some results from discrete nonlinear dynamics.

It is now widely recognized that the complex dynamic behavior is an inherent feature of many deterministic nonlinear systems. Numerous writers have already found interesting applications in economics. Most studies demonstrated the possibility of complex dynamics in either descriptive aggregate models (Bhaduri and Harris (1987), Day (1982, 1983)) or "short-lived" overlapping generations models (Grandmont (1985)). Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) found chaotic behavior in an infinite horizon optimizing model of growth. But their examples require a heavy discounting, and therefore, hard to interpret as a demonstration of high frequency fluctuations observed in the real world.[4]

This paper provides an example of how even a standard infinite horizon optimizing model of a monetary economy with a common specification of preferences and an arbitrarily small discounting can generate the complex behavior of the price level.[5] This exercise is partly motivated by the fact that the existing literature on endogenous fluctuations, such as Azariadis (1981), Azariadis and Guesnerie (1986) and Grandmont (1985), fails to generate the interest among macroeconomists that I believe it deserves. The reason may be that it considers a class of models that do not belong to the tool box of most macroeconomists. This consideration alone seems to provide a sufficient justification for the present analysis.

The rest of the paper is in five parts. Section 2 first expounds the Brock (1974, 1975) model and derives a two-parameter nonlinear first order difference equation of the price level. The parameters depend on the constant rate of money supply growth, the discount factor and the intertemporal elasticity of substitution. This section then considers the case of a large substitution, which yields global divergence. Section 3 demonstrates that a sufficiently small substitution implies global convergence, while there exists a medium range of elasticity of substitution that implies cycles and other nonconvergent equilibria. Section 4 considers the chaotic dynamics. It shows that an economy with a high rate of money supply growth is more likely to be in the chaotic region (an increase in the growth rate of money supply leads to a period-doubling transition to chaos) and that the strongly chaotic dynamics can emerge even with a constant money supply and an arbitrarily small discount rate. Section 5 provides an intuitive explanation of the conditions necessary for convergence, cycles and chaos. Section 6 presents some discussions. The main result is also summarized in Figure 4.

2. The Brock Model: Global Divergence

The economy is inhabited by a fixed large number of identical infinitely lived, utility maximizing agents with perfect foresight. Each agent maximizes the present discounted value of his utility stream,

$$W = \Sigma_{t=0}^{\infty} \beta^{t} U(c_{t}, m_{t}^{d}), \qquad 0 < \beta < 1 ,$$

subject to the flow budget constraint,

$$M_{t}^{d} = P_{t}(y-c_{t}) + H_{t} + M_{t-1}^{d}, \qquad \text{with } M_{-1}^{d} = M_{-1} \text{ given},$$

where β is the discount factor, y is his constant endowment of the perishable consumption good, c_t denotes his consumption, and m_t^d is real balances demanded, defined by the ratio of M_t^d, nominal money holdings, and P_t, the price level. At the beginning of period t, each agent receives H_t units of paper money from the government through a "helicopter drop", thus considered to be independent of his own money holdings. Consumption loans are not considered explicitly; The assumption that agents are identical means that in equilibrium the quantities of loans traded are zero, thus leaving equilibria intact. Of course, the model allows one to compute the real interest rate, as will be done in section 5, where an intuitive explanation is offered of the results obtained below.

The first order condition for the agent's problem, or the arbitrage condition, is given by,

$$U_c(c_t,m_t^d) = U_m(c_t,m_t^d) + \beta U_c(c_{t+1},m_{t+1}^d)P_t/P_{t+1}.$$

The total supply of the good in the economy is fixed and given by y. There is no government consumption and the money supply grows at a constant rate: that is, $M_{-1} + H_0 +\ldots+ H_t = \mu^t M_0 = \mu^t(H_0+M_{-1})$, where $\mu > \beta$.[6] The markets clear when $M_t^d = \mu^t M_0$ and $c_t = y$ for all t. This means that, along an equilibrium path, we have,

$$\beta U_c(y,m_{t+1})m_{t+1} = \mu m_t\{U_c(y,m_t) - U_m(y,m_t)\} \text{ , with } m_t = \mu^t M_0/P_t \text{ .} \tag{1}$$

Brock (1974, 1975) and Obstfeld and Rogoff (1983, 1986) restricted their analysis to the case of a separable utility function, $U(c,m) = u(c) + v(m)$. With reasonable assumptions on u and v, the first order difference equation, (1), possesses the unique steady state equilibrium, $m_t = \bar{m}$ or $P_t = \mu^t(M_0/\bar{m})$, where $\bar{m}$ is given by $(\mu-\beta)u'(y) = \mu v'(\bar{m})$, and it can be shown to be unstable.[7]

Any price sequence satisfying (1), if it starts with $P_0 > M_0/\bar{m}$, is explosive (hyperinflation) and, if it starts with $P_0 < M_0/\bar{m}$, is implosive (hyperdeflation). Brock and Obstfeld and Rogoff examined the conditions under which these divergent paths can be ruled out.

This paper drops the separability assumption and instead considers the following specification of the utility function,

$$(2) \qquad U(c,m) = \begin{cases} (c^{\alpha}m^{1-\alpha})^{1-\gamma}/(1-\gamma) , & \text{if } \gamma \neq 1,\ \gamma > 0 , \\ \alpha \log c + (1-\alpha)\log m , & \text{if } \gamma = 1 , \end{cases}$$

for $0 < \alpha < 1$. One interpretation would be that $c^{\alpha}m^{1-\alpha}$ is the one-period utility index and γ is the reciprocal of the intertemporal elasticity of substitution. This functional form satisfies all the standard properties of neoclassical utility functions. Namely, U_c, U_m, $U_{cc}U_{mm} - U_{cm}U_{mc} > 0$; U_{cc}, $U_{mm} < 0$, as well as the normality conditions and the Inada conditions. It has been frequently used in the literature; see, for example, Fischer (1979) and Obstfeld (1985). From (2), equation (1) can be written as, after some algebra,

$$(3) \qquad (m_{t+1})^{-\sigma} = (1+\delta)(m_t)^{-\sigma}[1-\{(1-\alpha)/\alpha\}(y/m_t)] ,$$

where

$$\sigma \equiv -(1-\alpha)(1-\gamma)-1 = (1-\alpha)\gamma + (\alpha-2) > \alpha-2 ,$$

$$\delta \equiv \mu/\beta - 1 > 0 .$$

(The economic interpretation of the two parameters, σ and δ, will be given in section 5.)

There exists the unique steady state equilibrium of (3), given by $\bar{m}$ =

$[(1-\alpha)(1+\delta)y]/(\alpha\delta)$ or $P_t = \mu^t(M_0/\bar{m})$. Let us normalize the price level by defining $p_t = \{(1-\alpha)y/\alpha\}(P_t/\mu^t M_0) = \{\delta/(1+\delta)\}(\bar{m}/m_t) = \{(1-\alpha)/\alpha\}(y/m_t)$. Then, (3) becomes,

$$(4) \qquad (p_{t+1})^\sigma = (1+\delta)(p_t)^\sigma(1-p_t) \ ,$$

with $\bar{p} \equiv (\mu-\beta)/\mu = \delta/(1+\delta)$ being the unique positive steady state. Two points should be made here. First, if $p_t \geq 1$, (4) cannot hold with any $p_{t+1} > 0$, violating the feasibility condition. Second, if $\sigma = 0$, (4) becomes $p_t = \bar{p}$ for all t. Thus, the steady state is the unique equilibrium path. In what follows, only the case of $\sigma \neq 0$ is considered. Then, from (4), any equilibrium path needs to satisfy,

$$(5) \qquad p_{t+1} = F(p_t) \equiv (1+\delta)^{1/\sigma} p_t(1-p_t)^{1/\sigma} \text{ , with } 0 < p_t < 1 \text{ for all t.}$$

Note that, although F has the two fixed points, 0 and $\bar{p}$, only the latter is economically meaningful.

Figure 1A depicts the case of a large elasticity of substitution (α-2 < σ < 0). It can be easily verified that F is increasing and convex in [0,1) with $F(0)=0$, $F \to \infty$ as $p \uparrow 1$, $0 < F'(0) = (1+\delta)^{1/\sigma} < 1$. The steady state is unstable and any sequence starting with $p_0 \neq \bar{p} = (\mu-\beta)/\mu = \delta/(1+\delta)$ will diverge. Whether these divergent sequences can be ruled out may be examined in a manner similar to Brock (1974, 1975) and Obstfeld and Rogoff (1983, 1986). Note that $\gamma = 1$, or $\sigma = -1$, means the separability so that it is a special case of what they examined. The possibility of hyperinflation and hyperdeflation is not the main concern of this paper, however. The next section turns to the case of $\sigma > 0$.

