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CONTINUOUS REPRESENTATION OF PREFERENCES

by

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1. INTRODUCTION

There are problems of economic analysis in which the preferences of agents are allowed to vary in some class. Such problems have been studied in connection with the so-called limit theorems, with the continuity properties of the Walras correspondence and in other contexts. In such problems it is often convenient to be able to represent a class of preferences by a utility function which is jointly continuous in preferences and commodities. Hildenbrand [1 p. 175] has constructed such a utility function for the class of continuous preferences which are monotonic, where the consumption set is the non-negative orthant of the (Euclidean) commodity space. Subsequently, Neuefeind [3] constructed such a utility function for the class of continuous preferences whose indifference surfaces have Lebesgue measure zero, where the consumption set is connected and is the closure of its interior in ℓ-dimensional Euclidean space.

In this paper we construct a utility function for the class of continuous preferences which satisfy the assumption of local non-satiation at non-bliss points (Assumption III) on a consumption set which is an arcwise connected subset of R^{ℓ}. We show that given a preference relation in this class and given a utility indicator for it which belongs to a certain subset of functions representing the given preference relation, there is a utility function for the class of preferences which has that indicator as its value at the given preference relation.

Our construction relies on the joint continuity in preferences and commodities of the distance between the upper contour sets determined by preference relations and commodity points. This is established in Lemmas 1 and 2 of Section 2. Our construction affords some insight into the basic structure of the problem of continuous representation of a class of preferences. [See Section 3.]

Finally, we provide examples to show that the assumption of local non-satiation cannot be dispensed with in Lemmas 1 and 2 (Example 1.), and that the class of preferences satisfying Neuefeind's assumptions does not include those satisfying the assumption of local non-satiation [Example 2]. It is clear that preferences whose indifferences classes are of Lebesgue measure zero do satisfy local non-satiation at non-bliss points.

2. CONTINUITY OF DISTANCE BETWEEN CONTOUR SETS

To avoid inessential complication we hold the consumption set constant for all agents and denote it by $X \subset R^{\ell}$. An agent is then characterized by his preference relation, a complete preordering $\precsim$ of X, or by its graph,

$$\{(x,y) \in X \times X \mid x \precsim y\}.$$

Assumption I. X is an arcwise connected subset of R^{ℓ}.

Assumption II. The preference relation $\precsim$ is a continuous preordering on X.

Assumption III. If $x \in X$ is not a greatest element for $\precsim$ on X, then for every $\epsilon > 0$ there exists $y \in X$ such that $y \succ x$ and $|y - x| < \epsilon$.

We denote by $\mathcal{P}$ the class of agents satisfying assumptions I, II, and III. The elements of $\mathcal{P}$ are the graphs of preference relations. We give $\mathcal{P}$ the topology of closed convergence. [1, p. 165].

We shall define

$$\mathcal{S}: \mathcal{P} \times X \to 2^X \text{ by}$$

$$\mathcal{S}(p,x) = \{\bar{x} \in X \mid \bar{x} \underset{p}{\gtrsim} x\}.$$

We use $|\cdot|$ to denote the norm in R^{ℓ} and $d(A,B)$ to denote the distance between sets A and B for A and B subsets of R^m.

We use the same symbol $d(\cdot,\cdot)$ for all m, relying on the context to distinguish $d(A,B)$ from $d(p,q)$ when A and B are subsets of X and p and q subsets of $X \times X$, i.e. $m = \ell$ vs. $m = 2\ell$.

We now establish the joint continuity of the distance between contour sets.*/

Lemma 1: If $p \in \mathcal{P}$ then for each $x \in X$ and each $\epsilon > 0$ there exists $\delta > 0$ such that

$$(x - \bar{x}) < \delta \text{ implies } d(\mathcal{S}(p,x), \mathcal{S}(p,\bar{x})) < \epsilon$$

on every compact set $K \subset X$.

