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# EFFICIENT SEQUENTIAL BARGAINING

by

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## Abstract

This paper establishes that efficient outcomes of static bargaining with two-sided incomplete information can be achieved in infinite-horizon, offer-counteroffer games. The entire ex ante Pareto frontier can often be reached by equilibria of the standard games in which a single player successively makes offers. Our equilibria are sequential and stationary, but do not utilize delay as a screening device; they have the property that most information revelation and most trade occurs in the initial two periods.

## 1. Introduction

In a remarkable paper, Myerson and Satterthwaite (1983) analyzed static bargaining with two-sided incomplete information, in direct mechanism terms. Two players simultaneously report their valuations to a mediator, who carries out trade with prescribed probabilities and expected transfers in a single trading round. Myerson and Satterthwaite characterized ex ante efficient bargaining for fairly general seller and buyer distribution functions. Moreover, they demonstrated that, when the supports overlap, ex ante efficiency requires ex post inefficiency: in particular, there exist pairs (s,b) of valuations such that positive gains from trade exist (i.e., $b > s$), yet the players trade with probability zero.

The direct mechanism approach can be criticized on account that, in real life, traders do not bargain in this way. First, bargaining better fits a dynamic rather than a static description. For example, after any given round of negotiations, players usually "are unable to commit to walking away from the bargaining table" (Cramton, 1984, pp. 579-580). Second, bargaining typically progresses through successive, rather than simultaneous, moves; actual mediatorless bargaining seldom involves simultaneous bids, and the role of real mediators is quite different from that posited by mechanism design. Thus, much of the bargaining literature has examined extensive form games which are dynamic and infinite-horizon, and in which offers and counteroffers are made successively (Rubinstein, 1982). See Wilson (1987), Rubinstein (1987), and our previous papers for reviews of this literature.

The belief in the bargaining literature has been that sequentiality comes at the expense of efficiency. Researchers have argued that, if the duration of bargaining is not artificially cut off, then delay is needed to credibly signal valuation. Since trading with delay corresponds (in mechanism terms) to a probability strictly between zero and one, this would imply an efficiency loss relative to the Myerson-Satterthwaite optimum. Additionally, the fact that simultaneous moves are not permitted would seem to preclude efficient information revelation, further contributing to waste.

The results of this article thus stand in stark contrast to most previous contributions to the sequential bargaining literature. We analyze the two simplest infinite-horizon, offer-counteroffer extensive forms--the seller-offer and buyer-offer games--for fairly general distributions of valuations. Surprisingly, despite the apparent restrictiveness of these games (especially as regards information revelation), we demonstrate that they admit extremely efficient outcomes. We construct sequential equilibria where essentially all usable information is revealed in the initial two trading rounds, and with the limiting property that trade occurs instantaneously or not at all.[1] In fact, we demonstrate that a portion of the Myerson-Satterthwaite efficient frontier is implementable in the seller-offer game and another portion is implementable in the buyer-offer game; together, often, the entire Pareto frontier is obtained.

(Insert Figure 1 about here)

Consider the well-known example where traders' valuations are both uniformly distributed on the unit interval. Our main theorem, when particularized to this special case, establishes that all ex ante efficient mechanisms which favor the seller (i.e., with seller weight $\lambda \in [1/2, 1]$)

are implementable in the seller-offer game. Symmetrically, all efficient mechanisms favoring the buyer (i.e., $\lambda \in [0, 1/2]$) are implementable in the buyer-offer game (see Figure 1). Thus, the trading rule that maximizes ex ante expected gains from trade (i.e., $\lambda = 1/2$), which is known to be an equilibrium of the sealed-bid double auction (Chatterjee and Samuelson, 1983), does not require the availability of simultaneous moves. Even more surprisingly, its implementation does not even depend on the ability of both players to make bids.

Successive exchanges of offers are probably the oldest and still most commonly used bargaining institutions. As economists, we should ask whether "these institutions survive because they employ trading rules that are efficient for a wide class of environments" (Wilson, 1987, p. 37). Our answer is affirmative. For fairly general distributions, offer-counteroffer games may collectively be viewed as unbiased institutions: changing the relative weights attached to traders, or changing the distributions of their valuations, does not require an alteration of the rules of the games. Rather, traders can simply play different equilibria of the same trading institutions and continue to realize efficiency. In fact, it is frequently the case that the two simplest offer-counteroffer games (where a single party has the exclusive ability to make offers) are sufficient to "span" the Pareto frontier.

It is illuminating to contrast the robustness of the seller- and buyer-offer games with the nonrobustness of the sealed-bid double auction. When the buyer's bid $p_b$ exceeds the seller's bid $p_s$, let trade occur at $kp_b + (1 - k)p_s$. Fixing $k = 1/2$, this double auction implements efficiency under equal weighting and uniform distributions. However, efficiency for

$\lambda \neq 1/2$ requires a change in k, and efficiency for other distribution functions generally requires other modifications in the trading institution.

Two other points should be emphasized. First, while efficient sequential bargaining is possible, tremendously inefficient sequential equilibria also exist.[2] Second, despite the multiplicity of equilibria, this is not a realm for folk theorems. There exist many static mechanisms which are not implementable in various offer-counteroffer games.[3]

The article is structured as follows. Section 2 defines the sequential game and static mechanism concepts. Sections 3 and 4 establish results on the splittability of mechanisms. Sections 5 and 6 develop results on two-price mechanisms. Section 7 states the implementation theorem for efficient sequential bargaining, with proof provided in the Appendix. We conclude in Section 8.