3. Global Convergence and Bounded Fluctuations

With $\sigma > 0$, F has the following properties.

(P.1) $F(0) = F(1) = 0$.

(P.2) F has a single peak at $p^* \equiv \sigma/(1+\sigma)$; F is strictly increasing on $[0,p^*)$, strictly decreasing on $(p^*,1]$.

(P.3) $F'(0) = (1+\delta)^{1/\sigma} > 1$.

(P.4) $F'(\bar{p}) = 1 - \delta/\sigma$.

(P.5) $F(p^*) < 1$ if $\delta < \Delta(\sigma) \equiv \sigma^{-\sigma}(1+\sigma)^{1+\sigma} - 1$; $F(p^*) = 1$ if $\delta = \Delta(\sigma)$; $F(p^*) > 1$ if $\delta > \Delta(\sigma)$. Thus, F maps [0,1] into itself if $\delta \le \Delta(\sigma)$. F maps (0,1) into itself if $\delta < \Delta(\sigma)$.

It is straightforward to verify (P.1)-(P.5). A map F: $[0,1] \to [0,1]$ satisfying (P.1) and (P.2) is called unimodal: see Devaney [1987,Definition 18.1]. Thus, if $\delta \le \Delta(\sigma)$, F is unimodal. Some properties of $\Delta(\sigma)$ are summarized below.

Lemma $\Delta(\sigma) \equiv \sigma^{-\sigma}(1+\sigma)^{1+\sigma} - 1$, defined on $(0,\infty)$, is strictly increasing, $\lim_{\sigma\to 0} \Delta(\sigma) = 0$ and $\Delta(\sigma) > 2\sigma$ for all $\sigma > 0$.

Proof See Appendix.

Let us now define F^t iteratively by $F^1(p) = F(p)$, $F^{i+1}(p) = F(F^i(p))$ for i=1,2,3,,,. From (P.5), if $\delta < \Delta(\sigma)$, F maps (0,1) into itself and therefore, $\{F^t(p_0)\}$ defines a sequence on (0,1) for any $p_0 \in (0,1)$. If $\delta \ge \Delta(\sigma)$, $\{F^t(p_0)\}$ defines a sequence on (0,1) only for some $p_0 \in (0,1)$. The nature of the dynamics crucially depends on the local stability of the steady state. From (P.4), it is monotonically stable if $0 < \delta \le \sigma$ (Figure 1B), and damped oscillations if $\sigma < \delta \le 2\sigma$ (Figure 1C). If $2\sigma < \delta$, then the steady state is locally oscillatory unstable (Figure 1D). The possibility of a locally stable steady state has been discussed by Gray (1984) and Obstfeld (1984). In fact, the steady state is globally stable when $0 < \delta \leq 2\sigma$. To prove it, it is convenient to define $G(p) \equiv \{F^2(p)/p\}^\sigma$. Clearly, $G(\bar{p}) = 1$. If $G(p') = 1$ for some $p' \in (0,1)/\bar{p}$, then there exists a period-2 cycle of F and p′ is a period-2 point.[8] From (5), G(p) has the following properties,

(P.6) $$G(p) = (1+\delta)^2(1-p)(1-F(p)) \ ,$$

(P.7) $$G'(p) = (1+\delta)^2[\{(1+\delta)(1-p)\}^{1/\sigma}\{(2+1/\sigma)p-1\}-1] \ ,$$

(P.8) $$G''(p) = (1+\delta)^{(2+1/\sigma)}(1+1/\sigma)(1-p)^{(1/\sigma-1)}\{2-(2+1/\sigma)p\} \ .$$

<u>Proposition 1</u>: If $0 < \delta \leq 2\sigma$, then all sequences starting at $p_0 \in (0,1)$ remain in (0,1) and $\lim_{t\to\infty} F^t(p) = \bar{p}$ for all $p \in (0,1)$.

<u>Proof</u>: From Lemma, $2\sigma < \Delta(\sigma)$ for $\sigma > 0$, thus $\delta < \Delta(\sigma)$, which proves the first half of the theorem. Next, let $\bar{\bar{p}} = 2\sigma/(1+2\sigma)$. Then, from (P.8), G′(p) is strictly increasing in $(0,\bar{\bar{p}})$, strictly decreasing in $(\bar{\bar{p}},1]$ and, from (P.7), $G'(\bar{\bar{p}}) = (1+\delta)^2[\{(1+\delta)/(1+2\sigma)\}^{1/\sigma}-1] \leq 0$. Thus, $G'(p) < 0$ in $(0,1)/\bar{\bar{p}}$, or G(p) is strictly decreasing in (0,1). Therefore, $\bar{p}$ is the only solution of $G(p) = 1$. In other words, F has no period-2 cycles. From the theorem of Coppel (1955), which states: if f is a continuous map of a compact interval to itself that has no period-2 cycles, the sequence $f^t(x)$ converges to a fixed point of f for every x in the interval, $\lim_{t\to\infty} F^t(p) = \bar{p}$ or $\lim_{t\to\infty} F^t(p) = 0$. But, (P.3) implies that no $p \in (0,1)$ cannot approach asymptotically to 0 and (P.1), (P.2), and $F(p^*) < 1$ jointly imply that it cannot be a preimage of 0, either. Therefore, $\lim_{t\to\infty} F^t(p) = \bar{p}$ for any $p \in (0,1)$. Q.E.D.

Therefore, when $\delta \leq 2\sigma$ so that the steady state is locally stable, a continuum of equilibria exist, but the price level will eventually stabilize at the steady state level in any equilibrium. On the other hand, whenever the steady state is unstable, endogenous price fluctuations can occur along equilibrium paths, as demonstrated below.

Proposition 2: If $0 < 2\sigma < \delta$, then a period-2 cycle exists.

Proof: It is sufficient to show that $G(p) = 1$ has a solution in $(0,1)/\bar{p}$. From (P.1) and (P.6), $G(+0) = (1+\delta)^2 > 1$ and $G(1) = 0$. Thus, the Intermediate Value Theorem implies that there exists $\hat{p} \in (0,\bar{p})$ such that $G(\hat{p}) = 1$ if $G'(\bar{p}) > 0$. From (P.7), $G'(\bar{p}) = (1+\delta)(\delta/\sigma-2) > 0$, or $2\sigma < \delta$. Q.E.D.

Thus, if $p_0 = \hat{p}$ or $F(\hat{p})$, the price level exhibits a cycle of period 2 forever. (Figure 2A suggests the existence of a period 2 cycle.) The condition of this theorem implies that, for any $\delta > 0$, there exists a range of intertemporal elasticity of substitution, $(2-\alpha)/(1-\alpha) < \gamma < \{\delta/2+(2-\alpha)\}/(1-\alpha)$, that implies cycles. On the other hand, a small substitution means global instability and a large substitution global stability. It also implies that, if $\sigma > 0$, a period-2 cycle can be generated with a sufficiently rapid money supply growth: $\mu > \beta(1+2\sigma)$.

Proposition 2 is sufficient to demonstrate that the possibility of endogenous price fluctuations. However, one might also be interested in how large the set of fluctuating equilibria is as a subset of all possible equilibria. For this purpose, we first show that any sequence satisfying (5) are bounded away from zero, if $F(p^*) < 1$.

Proposition 3: Suppose that $\sigma < \delta < \Delta(\sigma)$. Then,

(3.1) $0 < F^2(p^*) < \bar{p} < F(p^*) < 1$ and F maps $[F^2(p^*), F(p^*)]$ into itself.

(3.2) For any $p_0 \in (0,1)$, there exists $T > 0$ such that $F^T(p_0) \in$

$[F^2(p*),F(p*)]$.