Proof: Let $p \in \mathcal{P}$ and let $\lesssim$ denote the preordering of X given by p. Since p is fixed throughout the argument we shall write $\mathcal{S}(x)$ for $\mathcal{S}(p,x)$. To show that for $\epsilon > 0$ there exists $J_\epsilon = J$ such that $d(\mathcal{S}(x_j), \mathcal{S}(x)) < \epsilon$ for all $j > J$, since either $\mathcal{S}(x_j) \subset \mathcal{S}(x)$ or $\mathcal{S}(x) \subset \mathcal{S}(x_j)$, it suffices to show that

1) for each $j > J$ there is a point $y_j \in \mathcal{S}(x_j)$ such that $|y_j - \mathcal{S}(x)| < \epsilon$

2) for each $j > J$ there is a point $z \in \mathcal{S}(x)$ such that $|\mathcal{S}(x_j) - z| < \epsilon$,

*/ We are indebted to Leonid Hurwicz for useful comments on an earlier version of this result.

where $|y - \mathcal{Y}(x)| \equiv \min_{z \in \mathcal{Y}(x)} |y - z|$.

I. e., $\mathcal{Y}(x_j)$ is within ε of $\mathcal{Y}(x)$ and $\mathcal{Y}(x)$ is within ε of $\mathcal{Y}(x_j)$ for $j > J$.

Given $\varepsilon > 0$ and $x_j \to x$ we partition the set $\{x_j\}$ into three subsets;

(1) those elements for which $x_j \sim x$; (2) those such that $x_j > x$;

(3) those such that $x_j < x$.

For the terms (1), $\mathcal{Y}(x_j) = \mathcal{Y}(x)$ and hence $d(\mathcal{Y}(x_j), \mathcal{Y}(x)) = 0 < \varepsilon$.

So we consider first $A = \{x_j \mid x_j > x\}$ and suppose there are infinitely many elements in A. (If A contains only finitely many elements we may ignore it.) Since $x_j > x$ implies $\mathcal{Y}(x_j) \subset \mathcal{Y}(x)$ and hence if $z \in \mathcal{Y}(x_j)$ then $|z - \mathcal{Y}(x)| < \varepsilon$ for all $\varepsilon > 0$, it remains to show that $\mathcal{Y}(x_j)$ is within ε of $\mathcal{Y}(x)$, i.e., it remains to show that if $y \in \mathcal{Y}(x)$, then

$$| y - \mathcal{Y}(x_j) | < \varepsilon \quad \text{for all} \quad j > J(\varepsilon)$$

Suppose not. Then there exists $\varepsilon_o > 0$ such that for each integer J, there exists $k > J$ such that

$| y - \mathcal{Y}(x_k) | > \varepsilon_o$. (To lighten notation we shall write $\{x_k\}$ for sequence so constructed.)

Now, $y \in \mathcal{Y}(x) \Rightarrow y \succsim x$.

If $y \gtrsim x_k$ for some k

then $y \in \mathscr{L}(x_k)$. Hence

$$| y - \mathscr{L}(x_k) | \leqq | y - y | = 0,$$

contradicting $| y - \mathscr{L}(x_k) | > \varepsilon_o > 0$.

Hence we may conclude

$$y \prec x_k \text{ for all } k.$$

By continuity of preference, there exists an open sphere $\mathcal{O}_{\varepsilon'}(y)$ of radius $\varepsilon' > 0$ about y such that

$$y' \in \mathcal{O}_{\varepsilon'}(y) \text{ implies } y' \prec x_k \text{ for all } k.$$

Let $\varepsilon = \min(\varepsilon', \varepsilon_o)$. It follows from local non-satiation of preference, that there exists $z \in \mathcal{O}_{\varepsilon}(y)$ such that

$$z \succ y.$$

Now, if

$z \gtrsim x_k$, for some k, it would follow that $z \in \mathscr{L}(x_k)$ for the same k.