## 2. The Model

Two parties, a seller and a buyer, are negotiating over the price of a single indivisible object worth s to the seller and b to the buyer. At the time the bargaining commences, each trader is aware of his own valuation, but treats his opponent's as a random variable. These random variables are distributed independently on the common interval [0,1], according to the (commonly-known) distribution functions $F_1(s)$ and $F_2(b)$. Each distribution function $F_i(\cdot)$ possesses a density $f_i(\cdot)$ which is continuous and positive on the interior of the support. Traders are interested in maximizing their expected monetary gain.

A bargaining mechanism is a game in which both parties simultaneously report their reservation prices to a mediator, who then determines whether the good is transferred, and how much the buyer is to pay the seller. A bargaining mechanism is completely characterized by the two outcome functions, $p(\bullet,\bullet)$ and $x(\bullet,\bullet)$, where $p(s,b)$ denotes the probability of transfer given the reports of s and b, and $x(s,b)$ denotes the expected payment. With every bargaining mechanism $\{p,x\}$, we associate:

$$\text{(2.1)}\qquad \begin{aligned} \bar{p}_1(s) &= \int_0^1 p(s,v_2)f_2(v_2)dv_2 \qquad & \bar{p}_2(b) &= \int_0^1 p(v_1,b)f_1(v_1)dv_1 \\ \bar{x}_1(s) &= \int_0^1 x(s,v_2)f_2(v_2)dv_2 \qquad & \bar{x}_2(b) &= \int_0^1 p(v_1,b)f_1(v_1)dv_1, \end{aligned}$$

where $\bar{p}_1(s)$ is the probability of agreement and $\bar{x}_1(s)$ is the expected revenue to the seller of type s, and $\bar{p}_2(b)$ is the probability of agreement and $\bar{x}_2(b)$ is the expected payment for the buyer of type b. Thus, the seller's and buyer's (interim) expected payoffs are given by:

$$\text{(2.2)}\qquad U_1(s) = \bar{x}_1(s) - s\bar{p}_1(s) \quad\text{and}\quad U_2(b) = b\bar{p}_2(b) - \bar{x}_2(b).$$

A bargaining mechanism is <u>incentive compatible</u> if all player types have an incentive to report truthfully, i.e.,

$$\text{(2.3)}\qquad U_1(s) \geq \bar{x}_1(s') - s\bar{p}_1(s'), \quad U_2(b) \geq b\bar{p}_2(b') - \bar{x}_2(b'),$$

for all s, s', b and b' in [0,1]. A mechanism $\{p,x\}$ is <u>individually rational</u> if all player types want to participate voluntarily, i.e., $U_1(s) \geq 0$ and $U_2(b) \geq 0$, for every $s,b \in [0,1]$. Mechanisms satisfying both the individual rationality and the incentive constraint will be referred to as <u>incentive compatible bargaining mechanisms</u> (ICBM's).

The usual description of bargaining, rather than being static and direct in nature, involves players making repeated offers and counteroffers through time until an agreement is concluded, or an impasse is reached. By an <u>offer-counteroffer game</u>, we will mean any extensive form game (finite or infinite) in which at discrete moments in time one (and only one[4]) of the players is given an opportunity to make an offer, which the other player can then either accept or reject. Acceptances always conclude the bargaining; rejections may or may not do so. We also assume that the time between successive bargaining rounds is bounded away from zero, and that players discount the future with a common interest rate, r. Hence, if the good is traded at time t for the price $\pi$, the seller obtains a surplus of $e^{-rt}(\pi - s)$, and the buyer a surplus of $e^{-rt}(b - \pi)$. Note that implicit in the definition of offer-counteroffer game is the assumption that money only changes hands if the good changes hands. In particular, the rules of the game do not permit players to pay each other history-contingent transfers unrelated to the accepted offer. One example of an offer-counteroffer game is the seller-offer game, in which the seller gets to make all the offers at discrete moments in time, spaced z apart.

The revelation principle implies that to every Nash equilibrium of an offer-counteroffer extensive form there corresponds a <u>sequential bargaining mechanism</u>.[5] Such a mechanism specifies a pair of outcome functions $t(\cdot,\cdot)$ and $x(\cdot,\cdot)$, where $t(s,b)$ denotes the time at which the good will be transferred and $x(s,b)$ the discounted expected payment from the buyer to the seller, given respective reports of s and b. Letting $p(s,b) = e^{-rt(s,b)}$, we see that every Nash equilibrium maps into an ICBM $\{p,x\}$. Similarly, every sequential equilibrium of an offer-counteroffer game induces an ICBM after any history of the game.

It is interesting to know whether a given mechanism $\{p,x\}$ can be supported by sequential equilibria of some offer-counteroffer game (see also Ausubel and Deneckere, 1988a,b). If we restrict ourselves to equilibria that involve no randomization along the equilibrium path, then one necessary condition for this is that it be possible to repeatedly "split" the mechanism into incentive compatible "submechanisms," thereby tracing out the equilibrium histories of the game. We now turn to an investigation of this issue.