<u>Proof</u>: (3.1) Note that $\sigma < \delta$ implies that $p* < \bar{p} = F(\bar{p}) < F(p*)$. Also, $\delta < \Delta(\sigma)$ implies $F(p*) < 1$, so that $\bar{p} \in (p*,F(p*)) \subset [p*,1]$. From (P.2), $0 < F^2(p*) < \bar{p} = F(\bar{p}) < F(p*) < 1$. Since F maps $[p*,F(p*)]$ onto $[F^2(p*),F(p*)]$, nothing remains to be proved if $p* \leq F^2(p*)$. When $F^2(p*) < p*$, F maps $[F^2(p*),p*]$ onto $[F^3(p*),F(p*)]$. Since $F^2(p*) < p* < \bar{p}$, $F^2(p*) < F^3(p*)$. (If $F^2(p*) \geq F^3(p*)$, there exists a fixed point of F in $(0,F^2(p*)]$, which contradicts with the uniqueness of the fixed point.) Thus, F maps $[F^2(p*),p*]$ into $[F^2(p*),F(p*)]$, which proves (3.1).

(3.2) Divide (0,1) into three parts: $(0,F^2(p*))$, $[F^2(p*), F(p*)]$ and $(F(p*),1)$. A simple graphic analysis shows that (3.2) holds for $(0,F^2(p*))$. Since F maps $(F(p*),1)$ onto $(0,F^2(p*))$, (3.1) implies (3.2). Q.E.D.

All sequences starting in (0,1) will eventually be trapped into $[F^2(p*), F(p*)]$, and therefore, bounded away from zero. (The box in Figure 1D depicts this trapping interval.) Hence, neither the transversality condition nor the fractional backing of paper money proposed by Obstfeld and Rogoff (1983, 1986) can rule them out. The question is then; Do any of these equilibria exhibit persistent fluctuations? The following theorem states that almost all of them fluctuate forever if the steady state is locally unstable.

<u>Proposition 4</u>: Suppose that $0 < 2\sigma < \delta < \Delta(\sigma)$. Let N be the set of points $p \in (0,1)$ such that $F^t(p)$ converges. Then, N is at most countable, and therefore, has Lebesgue measure zero.

<u>Proof</u>: Let p be a point in N and $p\infty$ be the limit point of the sequence starting at p. From the continuity of F, $p\infty = \lim_{t\to\infty} F^{t+1}(p) = F(\lim_{t\to\infty} F^t(p)) = F(p\infty)$, therefore, the uniqueness of the fixed point implies $p\infty = \bar{p}$. Since

$2\sigma < \delta$ implies that $\bar{p}$ is locally unstable, $F^t(p)$ cannot approach it asymptotically. Therefore, p must be a preimage of the steady state: that is, $p \in \{ q \mid F^T(q) = \bar{p}$ for a finite T $\}$. This set is at most countable points, since that F is unimodal implies that, for any y, there are at most two x's that solve $y = F(x)$. Q.E.D.

It should be noted that this result is about the size of the set of fluctuating equilibria, not about the likelihood of such equilibria. Without a compelling theory of equilibrium selection, the analysis here has nothing to say about how often we observe price fluctuations.

4. The Chaotic Dynamics

The preceding analysis has shown that persistent fluctuations are regular rather than exceptional, when $F(p^*) < 1$ and the steady state is locally unstable. However, one might be also interested in knowing how the price level fluctuates, either periodically or erratically. Also the case of $F(p^*) \geq 1$ has not been fully analyzed. In order to address these questions, this section heavily borrows some results from the recent development in discrete nonlinear dynamics.[9]

A good starting point is the case of $\sigma = 1$, or $\gamma = (3-\alpha)/(1-\alpha)$. Equation (5) then becomes,

$$(6) \qquad p_{t+1} = F(p_t) = (\mu/\beta)p_t(1-p_t) = (1+\delta)p_t(1-p_t) \quad .$$

This first order difference equation, known as the logistic equation, has been applied to a variety of economic problems; see, for example, Day (1982, 1983), Jensen and Urban (1984) and Bhaduri and Harris (1987). From the results in the last section, one can easily verify that equation (6) is globally convergent if $0 < \delta \leq 2$; it has a period-2 cycle if $\delta > 2$; almost all sequences exhibit persistent fluctuations in the interval $[(1+\delta)^2(3-\delta)/16, (1+\delta)/4]$ if $2 < \delta < 3$. In fact, stronger results are known. As δ increases from 2, or as μ increases from 3β, period 2 points appear first, then they bifurcate to period 4 points, which in turn gives way to a cycle of period 8. This process of period-doubling bifurcations continue to generate period-2^n cycles until $\delta = 2.5699...$ For $\delta \geq 2.5699...$, a period-3 cycle exists. As shown by Sarkovskii (1964) and Li and Yorke (1975), the existence of period-3 cycles implies that there are cycles of every integer period and that there is an uncountable set of initial prices, $S \subset (0,1)$, which give rise to <u>chaotic</u> (i.e., aperiodic and not asymptotic to a cycle) price fluctuations in S. If $\delta = 3$ ($\mu = 4\beta$), or,

$$(7) \qquad p_{t+1} = F(p_t) = 4p_t(1-p_t) \quad ,$$

we have an analytic formula for the solution. Let $p_t = \sin^2\theta_t$. Then,

$$\sin^2\theta_{t+1} = p_{t+1} = 4p_t(1-p_t) = (2\sin\theta_t\cos\theta_t)^2 = \sin^2(2\theta_t) \quad ,$$

so that $\theta_t = 2^t\theta_0$, or,

$$(8) \qquad p_t = \sin^2(2^t\theta_0) \ .$$

Therefore, if $p_0 = \sin^2(k\pi/2^n)$ for some integer k and n, p_t eventually converge to 0, violating the feasibility condition. Otherwise, p_t remains in (0,1) forever, thus qualified for equilibria. In particular, by setting $p_0 = \sin^2[k\pi/(2^n-1)]$ with appropriate choices of k and n, one can generate a cycle of any period. If $p_0 \neq \sin^2[k\pi/(2^n-1)]$, then the solution is chaotic. In fact, one can show that the sample distribution of the price level will converge to the density function $1/\pi[p(1-p)]^{1/2}$ for almost all $p_0 \epsilon (0,1)$. Thus, along almost all equilibria, the price movement appears purely stochastic as if it is a random variable drawn from this density distribution. (A map F, when satisfying this property, is called <u>strongly chaotic</u>.) Furthermore, (8) shows that the dynamics of (7) has "sensitive dependence on initial conditions." A slight change in the initial price magnifies exponentially at the rate equal to log 2 > 0. Boldrin and Montrucchio (1986) and Deneckere and Pelikan (1986) also considered this strongly chaotic equation (7). Their example requires $\beta = 0.01$. On the other hand, the example here requires $\beta = \mu/4$, which is consistent with an arbitrarily small discounting. Jakobson (1981) proved that, for a set of parameter values δ having positive Lebesgue measure in (2.5699..., 3], (6) generates the dynamics similar to the case of $\delta = 3$. If $\delta > 3$ ($\mu > 4\beta$), most initial prices will violate the feasibility condition after finite periods, but there exists a totally disconnected set, $C \subset (0,1)$ such that, if $p_0 \epsilon C$, the price sequence remains in C forever.[10] See Devaney (1987) and Lauwerier (1986) for more discussion of the dynamics governed by (6).

Many properties that (6) possesses are also shared by (5) with $\sigma \neq 1$. First, let us demonstrate the existence of period 3 cycles, which is a sufficient condition for the existence of periodic equilibira of every integer period, as well as chaotic equilibria.

<u>Proposition 5</u>. For any $\sigma > 0$, there exists an open interval J containing $\Delta(\sigma)$ such that a period 3 cycles of F exists if $\delta \epsilon J$.

<u>Proof</u> Define $H(p) = \equiv \{F^3(p)/p\}^\sigma$. Then, $H(+0) = \{F'(0)\}^{3\sigma} = (1+\delta)^3 > 1$ and $H(\bar{p}) = 1$. From the Intermediate Value Theorem, it is sufficient to show that there exists $p \epsilon (0,\bar{p})$ such that $H(p) < 0$. If $\delta = \Delta(\sigma)$, then $p* < \bar{p}$ and

$F(p^*) = 1$ so $F^2(p^*) = F^3(p^*) = 0$ or $H(p^*) = 0 < 1$. From the continuity of H on δ, there exists a neighborhood of $\Delta(\sigma)$ such that $H(p^*) < 1$. Q.E.D.