But since

$$| y - z | < \varepsilon < \varepsilon_o,$$

and

$z \in \mathscr{L}(x_k)$, it follows that $| y - \mathscr{L}(x_k) | < \varepsilon < \varepsilon_o$,

contradicting

$$| y - \mathscr{L}(x_k) | > \varepsilon_o, \text{ for all } k.$$

Hence we may conclude that

$$z \prec x_k \text{ for all } k.$$

In summary, we have shown that

(*) $$x \lesssim y \prec z \prec x_k \text{ for all } k.$$

Let u be any continuous utility indicator of the preference relation $\precsim$. Since X is arcwise connected there is an arc, which we denote by $[x,x_k]$, connecting x and x_k and which is contained in X. It follows from continuity of preference, and the intermediate value theorem applied to u, that for each k there exists $v_k \in [x,x_k]$ such that

$$u(v_k) = u(z), \text{ (i.e., } v_k \sim z).$$

Hence

$$u(v_k) = u(z) < u(x_k) \quad \text{for all } k .$$

Now because $x_k \to x$ (since $x_j \to x$) and u is continuous, it follows that

$u(x_k) \to u(x)$. Furthermore, because $u(z) < u(x_k)$,

it follows that

(**) $$u(z) \leqq \lim u(x_k) = u(x).$$

Then, combining inequalities (*) and (**),

$$u(x) < u(z) \leqq u(x)$$

which is a contradiction.

The conclusion follows for the case $x_k > x$. Notice that no reference to compact sets was made in the argument so far.

Consider now the set $B = \{x_j \mid x_j \prec x\}$. Since $x_j \prec x$ implies $\mathcal{B}(x_j) \supset \mathcal{B}(x)$, and hence that $\mathcal{B}(x_j)$ is within ε of $\mathcal{B}(x)$, it remains to show that $\mathcal{B}(x) \cap K$ is within ε of $\mathcal{B}(x_j) \cap K$ for all j sufficiently large, and every compact set K. Suppose not. Then there exists a compact set K and $\varepsilon_o > 0$ such that for every J there is $k > J$ and $y_k \in \mathcal{B}(x_k)$ such that

$$|y_k - \mathcal{B}(x) \cap K| > \varepsilon_o.$$

I.e., there is an infinite set C of points $y_k \in \mathscr{P}(x_k) \cap K$ such that $|y_k - \mathscr{P}(x) \cap K| > \epsilon_o$ for all k.

$$C \equiv \{y_k \mid y_k \in \mathscr{P}(x_k) \cap K \text{ and } |y_k - \mathscr{P}(x)| > \epsilon_o\}$$

Since $C \subset K$, and K is compact, there is at least one point, y, in the closure of C such that for every $\delta > 0$ $\mathscr{I}_\delta(y) = \{x \in X \mid |x-y| < \delta\}$ contains infinitely many points of C.

If $y \in \mathscr{P}(x)$ then for $\delta < \frac{\epsilon_o}{2}$ there exist values of k such that

$$|y_k - y| < \delta < \frac{\epsilon_o}{2} < \epsilon_o,$$

which contradicts $|y_k - \mathscr{P}(x)| > \epsilon_o$ for all k.

Hence $y \notin \mathscr{P}(x)$. Since $\mathscr{P}(x)$ is closed, it follows that $y \prec x$, i.e.,

$u(y) < u(x)$.

Moreover, $|y - \mathscr{P}(x)| > 0$, and further

there exists a point $z' \in \mathscr{P}(x)$ such that

$$|y - \mathscr{P}(x)| = |y - z'| > 0$$

It follows that

$$u(y) < u(x) \leq u(z').$$

Let $[y,z']$ denote an arc in X connecting y and z'. By continuity of preference there is a point $z \in [y,z']$ such that

$$u(z) = u(x).$$

Hence $z \in \mathscr{L}(x)$. Thus,

$$u(y) > u(z) = u(x)$$

Let $a = u(\bar{z}) - u(y) > 0$

Again, by continuity of u, there is a point $v \in [y,z]$ such that

$$u(v) = u(y) + \frac{a}{2}$$

and hence there is a sphere $\mathcal{O}_\rho(y)$ with center y and radius $\rho > 0$ such that

$$u(\bar{y}) < u(v) \quad \text{for all } \bar{y} \in \mathcal{O}_\rho(y).$$

Now, since x_j converges to x the subsequence x_k

also converges to x.