## 3. Splittability and the B Functions

Consider any ICBM $\{p,x\}$, with the seller and buyer distributed according to the distributions $H_1(s)$ and $H_2(b)$. Let $S = \text{supp } H_1 \subset [0,1]$ and $B = \text{supp } H_2 \subset [0,1]$. Let $\{S^\alpha\}_{\alpha \in A}$ be a partition of S, and let $H_1^\alpha(s)$ be the conditional probability distribution of s given that $s \in S^\alpha$ (and the initial distribution is $H_1$).

Definition 3.1: A (seller) split of an ICBM $\{p,x\}$ with initial distributions $H_1$ and $H_2$ is a (nondegenerate) partition $\{S^\alpha\}_{\alpha \in A}$ of $S = \text{supp } H_1$ such that the submechanisms $\{p^\alpha, x^\alpha\}$, each defined by restricting $p(\bullet,\bullet)$ and $x(\bullet,\bullet)$ to the domain $S^\alpha \times B$, are ICBM's relative to the distributions $(H_1^\alpha, H_2)$.

This definition guarantees that incentive compatibility and individual rationality are maintained after the split. A mechanism for which there exists a seller split will be called <u>(seller)-splittable</u>. Obviously, there is an entirely analogous definition for buyer splits.

Consider now any sequential equilibrium of any offer-counteroffer game. First, observe that if only pure strategies are used along the equilibrium path, then the information revelation pattern along the equilibrium path takes the form of a successive refinement of the initial information partition. Second, let us denote by $H_1(s)$ and $H_2(b)$ players' posteriors after some (arbitrary) history. By sequential rationality, the equilibrium strategies induce a sequential equilibrium on the remainder of the game, and hence an ICBM $\{p,x\}$ relative to the $H_1$ and $H_2$. (For accounting reasons, p and x are discounted relative to time zero.) If the player who has the next move utilizes pure strategies, and if his action refines the current information partition, then $\{p,x\}$ is splittable.

In what follows, we only consider <u>binary splits</u>, where the index set A consists of exactly two elements. There is no loss of generality in doing so, since we can always add superfluous moves to the game (occurring in virtual time), where the player who induces a split first announces which of the two sets of actions he will subsequently pick from. For example, if the seller is about to induce the split $\{S^{\alpha}\}_{\alpha\in A}$ by taking the action $\{m^{\alpha}\}_{\alpha\in A}$, he can be convinced to first make one of the two announcements: "I am about to take an action in $M_1$" (or $M_2$), where $M_i = \{m^{\alpha}: \alpha \in A_i\}$ and $\{A_1,A_2\}$ is a binary partition of A.

One interesting example of a split is a <u>convex split</u>, where each element of the partition is convex. For example, in offer-counteroffer games, the accept/reject decision induces a convex split. In addition, all equilibria with which we are familiar in the literature (on infinite-horizon bargaining with incomplete information and a continuum of types) involve convex splits along the equilibrium path. It is therefore important to know under what conditions a mechanism is convexly splittable. As argued above, it will be sufficient to investigate the possibility of binary convex splits. It should also be observed that, although our theorem below is stated in terms of the initial mechanism {p,x} (relative to the distributions $F_1$ and $F_2$), our distributional assumptions permit a repeated application of the theorem (relative to the corresponding conditional distributions). Let us introduce some notation:

$$G_1(s) = \int_0^s \int_0^1 \{[v_2 - (1 - F_2(v_2))/f_2(v_2)] - [v_1 + F_1(v_1)/f_1(v_1)]\}p(v_1,v_2)f_2(v_2)f_1(v_1)dv_2dv_1$$

$$G_2(b) = \int_b^1 \int_0^1 \{[v_2 - (1 - F_2(v_2))/f_2(v_2)] - [v_1 + F_1(v_1)/f_1(v_1)]\}p(v_1,v_2)f_1(v_1)f_2(v_2)dv_1dv_2$$

$$B_1(s) = F_1(s)U_1(s) - G_1(s)$$

$$B_2(b) = [1 - F_2(b)]U_2(b) - G_2(b).$$

We will say that the mechanism {p,x} is <u>convexly splittable at s</u> if it is seller-splittable and if it induces the binary partition {[0,s),[s,1]}. Similarly, {p,x} is said to be <u>convexly splittable at b</u> if it is buyer-splittable and if it induces the binary partition {[0,b),[b,1]}.

<u>Theorem 3.2</u>: If the ICBM {p,x} is convexly splittable at s, then $B_1(s) = -F_1(s)U_2^I(0)$, where $U_2^I(0)$ refers to the utility of the lowest buyer type in the submechanism corresponding to [0,s). If it is convexly splittable at b then $B_2(b) = -[1 - F_2(b)]U_1^I(1)$, where $U_1^I(1)$ refers to the

utility of the highest seller type in the submechanism corresponding to $[b,1]$.