Next, let us ask how large the set of chaotic equilibria is as a set of all equilibria. We can apply the powerful mathematical results, if F satisfies, besides being unimodal, that its Schwartzian derivative, SF, exists and is negative on the relevant interval, whenever $p \neq p^*$, where,

$$SF \equiv F'''/F' - (3/2)[F''/F']^2 .$$

Note $SF < 0$ if $\log |F'|$ is concave. Some algebra can verify that $\log|F'|$ indeed has a negative second derivative on $[0,1]/p^*$ if $0 < \sigma \leq 1$. Therefore,

Proposition 6: Suppose $0 < \sigma < \delta \leq \Delta(\sigma)$ and $\sigma \leq 1$. Then,

(6.1) F has at most one weakly stable cycle.[11] This cycle lies entirely in the interval $[F^2(p^*), F(p^*)]$. When F has a weakly stable cycle, it attracts p^*, that is, the set of the weakly stable periodic points is the set of accumulation points of $\{F^t(p^*)\}$.

(6.2) If F has a weakly stable cycle, then the set of points p in (0,1) such that $F^t(p)$ does not tend to this cycle, E, has Lebesgue measure zero.

(6.3) If $\{F^t(p^*)\}$ converges to an unstable cycle, the empirical distribution of the price level generated by (5) converges weakly to a unique absolutely continuous distribution function on $[F^2(p^*), F(p^*)]$ for almost all $p_0 \in (0,1)$. That is, F is strongly chaotic.

Proof: Since F is a unimodal map with a negative Schwartzian derivative and that 0 is a unstable fixed point from (P.3), it satisfies the condition of Theorem 2, Proposition 4 and Corollary 6 of Grandmont (1984). (6.1) is the mere restatement of Theorem 2. From its Proposition 4, $E \cap (0,F(p^*))$ has

Lebesgue measure zero. Since F maps (F(p*),1) into (0,F(p*)), (6.2) results. Likewise, (6.3) follows from Corollary 6. Q.E.D.

Proposition 6 has some important implications. First, it states that almost all sequences have the same asymptotic behabior (see also Guckenheimer and Holmes [1986, p.270]). Second, (6.1) provides an "experimental" way of recognizing this asymptotic property. In particular, if $F^t(p^*)$ does not converge, then one has good reason to believe that F is strongly chaotic. Figures 2D and 2E plot the first one hundred observations of $F^t(p^*)$ for δ = 0.03, σ = 0.01 and δ = 0.015, σ = 0.005, respectively. Both sequences appear very chaotic.[12]

Third, (6.3) provides a sufficient condition for almost all equilibria to be chaotic and for all periodic equilibria to be unstable. In particular,

<u>Corollary</u>: Suppose that $\delta = \Delta(\sigma)$ and $0 < \sigma \leq 1$. Then, the sample distribution of the price level generated by (5) converges weakly to a unique absolutely continuous distribution function on (0,1) for almost all $p_0 \in$ (0,1).

<u>Proof</u>: Note that the condition implies that F(p*) = 1. Thus, $F^2(p^*) = 0$, which means that p* converges to an unstable cycle, 0, thereby satisfying the condition of (6.3). Q.E.D.

Several remarks about Corollary should be made here. First, not all $p_0 \in$ (0,1) is consistent with the equilibrium condition, since p* and its preimage will eventually converge to zero, violating the feasibility condition. Second, (7) is a special case that satisfies the condition of Corollary. Third, from Lemma, $\lim_{\sigma\to 0} \Delta(\sigma) = 0$, thus one can find σ satisfying the condition for an arbitrary small $\delta = \mu/\beta - 1$. A large discounting is no longer necessary for generating the strongly chaotic dynamics, even with a constant money supply ($\mu = 1$). Finally, although these chaotic price sequences satisfy the transversality condition, the fractional backing considered by Obstfeld and Rogoff can rule them out, since the price level will become arbitrarily close to zero along these equilibria. Of course, this does not imply no fluctuations. As seen in the logistic equation, there is a countable dense set, although with measure zero, of initial prices that lead to "unstable" cycles.

Next, note that $F(p^*) = p^*$ if $\delta = \sigma$ and $F(p^*) = 1$ if $\delta = \Delta(\sigma)$. Therefore, for any given $0 < \sigma \leq 1$, F as a one parameter δ family of maps with $\delta \in [\sigma, \Delta(\sigma)]$ is <u>a full family</u>.[13] Also, from Lemma, $\Delta(\sigma)$ is invertible and therefore, for any given $0 < \delta \leq 1$, F as one parameter family $-\sigma$ of maps with $\sigma \in [\Delta^{-1}(\delta), \delta]$ is also a full family. Therefore, from Theorems 7 and 8 in Grandmont (1984) (see also Devaney [1987, sec. 1.19] and Guckenheimer and Holmes [1986, pp.346-9]), one can conclude,

<u>Proposition 7</u> As one increases δ from σ to $\Delta(\sigma)$ (as the rate of money growth increases from $[1+\sigma]\beta$ to $[1+\Delta(\sigma)]\beta$) for any given $0 < \sigma \leq 1$, or decreases σ from δ to $\Delta^{-1}(\delta)$ for any given $0 < \delta \leq 1$, the dynamics given by equation (5) experiences "period-doubling transition to chaos." That is, for any $0 < \sigma \leq 1$, there exists a value $\Delta^*(\sigma)$ such that $2\sigma < \Delta^*(\sigma) < \Delta(\sigma)$ and if $\delta \in [2\sigma, \Delta^*(\sigma)]$, all cycles of F have a period that is a power of 2, and there exists an uncountable set of values of δ in $(\Delta^*(\sigma), \Delta(\sigma)]$ such that F has no weakly stable cycles. A similar statement can be made when one fixes δ and varies σ.

Figures 2A-2D demonstrate Proposition 7 for $\delta = 0.03$. As σ declines, a stable period 2 cycle appears first (Figure 2A), then a period 4 cycle (Figure 2B), and then a period 8 cycle (Figure 2C) and eventually a chaos (Figure 2D). (It should be noted that the results analogous to Propositions 6 and 7 also hold for the case of $\sigma > 1$. The difficulty is that, when $\sigma > 1$, the Schwartzian derivative is negative only on an interval $[0,p^+]$, where p^+ satisfies $p^* < p^+ < 1$. Therefore, one needs to restrict δ to guarantee $F(p^*) \leq p^+$ so that F has a negative Schwartzian derivative on the trapping interval.)

Finally, consider the case of $F(p^*) > 1$, or $\delta = \Delta(\sigma)$. See Figure 3. Note that there exists a closed interval of initial prices, I_0, that leave (0,1) after one period. There are two open intervals of initial prices that remain in (0,1) after one period. Note that F maps each of these intervals monotonically onto (0,1). This implies that there are two disjoint closed intervals of prices that leave (0,1) after two periods. This suggests that one can construct the set of initial prices that are consistent with equilibria by successively removing closed intervals from the "middles" of a set of open intervals. Using the technique explained in Devaney (1987), one can prove, when δ is sufficiently large or σ is sufficiently small, that the set of the initial prices consistent with equilibria is a totally disconnected set, that the number of periodic points with period less than or equal to n is 2^n-1, and that there exists an equilibrium path that comes arbitrarily close to other points in this set.[14] Again, we have very complex dynamics.

5. Making Sense of Cycles and Chaos

This section provides an intuitive explanation of the conditions necessary for generating convergence, cycles and chaos. First, note that equation (1) can be rewritten as follows, by defining $\Pi_t = P_{t+1}/P_t = \mu m_t/m_{t+1}$, the gross inflation rate and $R_t = U_c(y,m_t)/\beta U_c(y,m_{t+1})$, the gross real interest rate.

$$U_m(y,m_t)/U_c(y,m_t) = 1 - 1/(R_t\Pi_t) \ .$$

The above expression states that the marginal value of liquidity services is equal to the liquidity cost, which increases with the real interest rate and the inflation rate. When the consumption good is normal, the L.H.S. is decreasing in m_t so that we have $m_t = L(R_t\Pi_t)$, $L' < 0$. That is, demand for real balances negatively depends on the gross nominal interest rate. From $R_t = U_c(y,m_t)/\beta U_c(y,m_{t+1})$ and $m_{t+1} = \mu m_t/\Pi_t$, the real rate of interest generally depends on Π_t/μ and m_t, yielding the expression $R_t = A(\Pi_t/\mu,\ m_t)$ or,

$$m_t = L[A(\Pi_t/\mu,m_t)\Pi_t] \ . \tag{9}$$

The steady state is given by $\Pi_t = \mu$, $R_t = 1/\beta$, $R_t\Pi_t = \mu/\beta = 1+\delta$ and $m_t = \bar{m} = L[\mu/\beta]$. Thus, δ is the nominal interest rate in the steady state.