By continuity of u, for $\varepsilon' = \frac{a}{2} > 0$, there exists $J(\varepsilon')$

such that $j > J(\varepsilon')$ implies

$$| u(x_j) - u(x) | < \frac{a}{2}$$

Since $w \in \mathscr{L}(x_j)$ implies $u(w) \geqq u(x_j)$

it follows that for $j > J(\varepsilon')$

$$u(w) \geqq u(x_j) > u(x) - \frac{a}{2}$$

and, since

$$w \in \mathcal{O}_\rho(y) \quad \text{implies}$$

$$u(w) < u(x) - \frac{a}{2},$$

it follows that

$$\mathscr{L}(x_j) \cap \mathcal{O}_\rho(y) = \phi$$

for all $j > J(\varepsilon')$. Hence, $\mathcal{O}_\rho(y)$ contains at most finitely many points of C. This contradicts the statement that every sphere about y contains infinitely many points of C.

Hence the assumption that $\mathcal{S}(x)$ is not within ε of $\mathcal{S}(x_j)$ for all j sufficiently large is false.

This establishes the conclusion that if $x_j \to x$

$d(\mathcal{S}(x_j), \mathcal{S}(x)) \to 0$.

Lemma 2: For every $\varepsilon > 0$ there exists $\delta > 0$ such that if p and p' belong to $\mathcal{P}$ then $d(p,p') < \delta$ and $|x - x'| < \delta$ imply $d(\mathcal{S}(p,x), \mathcal{S}(p',x')) < \varepsilon$.

Proof: Given $\varepsilon > 0$ let p and p' belong to $\mathcal{P}$ and be such that $d(p,p') < \frac{\varepsilon}{2}$. We choose $\delta > 0$ so that if $|x - x'| < \delta$, then $d(\mathcal{S}(p,x), \mathcal{S}(p,x')) < \frac{\varepsilon}{2}$. Now assume that $y \in \mathcal{S}(p,x)$. Because $|x - x'| < \delta$, there is a $y' \in \mathcal{S}(p,x')$ such that $|y - y'| < \frac{\varepsilon}{2}$. But $d(p,p') < \frac{\varepsilon}{2}$, hence the set distance $d(\mathcal{S}(p,x'), \mathcal{S}(p',x')) < \frac{\varepsilon}{2}$. Thus there exists a $y'' \in \mathcal{S}(p',x')$ such that $|y' - y''| < \frac{\varepsilon}{2}$. Hence $|y - y''| < \varepsilon$, i.e. $\mathcal{S}(p,x)$ is within an ε-neighborhood of $\mathcal{S}(p',x')$. Conversely, assume that $z \in \mathcal{S}(p',x')$. Then $d(\mathcal{S}(p',x'),\mathcal{S}(p,x')) < \frac{\varepsilon}{2}$. Thus there is a $z' \in \mathcal{S}(p,x')$ such that $|z - z'| < \frac{\varepsilon}{2}$. But $|x - x'| < \delta$, hence $d(\mathcal{S}(p,x), \mathcal{S}(p,x')) < \frac{\varepsilon}{2}$. Thus there is a $z'' \in \mathcal{S}(p,x)$ such that $|z' - z''| < \frac{\varepsilon}{2}$. Thus $|z - z''| < \varepsilon$, i.e. z is within an ε-neighborhood of $\mathcal{S}(p,x)$.

3. CONSTRUCTION OF UTILITIES

In this section we construct an indicator of preference on the class $\mathcal{P}$ which is jointly continuous in preferences and commodities. I.e., we construct a function

$$u: \mathcal{P} \times X \to R$$

which is continuous on $\mathcal{P} \times X$ and such that for each $p \in \mathcal{P}$, $u(p,\cdot)$ is an indicator of the preference relation p.

Let $p \in \mathcal{P}$; we shall also write $\succsim$ for the preference relation on X, corresponding to p.

Choose $x_o \in X$ arbitrarily, and define $u(x_o) = 1$. */

For $x \in X$ either $x \succsim x_o$ or $x \prec x_o$. If $x \sim x_o$, let $u(x) = u(x_o) = 1$.

If $x \succ x_o$ let $u(x) = u(x_o) + d(\mathcal{S}(x_o), \mathcal{S}(x))$, where $\mathcal{S}(x)$ is, as in Section 2, the upper contour set on x determined by the preference relation $\precsim$.