Proof: Let $I = [0,s)$ and $II = [s,1]$, so that $F_1^I(v_1) = F_1(v_1)/F_1(s)$ and $F_1^{II}(v_1) = [F_1(v_1) - F_1(s)]/[1 - F_1(s)]$ are the conditional probability distributions. Also, let $U_1^I(v_1)$ and $U_2^I(v_2)$ refer to the seller's and the buyer's utility in the mechanisms $\{p^I, x^I\}$ (relative to the distributions $F_1^I$ and $F_2$). By incentive compatibility of $\{p^I, x^I\}$ and Myerson and Satterthwaite (1983, Theorem 1):

$$(3.1) \qquad U_1^I(s) + U_2^I(0) = \int_0^s \int_0^1 \{[v_2 - (1 - F_2(v_2))/f_2(v_2)] - [v_1 + F_1^I(v_1)/f_1^I(v_1)]\}p^I(v_1,v_2)f_2(v_2)f_1^I(v_1)dv_2dv_1.$$

Now $U_1^I(s) = U_1(s)$ by Definition 3.1. Also, $p^I(v_1,v_2) = p(v_1,v_2)$ on $I \times [0,1]$, and hence (3.1) may be rewritten as:

$$U_1(s) + U_2^I(0) = G_1(s)/F_1(s),$$

from which the desired result follows. The derivation for a buyer split proceeds entirely analogously. []

We can add some more content to Theorem 3.2 by assuming that the split is induced by a move in an offer-counteroffer game. First, we need to prove the following result:

Theorem 3.3: Let $\{p,x\}$ be an ICBM induced by a sequential equilibrium of an offer-counteroffer game (in which the discount factor between periods is bounded away from 1). Then $U_2(0) = U_1(1) = G_2(0) = G_1(1) = B_2(0) = B_1(1) = 0$.

Proof: We will argue that $U_2(0) = 0$. The argument for $U_1(1)$ proceeds in an entirely analogous fashion.

First, since acceptances are ex post IR, (strictly) negative buyer offers are rejected by the seller with probability one. It will follow that $U_2(0) = 0$ if we can show that the seller never offers a (strictly) negative price. This will certainly be the case if negative prices have probability one of acceptance. Let Q be the infimum of all seller offers which are rejected with positive probability (the infimum taken over all seller types, all sequential equilibria, and all histories). Using an argument entirely analogous to Fudenberg, Levine and Tirole (1985, Lemma 2) we can establish that the lowest buyer's equilibrium utility is bounded above by 1. Using the Lipschitz continuity of utility (with Lipschitz constant of one), it follows that a buyer of type b can earn no more than (1 + b); consequently, any offer below -1 has probability one of acceptance. We have just shown that $Q \geq -1$. Next, we claim that $Q = 0$. Suppose instead that $Q < 0$, and let $\delta < 1$ be the minimal discount factor between periods. Then there exists an offer $q < \delta Q$, made by some seller type in some sequential equilibrium after some history, which is rejected with positive probability. But any buyer rejecting such an offer is irrational, since he cannot hope to receive a future offer below Q, and since any negative offer the buyer might make will be rejected with probability one. Consequently, $Q = 0$, and $U_2(0) = 0$.

Observe, furthermore, that $G_1(1) = G_2(0)$, and that $G_1(1) =$

$U_2(0) + U_1(1)$ (Myerson and Satterthwaite, 1983, Theorem 1). This establishes that $G_1(1) = G_2(0) = U_2(0) = U_1(1) = 0$, and hence also that $B_1(1) = B_2(0) = 0$. []

A mechanism for which $U_1(1) = 0 = U_2(0)$ will be called <u>balanced</u>. Theorems 3.2 and 3.3 have an immediate implication:

<u>Corollary 3.4</u>: Suppose $\{p,x\}$ is an ICBM induced by a sequential equilibrium in an offer-counteroffer game, and suppose that $\{p,x\}$ is convexly splittable at s (b). Then $B_1(s) = 0$ ($B_2(b) = 0$).

<u>Proof</u>: Consider the continuation game after the seller splits and reveals that his valuation belongs to $[0,s)$. As argued in the proof of Theorem 3.3, the players never trade at a negative price; individual rationality thus implies $U_2^I(0) = 0$. Applying Theorem 3.2, we conclude that $B_1(s) = 0$ (and analogously for the buyer). []

There are many mechanisms which are convexly splittable. In particular, we have:

<u>Theorem 3.5</u>: Every seller-first mechanism is convexly seller-splittable at all s, i.e., $B_1(s) \equiv 0$.

<u>Proof</u>: A seller-first mechanism is a mechanism which remains incentive compatible if the seller publicly announces his type before the buyer announces his type to a mediator (see Ausubel and Deneckere, 1988b).

Consequently, the mechanism is (seller)-splittable at every s. Hence, $B_1(s) \equiv 0$. []

In general, however, it need not be true that $B_1(s)$ (or $B_2(b)$) equals zero. Consider, for example, the Chatterjee-Samuelson (1983) mechanism:

$$(3.2) \qquad p(s,b) = \begin{cases} 1 \\ \\ 0 \end{cases} \qquad x(s,b) = \begin{cases} (s + b + 1/2)/3 & \text{if } b \geq s + 1/4 \\ \\ 0 & \text{otherwise,} \end{cases}$$

where s and b are uniformly distributed on [0,1]. Myerson and Satterthwaite (1983) established that this mechanism maximizes expected total gains from trade over all ICBM's. Some direct computations show that $B_1(s) = (s/6)(3/4 - s)^2$, which is a strictly quasiconcave function on [0, 3/4] with $B_1(0) = B_1(3/4) = 0$, and with peak $s^* = 1/4$. Since $p(s,b) \equiv 0$ for all $s \geq 3/4$, only seller splits with $s < 3/4$ have any relevance. However, there exists no $s \in (0, 3/4)$ such that $B_1(s) = 0$, and consequently the above mechanism is not convexly splittable. In fact, since splittability depends only on $p(\bullet,\bullet)$ and not on $x(\bullet,\bullet)$, no other transfer function $x'(\bullet,\bullet)$ will yield a convexly (seller)-splittable $\{p,x'\}$. Finally, since $B_2(b)$ = ----, the mechanism is not convexly (buyer)-splittable either. In fact, as we will argue in the next section, the impossibility of convex splits in ex ante efficient mechanisms (not coinciding with either monopoly or monopsony) holds quite generally. Before moving to this topic, it is of some interest to investigate the possibility of nonconvex (binary) splits. In what follows, we let $0 = s_0 < s_1 < s_2 \ldots < s_n = 1$, and we denote by