From equation (9), it is easy to see why the separability leads to a global convergence. The separability implies that the real interest rate is constant and given by $(1/\beta)-1$. Now, suppose that $m_t < \bar{m}$. Then, $L(\mu/\beta) = \bar{m} > m_t = L(\Pi_t/\beta)$ so that $\Pi_t > \mu$, or $\bar{m} > m_t > m_{t+1}$. In order to make the agent willing to hold lower real balances, the nominal interest rate should be higher than its steady state level. With the real interest rate being constant, this implies that the inflation rate must be higher than the money growth rate. Repeating this shows that real balances will continue to decline. Likewise, if $m_t > \bar{m}$, real balances will continue to rise forever.

With a nonseparable utility function, this need not be the case since a deviation of the inflation rate from its "fundamentals", Π/μ, affects the real rate of interest. If an inflation rate higher than its steady state level reduces the real interest rate so much that the nominal interest rate declines, then an inflation rate <u>lower</u> than the money growth rate is necessary to reduce demand for real balances. (If $m_t < \bar{m}$, $\Pi_t/\mu < 1$ so that $m_t < m_{t+1}$.) If this "perverse" effect on the real interest rate is very strong, only a small deviation of the inflation rate would suffice, therefore $m_t < m_{t+1} < \bar{m}$, generating a convergent path. On the other hand, if the effect is relatively small, then a large deviation of the inflation rate is necessary, therefore $m_t < \bar{m} < m_{t+1}$, generating an oscillatory movement.

Under the specification of the utility functions considered in the previous sections, $R_t = (1/\beta)(\mu/\Pi_t)^{\sigma+1}$ and thus,

$$R_t\Pi_t/(\mu/\beta) = R_t\Pi_t/(1+\delta) = (\Pi_t/\mu)^{-\sigma} \quad ,$$

so that $-\sigma$ is the elasticity of a deviation of the nominal interest rate from its steady state level with respect to a deviation of the inflation rate from its steady state level. Therefore, a small, positive σ gives rise to cycles and chaos, while a large, positive σ implies convergence. The reason for a large δ implying cycles and chaos and a small δ implying convergence when $\sigma > 0$ is that the nominal interest rate elasticity of money demand, $|d\log L/d\log(R\Pi)|$, evaluated at the steady state, is equal to $(1+\delta)/\delta$, negatively depending upon δ.

The above argument has two important implications. First, it suggests that endogenous price fluctuations are not necessarily unique to the Brock model. Any model in which the real interest rate declines when the inflation rate is higher than the growth rate of money supply would have the possibility of generating complex price dynamics.[15] Second, equation (9) implies that periodic and chaotic movements of price and real balances are accompanied by periodic and chaotic movements of the nominal and real interest rates. This is consistent with the findings of strong evidence of chaos in stock and T-bill returns by Scheinkman and LeBaron (1987) and Brock (1988).

6. Discussions

This section addresses some of the issues that the preceding analysis might provoke.

a. Welfare Implications.

The analysis here can be considered as a demonstration of price destabilizing speculation consistent with perfect competition and rational expectations in a general equilibrium setting, since intertemporal arbitrage by agents is a driving force behind the price dynamics. One may feel intuitively that such a price destabilizing speculation is "bad," but, if so, one's intuition is faulty. To see this, note that, from (2) and the definition of σ and p_t,

$$U(y,m_t) = -\{B/(1+\delta)\}(p_t)^{1+\sigma} = \{B/(1+\sigma)\}\{(\beta/\mu)(p_{t+1})^{\sigma}-(p_t)^{\sigma}\} ,$$

where B is a positive constant, independent of μ, and use has been made of equation (4). Therefore, one can easily calculate the welfare level under a constant money supply ($\mu = 1$) as follows:

$$\begin{aligned} W = \Sigma_{t=0}^{\infty} \beta^t U(y,m_t) &= \{B/(1+\sigma)\}[\Sigma_{t=0}^{\infty} \beta^t\{\beta(p_{t+1})^{\sigma}-(p_t)^{\sigma}\}] \\ &= \{B/(1+\sigma)\}[\lim_{T\to\infty}\beta^T(p_T)^{\sigma} - (p_0)^{\sigma}] . \end{aligned}$$

Consider the case of $\sigma > 0$, where there are multiple bounded equilibria. From the boundedness, $\lim_{T\to\infty}\beta^T(p_T)^{\sigma} = 0$ so that,

$$W = \Sigma_{t=0}^{\infty} \beta^t U(y,m_t) = -\{B/(1+\sigma)\}(p_0)^\sigma .$$

Therefore, an equilibrium with a lower initial price gives higher welfare under a constant money supply. In particular, when $\delta > 2\sigma > 0$, all fluctuating equilibria starting with $p_0 < \bar{p}$ are better than the steady state equilibrium.

Pareto dominance is sometimes suggested as a criterion for equilibrium selection in a model with multiple equilibria. The above result implies that even if one subscribes to such a criterion, fluctuating equilibria cannot be eliminated.

b. Money in the Utility Function

In the model discussed above, demand for money arises through the assumption that an agent's utility depends on his real balances. This approach has been adopted here, not because it is the best way of putting money in an optimizing model, but because it is the approach favored by many "mainstream" macroeconomists. The results here should be interpreted as a caution to the proponents of this approach that even a simple model of money can generate complex dynamics.

The money-in-the-utility-function approach is often considered as a way of capturing the role of money in reducing transaction costs associated with imperfect markets: see, for example, Brock (1974, pp.768-9) and Feenstra (1986). Viewed in this way, the analysis here is akin to Woodford's (1987) study. He demonstrated that the chaos can arise with a small discounting if some agents face liquidity constraints. It seems therefore hard to ignore the possibility of complex dynamics in an economy with market imperfection.[16]

One potential problem is that, if agents derive utility from holding money because it helps them to save resources such as leisure, then it is hard to justify unbounded utility functions. Equation (2) implies that $u(c,m) \to \infty$ as $m \to \infty$ when $\gamma \geq 1$, and $u(c,m) \to -\infty$ as $m \to 0$ when $0 < \gamma \leq 1$. However, this problem does not reduce the validity of the main result; i.e., bounded equilibrium fluctuations of prices and real balances. All we really need is that the utility function has the form given in (2) only over the trapping interval.

c. Discrete Time Specification

Most studies of complex dynamics in economics deal with discrete time models. This is partly because we can generate cycles and chaos much easier in discrete time systems than in continuous time ones. Therefore, critics may argue that endogenous fluctuations are an artifact of a coarse discrete time approximation: *Natura non facit saltus*.

The present analysis seems less prone to such a criticism than other studies in the literature. First of all, it is hard to take continuous time models literally. Although financial markets may open anytime, nobody goes shopping every second nor eats incessantly. The neoclassical (Fishian) approach to the consumption-saving decision and its implication of consumption smoothing make sense only if the time interval between consumption decisions is interpreted to be sufficiently long. This is particularly true in the Brock model, where the single consumption good actually represents a composite of various consumption goods. Moreover, if one thinks that economic activities of a typical household consist of production (working), consumption, and transaction (shopping), then it seems reasonable to take the time interval to be at least as long as a week or a month. And it is no accident that most studies of money, including the voluminous literature on the transactions demand for money and the cash-in-advance constraints, adopt a discrete time specification. Although a continuous time specification is analytically convenient and innocuous for many purposes, a discrete time model seems to be closer to the reality. One appealing feature of the present analysis is that cycles and chaos can emerge with an arbitrarily small discounting or with an arbitrarily short time interval. Therfore, the restriction imposed by a discrete time specification is considered negligible.

It should be emphasized, however, that complex dynamics are not unique to the discrete time system. It is known that nonlinear differential equations are also capable of generating chaotic behavior if the number of variables is at least three (see Guckenheimer and Holmes (1986)). Although the theoretical literature on high dimensional nonlinear systems is still sparse, the limited results suggest that higher order systems, both discrete and continuous, can produce chaotic dynamics with less nonlinearity. Introducing some state variables, such as consumer durable goods and consumer's habit formation, would be an important extension.