If $x \prec x_o$ let $u(x) = u(x_o) - d(\mathcal{S}(x_o), \mathcal{S}(x))$.

It follows from Lemma 1 that $d(\cdot,\cdot)$ is continuous on X for any $p \in \mathcal{P}$ and hence that $u(p,\cdot)$ is continuous on X. It follows from Lemma 2 that u is jointly continuous on $\mathcal{P} \times X$.

It is clear from the construction that u is increasing with respect to $\precsim$ since $x \precsim y$ implies $\mathcal{S}(y) \subset \mathcal{S}(x)$.

We may summarize this construction in the following Lemma.

<u>Lemma 3.</u> If $\mathcal{P}$ is a class of agents satisfying Assumptions I, II, and III, then there exists an indicator of preferences $u: \mathcal{P} \times X \to R$ continuous on $\mathcal{P} \times X$.

*/ If the consumption set X was permitted to vary we would choose x_o in the intersection (assumed non-empty) of all the consumption sets.

For the fixed consumption set X let R^X denote the class of continuous real valued functions on X.

Define the mapping

$$\varphi : R^X \to X \times X$$

by

$$\varphi(q) = p = \{(x,y) \in X \times X \mid q(x) \leq q(y)\}$$

Thus, φ associates to each function q the continuous preference relation $\varphi(q)$ given by the contour sets of q.

Let Q be the inverse image of $\mathcal{P}$ under φ, i.e., $Q \subset R^X$ is the set of utility functions whose contours give preference relations in $\mathcal{P}$.

We note first that φ is continuous on Q in any topology such that convergence of functions $q_j \to q$ implies convergence of preferences in $\mathcal{P}$ and also of the contour sets.

The topology of uniform convergence is sufficient for this, in the presence of local non-satiation. To see this it suffices to show that for any $\varepsilon > 0$ there exists an integer J such that for any $x \in X$, $j > J$ implies

$$d(\mathcal{B}(u_j,x), \mathcal{B}(u,x)) < \varepsilon .$$

Since $\mathcal{B}(u_j,x) = \{y \in X \mid u_j(y) \geq u_j(x)\}$

and

$$\mathcal{B}(u,x) = \{y \in X \mid u(y) \geqq u(x)\}$$

it suffices to show that if $y \in \mathcal{B}(u_j,x)$ for all $j \in J$ then there exists $y' \in \mathcal{B}(u,x)$ such that $|y' - y| < \varepsilon$ and if $y \in \mathcal{B}(u,x)$ there exists $y' \in \mathcal{B}(u_j,x)$ for all $j > J$ such that $|y' - y| < \varepsilon$.

Let $y \in \mathcal{B}(u_j,x)$ for all $j > J(\varepsilon)$

Then $u_j(y) \geqq u_j(x)$, for all $j > J(\varepsilon)$.

We consider first the case $u_j(y) > u_j(x)$, and taking $y' = y$, compute $u(y) - u(x)$.

$$u(y) - u(x) = u(y) - u_j(y) + u_j(y) - u_j(x) + u_j(x) - u(x)$$

$$\geqq u_j(y) - u_j(x) + 2\delta > 0$$

where $j > J(\delta)$ implies $|u_j(z) - u(z)| < \delta$ for all $z \in X$. Since this argument is symetric in u_j and u, it can be applied to show that

$$u(y) - u(x) > 0 \quad \text{implies} \quad u_j(y) - u_j(x) > 0$$

for j sufficiently large.

Now consider the case $u_j(y) = u_j(x)$. If $u(y) < u(x)$, then the argument above implies $u_j(y) < u_j(x)$ contrary to hypothesis. It follows that $u_j(y) \geqq u_j(x)$ implies $u(y) \geqq u(x)$. Similarly,

$$u(y) \geqq u(x) \quad \text{implies} \quad u_j(y) \geqq u_j(x)$$

for all j sufficiently large.

The correspondence $\varphi^{-1}: \mathcal{P} \rightarrow Q$ assigns to each (continuous) preference relation in $\mathcal{P}$ the set of (continuous) representations of it.