$I = [0, s_1) \cup [s_2, s_3) \cup \ldots$ and $II = [s_1, s_2) \cup [s_3, s_4) \cup \ldots$, a binary partition of $[0,1]$.

<u>Theorem 3.6</u>: If the ICBM {p,x} is seller-splittable in the binary partition {I,II}, then $\Sigma_{i=0}^{n} (-1)^i B_1(s_i) = -\alpha(n) U_2^I(0)$, where $\alpha(n) = a_{n-1}$ if n is even, $\alpha(n) = a_n$ if n is odd, and $a_k = F_1(s_k) - F_1(s_{k-1}) +$ $F_1(s_{k-2}) - F_1(s_{k-3}) + \ldots \pm F_1(s_1)$.

<u>Proof</u>: For the sake of brevity, we will only consider here the case where n is even. Define, for $k = 0, \ldots, n/2 - 1$:

$$F_1^I(v_1) = \begin{cases} (F_1(v_1) - a_{2k})/a_{n-1} \\ a_{2k+1}/a_{n-1} \end{cases} \qquad f_1^I(v_1) = \begin{cases} f_1(v_1)/a_{n-1} \\ 0 \end{cases} \qquad \text{if} \quad \begin{matrix} s_{2k} \le v_1 < s_{2k+1} \\ \\ s_{2k+1} \le v_1 < s_{2k+2}. \end{matrix}$$

Then:

$$\int_0^1 \int_0^1 (v_2 - v_1) p(v_1, v_2) f_1^I(v_1) f_2(v_2) dv_1 dv_2 =$$

$$= U_1(1) + U_2^I(0) + \int_0^1 \int_0^1 \{F_1^I(t_1) f_2(t_2) + [1 - F_2(t_2)] f_1^I(t_1)\} p(t_1, t_2) dt_1 dt_2$$

$$= U_1(1) + U_2^I(0) + \int_0^1 \int_I \{F_1^I(t_1) f_2(t_2) + [1 - F_2(t_2)] f_1^I(t_1)\} p(t_1, t_2) dt_1 dt_2$$

$$+ \int_0^1 \int_{II} F_1^I(t_1) f_2(t_2) p(t_1, t_2) dt_1 dt_2.$$

where the first equality follows from manipulations similar to those of Myerson and Satterthwaite (1983, p. 270). Equating the two expressions yields:

$$\int_0^1 \int_I \{[v_2 - (1 - F_2(v_2))/f_2(v_2)] - [v_1 + F_1^I(v_1)/f_1^I(v_1)]\}$$

$$p(v_1, v_2) f_1^I(v_1) f_2(v_2) dv_1 dv_2$$

$$= U_1(1) + U_2^I(0) + \Sigma_{k=0}^{n/2-1} F_1^I(S_{2k+1}) \int_{s_{2k+1}}^{s_{2k+2}} \bar{p}_1(t_1) dt_1$$

$$= U_1(1) + U_2^I(0) + \Sigma_{k=0}^{n/2-1} F_1^I(S_{2k+1})[U_1(s_{2k+1}) - U_1(s_{2k+2})].$$

Now:

$$a_{n-1} \int_0^1 \int_I \{v_2 - (1 - F_2(v_2))/f_2(v_2) - [v_1 + F_1^I(v_1)/f_1^I(v_1)]\}$$

$$p(v_1, v_2) f_1^I(v_1) f_2(v_2) dv_1 dv_2 =$$

$$= \int_0^1 \int_I \{[v_2 - (1 - F_2(v_2))/f_2(v_2)]$$

$$- [v_1 + F_1(v_1)/f_1(v_1)]\} p(v_1, v_2) f_1(v_1) f_2(v_2) dv_1 dv_2$$

$$+ \int_0^1 \Sigma_{k=0}^{n/2-1} a_{2k} \int_{s_{2k}}^{s_{2k+1}} p(t_1, t_2) f_2(t_2) dt_2 dt_1$$

$$= \{G(s_1) - G(s_2) + G(s_3) \ldots - G(s_{n-2}) + G(s_{n-1})\}$$

$$+ \Sigma_{k=0}^{n/2-1} a_{2k} \{U_1(s_{2k}) - U_1(s_{2k+1})\}.$$

We conclude that:

$$G(s_1) - G(s_2) + \ldots + G(s_{n-1}) =$$

$$= a_{n-1}[U_1(1) + U_2^I(0)] + \Sigma_{k=1}^{n-1} (-1)^{k-1}(a_k + a_{k-1})U_1(s_k) - a_{n-1}U_1(s_n)$$

$$= a_{n-1}U_2^I(0) + \Sigma_{k=1}^{n-1} (-1)^{k-1}F_1(s_k)U_1(s_k),$$

where the last equality follows from the fact that $a_k + a_{k-1} = F_1(s_k)$ and that $s_n = 1$. The result now follows from noting that $B_1(s_k) = F_1(s_k)U_1(s_k) - G_1(s_k)$. []