Appendix

Proof of Lemma

First, note that $1+\Delta(\sigma) = \sigma^{-\sigma}(1+\sigma)^{1+\sigma} > 0$. By differentiating $\log[1+\Delta(\sigma)]$, we have,

$$\Delta'(\sigma) = [1+\Delta(\sigma)]\log(1+1/\sigma) > 0 ,$$

so that $\Delta(\sigma)$ is strictly increasing. Next,

$$\lim_{\sigma\to 0} [1+\Delta(\sigma)] = [\lim_{\sigma\to 0}(1+\sigma)][\lim_{\sigma\to 0}(1+1/\sigma)^{\sigma}] = 1 ,$$

thus $\lim_{\sigma\to 0}\Delta(\sigma) = 0$. To prove that $\Delta(\sigma) > 2\sigma$, define

$$Z(\sigma) = \log[(1+\Delta(\sigma))/(1+2\sigma)] .$$

It is sufficient to show that $Z(\sigma) > 0$. Since $\lim_{\sigma\to 0} Z(\sigma) = \log(1) = 0$, it is sufficient to show,

$$Z'(\sigma) = \Delta'(\sigma)/[1+\Delta(\sigma)] - 2/(1+2\sigma)$$

$$= \log(1+1/\sigma) - 2/(1+2\sigma) > 0 .$$

Since $\lim_{\sigma\to\infty} Z'(\sigma) = 0$, this is proven by demonstrating $Z''(\sigma) < 0$, as below;

$$Z''(\sigma) = -[\sigma(\sigma+1)(1+2\sigma)^2]^{-1} < 0 . \qquad \text{Q.E.D.}$$

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Footnotes

1. For the closely related literature on stochastic bubbles, see Blanchard (1979), Blanchard and Watson (1982), Diba and Grossman (1988) and Flood and Garber (1980).

2. It is also possible to have multiple steady states in a nonseparable utility function; see Brock (1974, 1975). However, this possibility can be eliminated if real balances are normal.

3. Obstfeld and Rogoff (1983) introduced a nonproducible capital in the Brock model and showed that, if the government sells capital for money at an arbitrarily high pre-set nominal price with some probability, explosive price paths can be ruled out. A similar analysis can be easily made in the present context so that capital will not be introduced explicitly below to simplify the notation. The government can replicate Obstfeld and Rogoff's scheme if it prints money on the paper that has small intrinsic value. Farmer (1984) considered an alternative way of ruling out hyperinflations in an overlapping generations economy. It is not clear whether his scheme can eliminate fluctuating equilibria considered here.

4. After the first draft of this paper had been written, I became aware that Woodford's (1987) model of capital market imperfection and Deneckere and Judd's (1987) model of product development also generate the chaos without a heavy discounting. I am indebted to Prof. Brock for bringing my attention to Woodford's work.

5. Baumol and Benhabib (1987) anticipated the possibility of chaos in the Brock model, but the reason for the emergence of chaos they suggested seems quite different from what we discovered here.

6. If $\mu \leq \beta$, then no equilibrium exists: see Brock (1974, pp.764-5).

7. Strictly speaking, $P_t = \infty$ is another candidate of the steady state equilibrium. This is the situation where paper money has no value. One can show that, for the class of utility functions assumed below, $P_t = \infty$ can be ruled out as an equilibrium.

8. A period-k cycle is defined by $(p_0, p_1, \ldots, p_{k-1})$ such that $F^k(p_0) = p_0$ and $F^i(p_0) = p_i \neq p_0$ for all $i = 1,2,\ldots,k-1$, and p_i is a period-k point.

9. See Devaney (1987, part I), Grandmont (1984) and Guckenheimer and Holmes (1986, sections 5.6, 6.3 and 6.8). Grandmont's paper provides a summary of the results contained in Collet and Eckmann (1980), which I have not yet seen.

10. One can show that the union of C and the set of $\{p \mid F^T(p) = 0 \text{ for some } T\}$ is a Cantor set (i.e., a closed, totally disconnected, perfect set).

11. A cycle $(p_0, p_1, , , p_{k-1})$ is weakly stable if $|DF^k(p_0)| \leq 1$. It is unstable if $|DF^k(p_0)| > 1$.

12. Since iterations must be stopped after a finite time, this experimental way is unable to distinguish between chaotic behavior and the periodic behavior of a long period.

13. This definition of a full family is due to Guckenheimer and Holmes (1986, p.272). It is slightly (and without significance) different from Grandmont's (1984) or Devaney's (1987, p.149) definition of <u>a transition family</u>.

14. Devaney (1987, section 1.7) proves the essentially same results for (6). His proof only requires that F is <u>everywhere expanding</u>, that is, $|F'(p)| > 1$ for any p such that $|F(p)| < 1$. Although this condition does not hold in our example, if one defines $q = (p)^\sigma$ and considers the first order difference equation in q, then it can be shown that the condition holds if $\{\delta/(1+\delta)\}^\sigma[(1+\delta+\sigma)/(1+\Delta(\sigma))] > 1$. This is guaranteed by choosing δ large enough or σ sufficiently close to zero. The qualitative dynamics of p and q are equivalent since they are <u>topologically conjugate</u>. The results here differ from Devaney's only because Devaney considers the removal of open intervals in the middles from [0,1], while we consider the removal of closed intervals from (0,1).

15. This condition should not be confused with the so-called Mundell (1963)-Tobin (1965) effect: an inflation due to a higher money growth reduces the real interest rate in the long run. In the Brock model the steady state real interest rate is $1/\beta$, independent of the money growth rate and the steady state inflation rate.

16. Given the Turnpike properties of efficient allocations with a small discounting, such as Brock and Scheinkman (1976), Cass and Shell (1976) and McKenzie (1976), it seems hard, though not impossible, to obtain complex dynamics with a small discounting in an economy where the first welfare theorem holds.

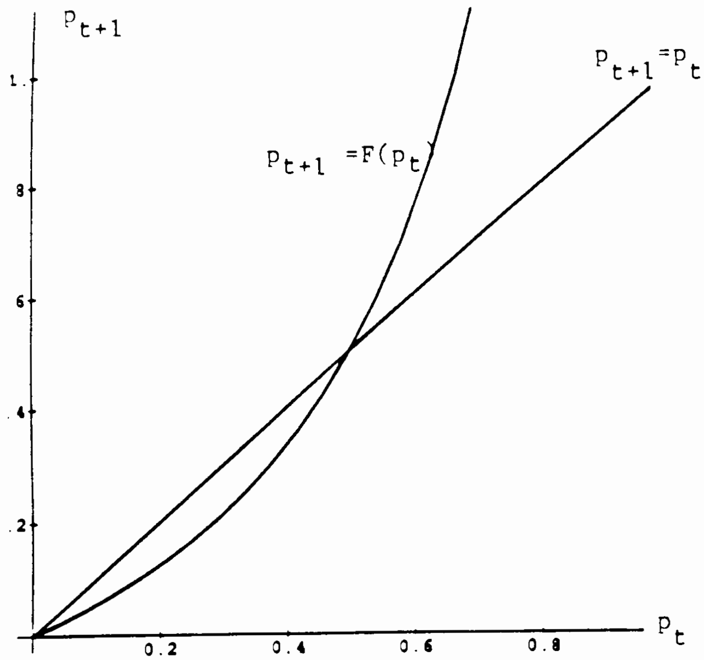


Figure 1A
(δ = 1, σ = -1)

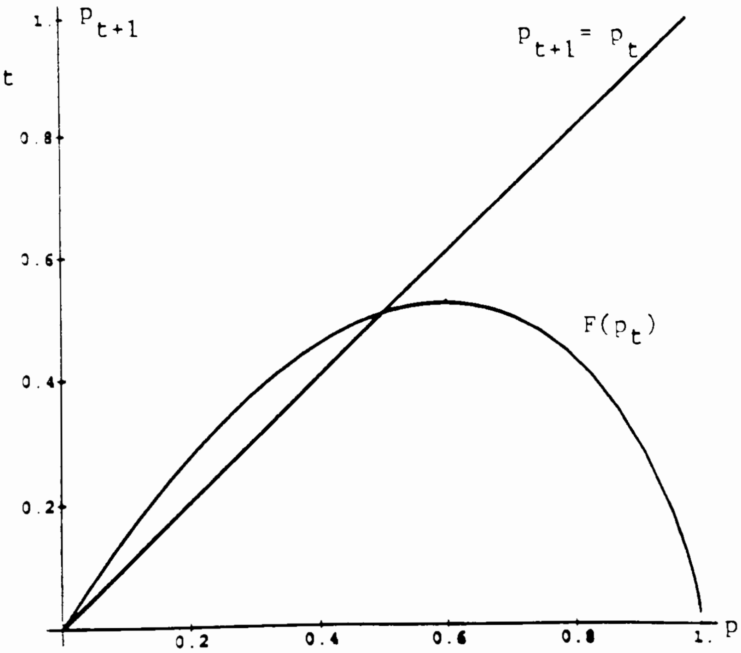


Figure 1B
(δ = 1, σ = 1.5)