Let Q/φ denote the quotient space of Q under φ, (i.e., an equivalence class in Q consists of all functions representing the same preference relation) and let Q/φ have the quotient topology.

The situation may be summarized in the following diagram.

$$\begin{tikzcd} Q \arrow[d, dashed, "\pi"'] \arrow[dr, "\varphi"] & \\ \frac{Q}{\varphi} \arrow[r, "h"] & \mathcal{P} \end{tikzcd}$$

Figure 3.1

Here π is the projection of Q on $\frac{Q}{\varphi}$ and $h \equiv \varphi \cdot \pi^{-1}$. We note first that h is 1 - 1, since elements of Q are equivalent if and only if they have the same φ-image. Further, h is onto because φ is onto, (i.e. every preference relation in $\mathcal{P}$ is continuously representable.)

The function h is continuous, since for $\theta \subset \mathcal{P}$, θ open, $\varphi^{-1}(\theta)$ is open by continuity of φ, and π is open by definition of the quotient topology on $\frac{Q}{\varphi}$.

This argument is symetric in π and φ. Hence it follows that h^{-1} is continuous if and only if φ is open. It then follows that φ open is equivalent to $\frac{Q}{\varphi}$ and $\mathcal{P}$ homeomorphic.

Let

$$\overline{Q} = \{q \in Q \mid q = F \circ u(p,\cdot) \text{ for some } p \in \mathcal{P}\}$$

where F is a uniformly continuous strictly increasing function from R to R, and $u(p,\cdot)$ denotes the standard representation of p according to Lemma 3.

<u>Theorem</u> If $q_o \in \overline{Q}$, there is a continuous selection ξ_o from φ^{-1} which has q_o in its range.

<u>Proof</u>: Let $q_o \in \overline{Q}$ and $p_o = \varphi(q_o)$. $p_o \in \mathcal{P}$ since $q_o \in Q$. Since $q_o \in \overline{Q}$ there exists a uniformly continuous strictly increasing function $F_o : R \to R$ such that

$$q_o = F_o \circ u(p_o, \cdot)$$

Let $\xi_o(p) = F_o \circ u(p,\cdot)$ for $p \in \mathcal{P}$. Then ξ_o maps $\mathcal{P}$ onto $\overline{Q}$, and takes the value q_o at p_o. It is well-known that if $u(p,\cdot)$ is a utility indicator for p then so is $F_o \circ u(p,\cdot)$ and hence ξ_o is a selection from φ^{-1}.

It remains to show that ξ_o is continuous on $\mathcal{P}$. To see this, let $p \in \mathcal{P}$, let $p_j \to p$. Then it follows from Lemma 2 that

$$u(p_j,\cdot) \to u(p,\cdot).$$

I.e., for $\delta > 0$ there exists $J(\delta)$ such that

$$|u(p_j,x) - u(p,x)| < \delta \quad \text{for all} \quad j > J(\delta)$$

and all $x \in X$. Since F_o is uniformly continuous, for every $\varepsilon > o$ there exists $\delta(\varepsilon)$, such that for r and s in R,

$$|r - s| < \delta(\varepsilon) \quad \text{implies} \quad |F(r) - F(s)| < \varepsilon.$$

For $\varepsilon > o$, take $\delta = \delta(\varepsilon)$. Then $j > J(\delta(\varepsilon))$ implies

$$|F_o(u(p_j,x)) - F_o(u(p,x))| < \varepsilon, \text{ for all } x \in X.$$

Thus, $\xi_o(p_j) \to \xi_o(p)$.

Thus, if q is a utility function obtained by a uniformly continuous strictly monotone transformation from the standard representation of the preference relation p_o, then there is a utility indicator continuous in preferences which hits q_o.