## 4. Ex Ante Efficiency and the Quasiconcavity of the B Functions

Direct inspection of the B functions reveals that they are differentiable almost everywhere, and that $B_1(0) = B_2(1) = 0$. Furthermore, we have seen that if {p,x} is supported by a sequential equilibrium in some offer-counteroffer game then the mechanism is balanced and so $B_1(1) = B_2(0) = 0$. We would now like to understand what (if any) extra structure is imposed on the B functions associated with ex ante efficient mechanisms. Recall that a mechanism {p,x} is ex ante efficient if it maximizes $\lambda\int_0^1 U_1(s)dF_1(s) + (1 - \lambda)\int_0^1 U_2(b)dF_2(b)$ (over all ICBM's), for some weight $\lambda \in [0,1]$. Ex ante efficient mechanisms have a nice characterization only under the distributional assumption:

<u>Assumption 4.1</u>:

(a) $v_1 + F_1(v_1)/f_1(v_1)$ is strictly increasing on [0,1],

(b) $v_2 - [1 - F_2(v_2)]/f_2(v_2)$ is strictly decreasing on [0,1], which we will maintain henceforth. This assumption has a clear economic interpretation. Assumption 4.1(b), for example, holds if and only if for every $s < 1$, the static payoff function from charging a single price $q$, $\pi(q,s) = [1 - F_2(q)][q - s]$, is strictly quasiconcave in $q$. Before stating the next result, it is necessary to make the following definition:

Definition 4.2: A mechanism $\{p,x\}$ is a 0-1 mechanism if it is an ICBM and if there exists a nondecreasing function $g(s)$ such that $p(s,b) = 0$ for $0 \le b < g(s)$ and $p(s,b) = 1$ for $g(s) \le b \le 1$.

Theorem 4.3 (Williams, 1987): Suppose that Assumption 4.1 holds. Then if $\{p,x\}$ is ex ante efficient,[6] it is a balanced 0-1 mechanism.

Balanced 0-1 mechanisms often induce strictly quasiconcave $B_1(\cdot)$ functions on $[0,\hat{s}]$, where $\hat{s} \equiv \sup \{s \in [0,1]: \bar{p}_1(s) > 0\}$. When this is the case, the fact that $B_1(0) = 0 = B_1(\hat{s})$ implies that $B_1(s) \neq 0$ for all $s \in (0,\hat{s})$, and so the mechanism is not convexly seller-splittable. For future reference, observe that $B_1(\cdot)$ is a $C^1$ function for any 0-1 mechanism having a continuous boundary and its derivative, $B_1'(\cdot)$, satisfies:

$$(4.1)\quad B_1'(s)/f_1(s) = U_1(s) - [g(s) - s][1 - F_2(g(s))] = \bar{x}_1(s) - g(s)\bar{p}_1(s),$$

for all $s \in [0,1]$.

Also for use in the next theorem, define $g^*(\cdot)$ implicitly by:

$$(4.2)\qquad g^*(s) - s - [1 - F_2(g^*(s))]/f_2(g^*(s)) = 0, \text{ for all } s \in [0,1].$$

Assumption 4.1(b) guarantees that $g^*(s)$ is uniquely defined for all $s \in [0,1]$. We refer to $g^*(\cdot)$ as the <u>monopoly boundary</u>. The monopsony boundary is defined analogously, and analogous results to those below can be proved for the function $B_2(\cdot)$:

<u>Theorem 4.4</u>: Suppose {p,x} is a balanced 0-1 mechanism with strictly increasing continuous boundary $g(s) \geq s$. Suppose also that Assumption 4.1(b) holds and that there exists $\tilde{s} \in (0,1)$ such that $g(s) < g^*(s)$ for $s \in [0,\tilde{s})$ and $g(s) > g^*(s)$ for $s \in (\tilde{s},\hat{s})$. Then $B_1(s)$ is strictly quasiconcave on $[0,\hat{s}]$.

<u>Proof</u>: Define $\mu_1(s) \equiv B_1'(s)/f_1(s)$ (see (4.1)). We will show that there exists a unique $s^* \in (0,\hat{s})$ such that $\mu_1(s^*) = 0$. Since $g(\cdot)$ is monotone, $\mu_1'(s)$ exists a.e., and: $\mu_1'(s) = g'(s)f_2(g(s))\{g(s) - s - [1 - F_2(g(s))]/f_2(g(s))\}$. Observe that $\mu_1'(s) > 0$ if $s > \tilde{s}$, and $\mu_1'(s) < 0$ if $s < \tilde{s}$. Then since $\mu_1(\cdot)$ is increasing on $[\tilde{s},\hat{s}]$, and since $\mu_1(\hat{s}) = 0$ by definition, it follows that $\mu_1(s) < 0$ on $[\tilde{s},\hat{s}]$. On $[0,\tilde{s}]$, on the other hand, $\mu_1$ is strictly decreasing, and hence can have at most one zero. Consequently, $B_1$ can have at most one stationary point in $[0,\hat{s}]$. Since $B_1(0) = B_1(\hat{s}) = 0$, and since $B_1(s) > 0$ on $[\tilde{s},\hat{s}]$, such a stationary point exists and it is a maximum. []

Finally, we present a sufficient condition under which ex ante efficient mechanisms satisfy the conditions of Theorem 4.4, and hence are not convexly splittable.