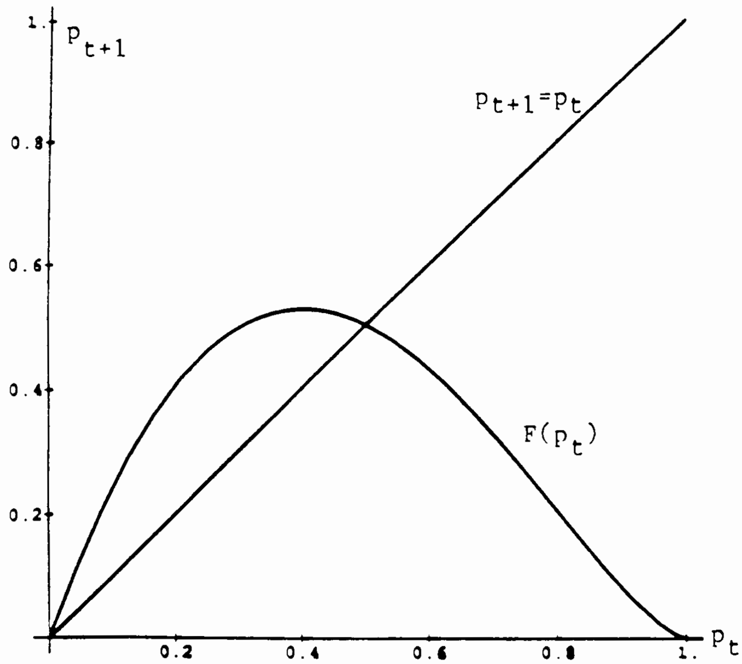


Figure 1C
(δ = 1, σ = 2/3)

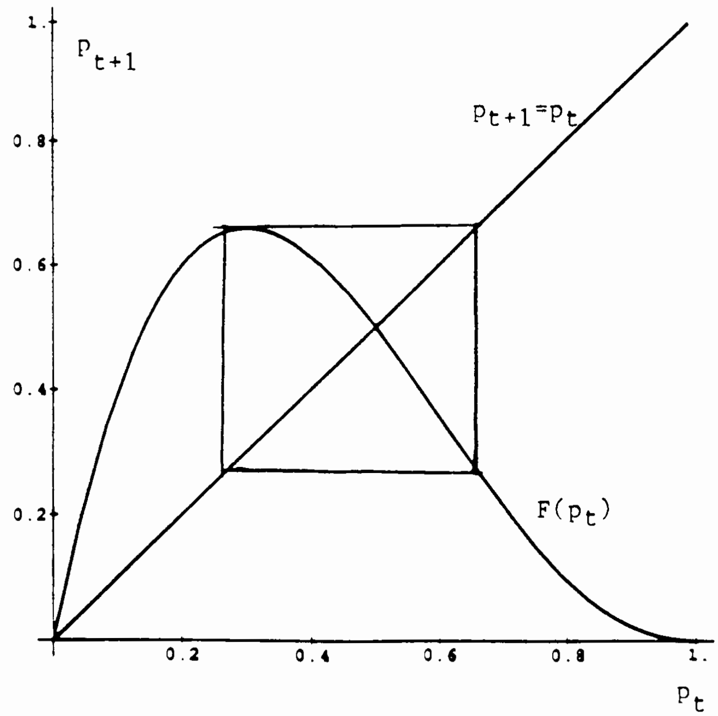


Figure 1D
(δ = 1, σ = 3/7)

Figure 2A
δ = 0.03
σ = 1/75

Period 2 Cycles

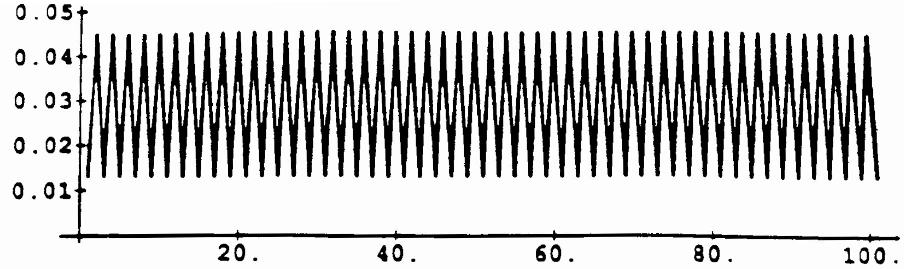


Figure 2B
δ = 0.03
σ = 1/85

Period 4 Cycles

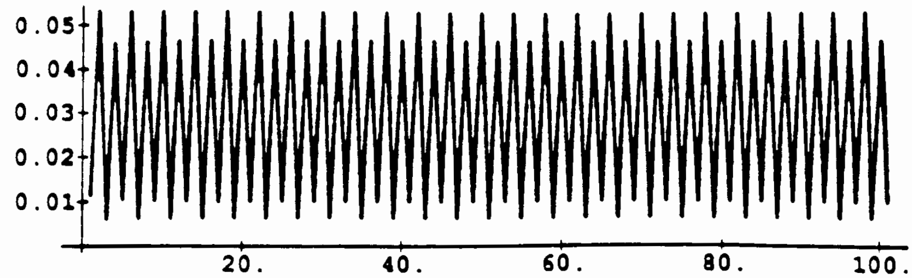


Figure 2C
δ = 0.03
σ = 1/89

Period 8 Cycles

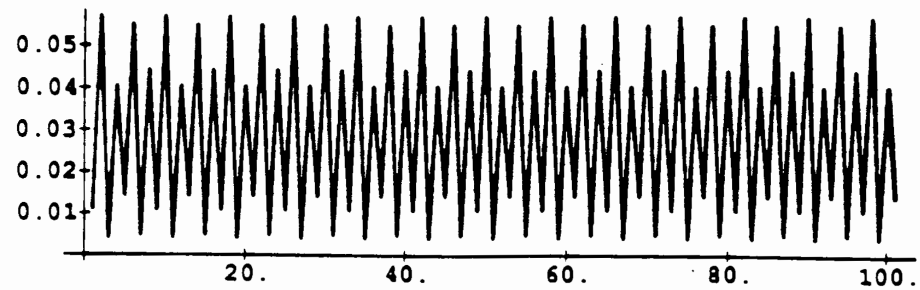


Figure 2D
δ = 0.03
σ = 0.01

Chaos

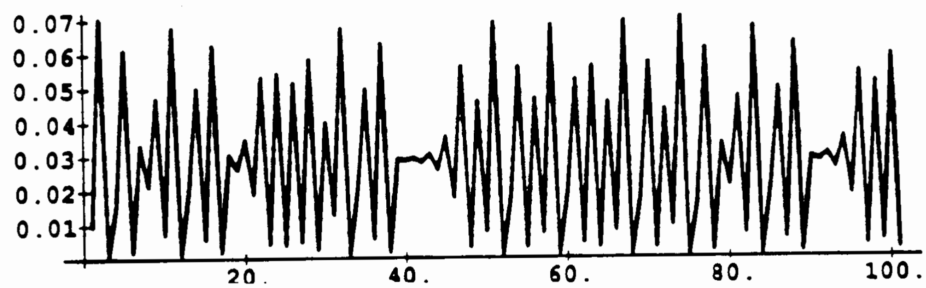
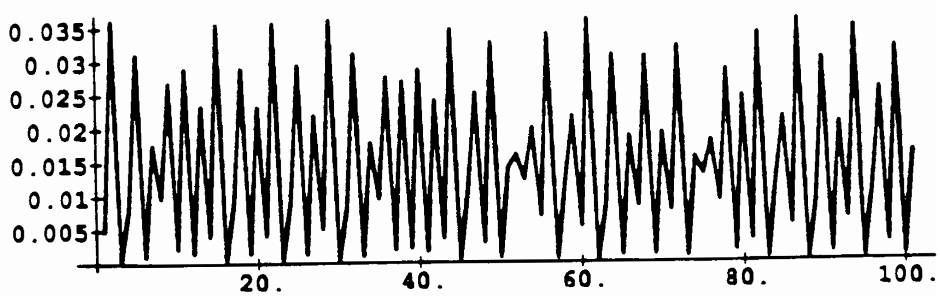