It follows from a result of Michael [2 Proposition 2.2, p. 362] that the correspondence $\bar{\varphi}^{-1}$ given by

$$\bar{\varphi}^{-1}(p) = \varphi^{-1}(p) \cap \bar{Q},$$

is lower semi-continuous. According to another remark of Michael's [2, Example 1.1* p. 362] $\bar{\varphi}^{-1}$ is lower semi-continuous if and only if the restriction of φ to $\bar{Q}$ is open. Denote by $\bar{\varphi}$ the restriction of φ to $\bar{Q}$. Then, supplementing Figure 3.1 we have Figure 3.2.

$$\begin{tikzcd}
\bar{Q} \arrow[d, "\pi"'] \arrow[dr, "\bar{\varphi}"] & \\
\dfrac{\bar{Q}}{\bar{\varphi}} \arrow[r, shift left, "h"] & \mathcal{P} \arrow[l, shift left, "h^{-1}"]
\end{tikzcd}$$

Figure 3.2

Now, it follows from the fact that $\bar{\varphi}$ is open, that $\dfrac{\bar{Q}}{\bar{\varphi}}$ and $\mathcal{P}$ are homeomorphic since the argument used above to establish the continuity of h is symmetric in π and $\bar{\varphi}$.

4. EXAMPLES

We show first that Assumption III,(local non-satiation) cannot be dispensed with for continuity of the contour set correspondence $\mathscr{L}$.

Example 1:

Let $X = R^2_+$, (the non-negative quadrant of R^2),and let $x = (0,0)$ $y = (1,1)$

$$I(x) = \{z \in X \mid z \sim x\} = \{(z_1 z_2) \in R^2_+ \mid z_1 = z_2\} \cup \{(z_1 z_2) \in R^2_+ \mid z_2 = 2z_1\}$$

$$\cup \{(z_1 z_2) \in R^2_+ \mid z_2 = \tfrac{1}{2} z_1\}.$$

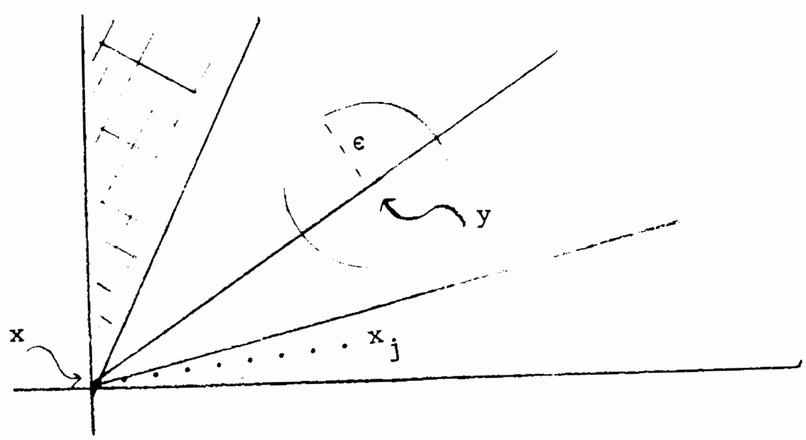
Let

$$\mathscr{L}(x) = \{(z_1 z_2) \in R^2_+ \mid z_2 \geq 2\, z_1\} \cup \{(z_1 z_2) \in R^2_+ \mid z_2 < \tfrac{1}{2} z_1\}$$

$$\cup\; I(x)$$

[and, of course, $z < x$ for all $z \notin \mathscr{L}(x)$.]

Let $x_j \to x$, such that $x_j > x$ for all j.



Then, for $\varepsilon = \frac{1}{4}$, there is no point of $\mathscr{L}(x_j)$ closer to y than ε for any j.

We show next that the class $\mathcal{P}$ of preferences satisfying Assumptions I, II, and III is not included in the class of preferences having indifferent surfaces of Lebesgue measure zero. We take the consumption set to be R_+ and hence to satisfy Neuefeind's Assumption I, i.e. it is connected and the closure of its interior in R.

Example 2:

Preferences satisfying the condition that indifference sets have Lebesgue measure 0 do not necessarily satisfy local non-satiation at non-bliss points.

Let $X = R_+$ and let the preference relation be represented by the utility function

$$u(x) = x \sin x \qquad x \geq 0 .$$

Then,

1) there is no absolute maximum of u on X;

2) indifference sets have Lebesgue measure zero, since they are of the form

$$\{x \in R^+ \mid x \sin x = c\},$$

which consists of isolated points;

3) the local maxima of u are points at which local non-satiation fails.