Theorem 4.5: Let $\{p,x\}$ be any ex ante efficient mechanism with boundary $g(\bullet)$ (not coinciding with $g^*(\bullet)$). Also suppose that $F_1(s)/f_1(s)$ and $[F_2(b) - 1]/f_2(b)$ are strictly increasing functions. Then $B_1(s)$ is strictly quasiconcave on $[0,\hat{s}]$.

Proof: By Theorem 4 of Williams (1987), $b = g(s)$ satisfies the equation:

$$b - \sigma[1 - F_2(b)]/f_2(b) = s + \tau[F_1(s)/f_1(s)],$$

for some $\sigma,\tau \in [0,1]$ with $\sigma \neq 1$ and $\tau \neq 0$. Subtracting (4.2) yields:

$$(1 - \sigma)[1 - F_2(g^*(s))]/f_2(g^*(s)) = \tau[F_1(s)/f_1(s)],$$

at any s such that $b = g(s) = g^*(s)$. Since $g^*(s)$ is strictly increasing in s, the left side of this equation is strictly decreasing in s, while the right side is strictly increasing. Consequently, there is at most one value $\tilde{s}$ such that $g(\tilde{s}) = g^*(\tilde{s})$. At the same time, at $s = 0$, the left side strictly exceeds the right side, and vice versa for $s = 1$. Consequently, the existence of $\tilde{s} \in (0,1)$ is guaranteed. Meanwhile, using the above equations for the boundary, it is straightforward to show that $g(s) < g^*(s)$ for $s < \tilde{s}$, and the reverse for $s > \tilde{s}$. []

## 5. Two-Price Mechanisms

In this and the next two sections, we will be interested in mechanisms associated with sequential equilibria of the seller-offer game. Consider any balanced 0-1 mechanism $\{p,x\}$ with strictly increasing boundary $g(\cdot)$ and strictly quasiconcave $B_1(\cdot)$ on $[0,\hat{s}]$. We have shown in Corollary 3.4 that the first revealing move in an equilibrium inducing this $\{p,x\}$ must be a <u>nonconvex seller split</u>. Observe that any nonconvex seller split necessarily involves some amount of pooling between different seller types. This pooling cannot continue indefinitely, since (by strict monotonicity of the boundary) sellers with high valuations transact with strictly fewer buyer types than do sellers with lower valuations. In fact, since $p(s,b) = 1$ for $b \geq g(s)$ (and $p(s,b) = 0$ for $b < g(s)$) buyers must know almost instantaneously whether or not to transact. Any further revelation must thus occur very quickly. In this section, we will consider mechanisms with the simplest possible description satisfying this requirement (termed two-price mechanisms): information is revealed in exactly two stages. In the first revelation stage, sellers of different valuations pairwise pool: for every $a \in [0,s^*)$, there is associated a unique $c(a) \in [s^*,\hat{s})$, offering the same initial price $p_0$. More precisely, sellers with valuation $s \in [0,\hat{s})$, by naming the initial price $p_0(s)$, reveal which doubleton $\{a,c(a)\}$ they belong to; sellers with valuations in $[\hat{s},1]$ merely pool by making nonserious offers. In the second revelation stage, the doubletons split into singletons: seller $s \in \{a,c(a)\}$ reveals whether $s = a$ or $s = c(a)$ by offering the boundary price $g(s)$.[7] Buyers merely optimize given their knowledge of the offer structure. Let $\beta(s)$ be the lowest buyer valuation accepting $p_0(s)$. Then, since the seller types a and $c(a)$ are indistinguishable at the first offer:

$$(5.1) \qquad p_0(c(a)) = p_0(a) \quad \text{and} \quad \beta(c(a)) = \beta(a), \text{ for all } a \in [0,s^*).$$

For sales to be nonnegative at both the first and second offer, we require:

$$g(s) \le \beta(s) \le 1, \text{ for all } s \in [0,1]. \tag{5.2}$$

Let $\theta(s)$ be the probability that $s = a$, conditional on the event $s \in \{a,c(a)\}$. For $\theta(\bullet)$ to be consistent with the pairing function $c(\bullet)$ and the distribution function $F_1(\bullet)$, it must be that for all $a \in [0,s^*)$ where $c(\bullet)$ is differentiable:

$$\theta(a) = f_2(a)/\{f_1(a) + f_2(c(a))|(dc/da)(a)|\}. \tag{5.3}$$

Finally, the buyer with valuation $\beta$ should be indifferent between the first offer and the expectation of second offers:

$$\beta(a) - p_0(a) = \delta\{\beta(a) - \theta(a)g(a) - [1 - \theta(a)]g(c(a))\}, \quad \text{for all } a \in [0,s^*). \tag{5.4}$$

Note that (5.4) requires the buyer to discount future offers at the rate $\delta$. The reason for this departure from strict 0-1 mechanisms is the desire to construct (in Section 7) equilibria in the seller-offer game with discounting. The mechanism determined by the sextuplet $\{p_0(\bullet),\beta(\bullet),g(\bullet),c(\bullet),\theta(\bullet),\delta\}$ is then:

$$\text{(5.5)} \qquad p(s,b) = \begin{cases} 1 \\ \delta \\ 0 \end{cases} \qquad x(s,b) = \begin{cases} p_0(s), & \text{if } \beta(s) < b \le 1 \\ \delta g(s), & \text{if } g(s) < b \le \beta(s) \\ 0, & \text{otherwise.} \end{cases}$$