Figure 2E
δ = 0.015
σ = 0.005

Chaos



Figures 2A-2E: The equilibrium price sequence starting with $p_0 = p^*$

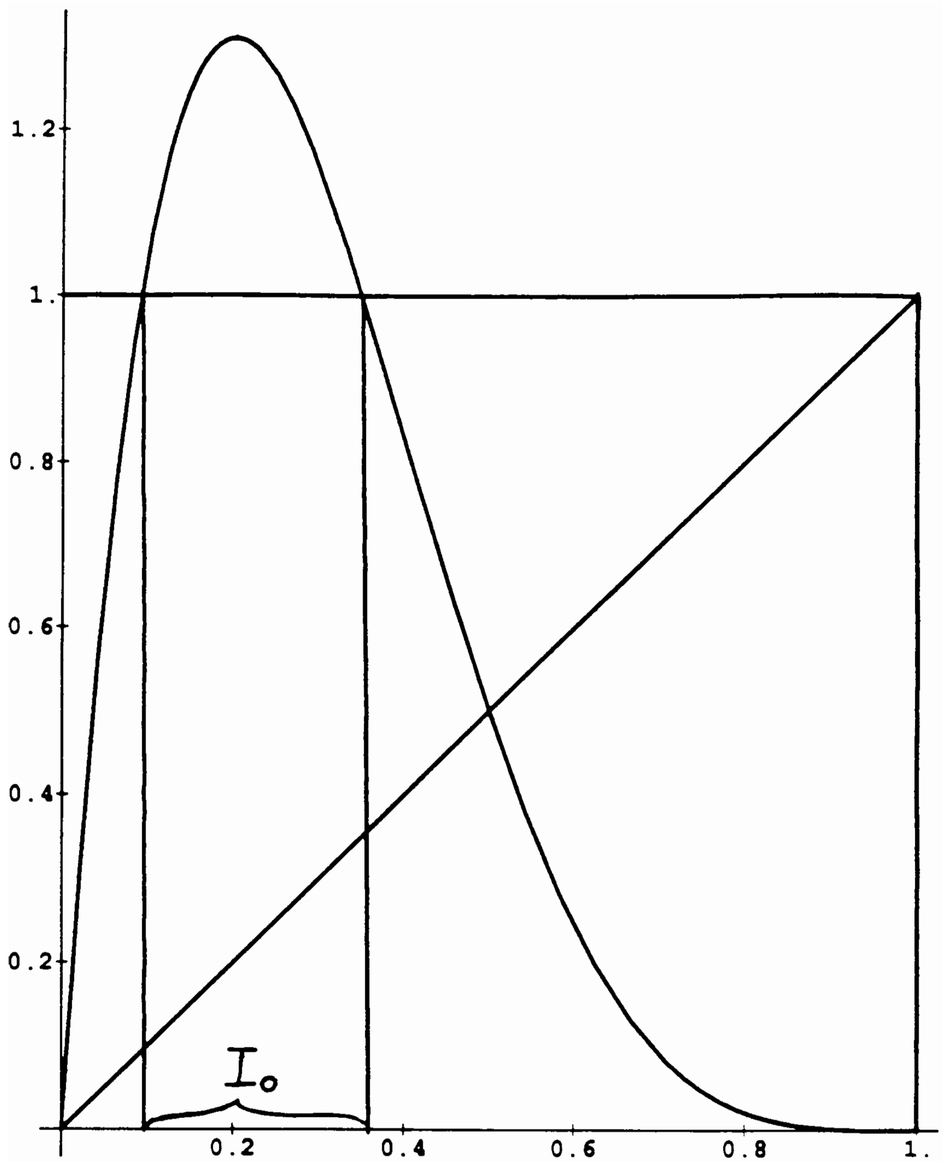


Figure 3

($\delta = 1$, $\sigma = 0.25$)

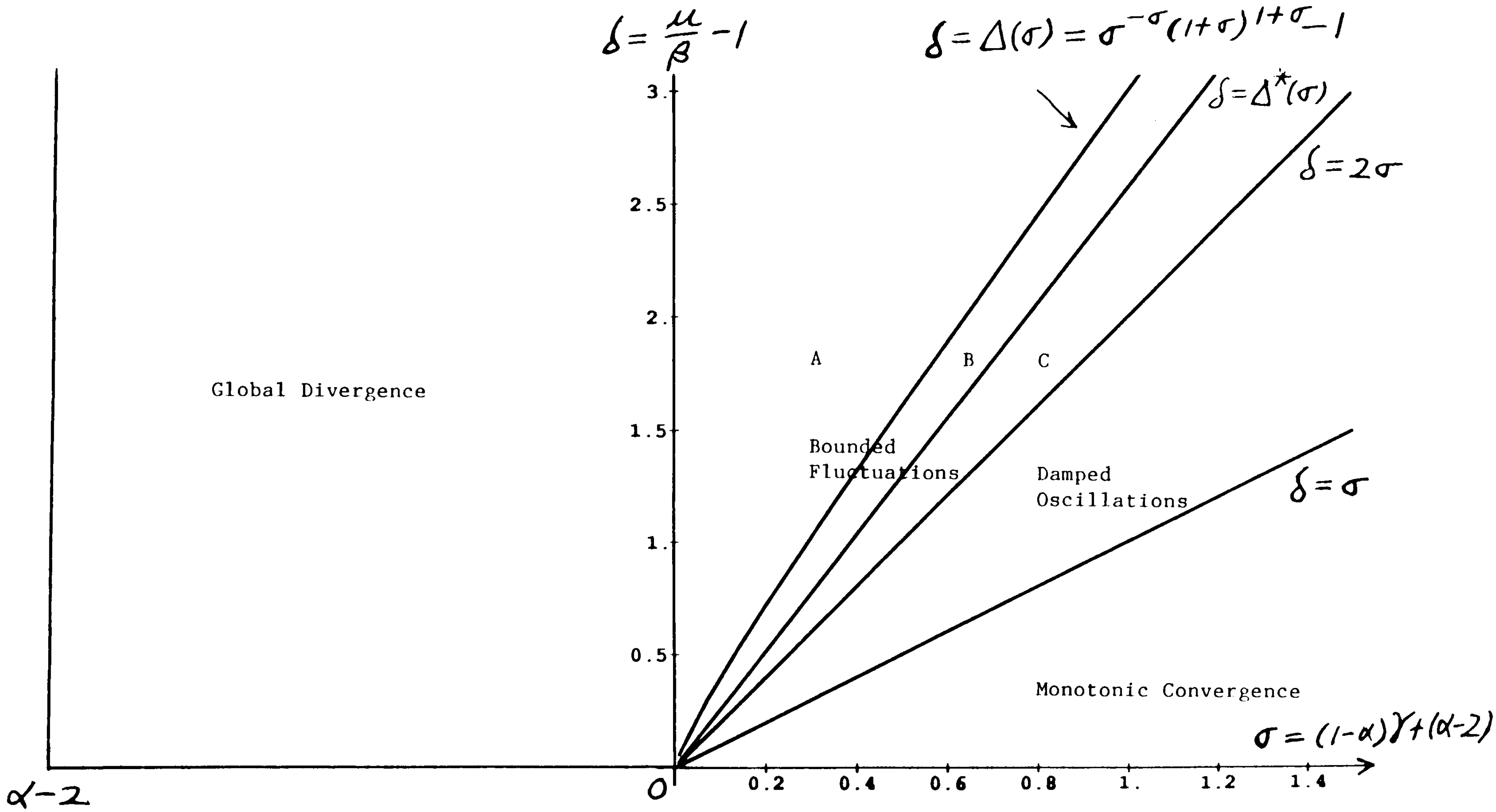


Figure 4:

A: Price dynamics are confined to a totally disconnected set.
B: For many (but not all) values of (δ, σ), there exists no weakly stable cycles, so that most initial prices lead to chaotic fluctuations.
C: There are cycles of period 2^n.