We are now ready to define:

<u>Definition 5.2</u>: The sextuplet $\{p_0(\bullet),\beta(\bullet),g(\bullet),c(\bullet),\theta(\bullet),\delta\}$ will be called a <u>differentiable two-price mechanism</u> if:

(i) $c\colon [0,s^*) \to [s^*,\hat{s})$ is an a.e. differentiable bijection;

(ii) (5.1), (5.2), (5.3) and (5.4) are satisfied; and

(iii) $\{p,x\}$ determined by (5.5) is a balanced ICBM.

Conversely, suppose we are given a 0-1 mechanism with boundary $g(\bullet)$. We will say that $g(\bullet)$ induces a differentiable two-price mechanism if there exists $s^* \in (0,\hat{s})$ and functions $p_0(\bullet)$, $\beta(\bullet)$, $c(\bullet)$ and $\theta(\bullet)$ such that $\{p_0(\bullet),\beta(\bullet),g(\bullet),c(\bullet),\theta(\bullet),1\}$ satisfies (i), (ii) and (iii) above.

We can now state the following remarkable result on two-price mechanisms:

<u>Theorem 5.3</u>: Let $g(\bullet)$ be strictly monotone on $[0,\hat{s})$ and suppose that $g(\bullet)$ induces a differentiable two-price mechanism with pairing $c(\bullet)$. Then $B_1(\bullet)$ is quasiconcave and:

(5.6) $dc/da = B_1'(a)/B_1'(c(a))$, for almost every $a \in [0,s^*)$.

Conversely, let $g(\bullet)$ be a strictly increasing continuous boundary associated with a balanced 0-1 mechanism having a quasiconcave $B_1(\bullet)$ on $[0,\hat{s}]$. Let $c(\bullet)$ be any bijection from $[0,s^*)$ to $[s^*,\hat{s})$ satisfying (5.6). Then $g(\bullet)$ induces a differentiable two-price mechanism with pairing $c(\bullet)$, provided the implied $\beta(\bullet)$ satisfies (5.2).

<u>Proof</u>: Note from (5.5), for a 0-1 mechanism, that $\bar{p}_1(s) = 1 - F_2(g(s))$ and that $\bar{x}_1(s) = [1 - F_2(\beta(s))]p_0(s) + [F_2(\beta(s)) - F_2(g(s))]g(s) = [1 - F_2(\beta(s))]\ [p_0(s) - g(s)] + g(s)\bar{p}_1(s)$. Hence, using (4.1) we have for almost every s:

(5.7) $[1 - F_2(\beta(s))][p_0(s) - g(s)] = B_1'(s)/f_1(s)$.

Consequently, using (5.7) and (5.1):

(5.8)
$$[p_0(a) - g(a)]/[p_0(a) - g(c(a))] =$$
$$= [B_1'(a)/f_1(a)]/[B_1'(c(a))/f_1(c(a))], \text{ a.e. } a \in [0,s^*).$$

In order for (5.1) to be satisfied with $\delta = 1$, we must have $p_0(a) = \theta(a)g(a) + [1 - \theta(a)]g(c(a))$, and hence:

(5.9) $1 - 1/\theta(a) = [p_0(a) - g(a)]/[p_0(a) - g(c(a))]$, a.e. $a \in [0,s^*)$.

Combining (5.8) and (5.9), this implies:

$$(5.10) \qquad \theta(a) = -[B_1'(c(a))/f_1(c(a))] \;/\; [B_1'(a)/f_1(a) - B_1'(c(a))/f_1(c(a))], \quad \text{a.e. } a \in [0,s^*).$$

From Assumption 4.1, $f_1(s) \neq 0$ on $(0,1)$; (5.3) and (5.10) thus imply that (5.4) holds a.e. Moreover, since $g(a) \leq p_0(a) = p_0(c(a)) \leq g(c)$, (5.7) and the second inequality in (5.2) imply that $B_1'(a) \geq 0$ and $B_1'(c(a)) \leq 0$ for all $a \in [0,s^*)$, establishing quasiconcavity and proving the first part of the theorem.

Suppose conversely that $c(\bullet)$ satisfies (5.6). Since $g(\bullet)$ is continuous, $B_1(\bullet)$ is differentiable everywhere. Evaluating (5.7) at a and c(a), and subtracting the resulting expressions, yields a unique candidate $\beta(a) = \beta(c(a))$:

$$(5.11) \qquad 1 - F_2(\beta) = [B_1'(a)/f_1(a) - B_1'(c(a))/f_1(c(a))] \;/\; [g(c(a)) - g(a)].$$

By quasiconcavity of $B_1(\bullet)$ and monotonicity of $g(\bullet)$, the right side of (5.11) is nonnegative and hence any solution satisfies $\beta \leq 1$. We will now show that if $\beta(s) \geq g(s)$, then we can define functions $p_0(\bullet)$ and $\theta(\bullet)$ such that all the other requirements for a differentiable two-price mechanism are established. First, if $\beta(s) = 1$, we can define $\theta(s)$ from (5.3) if $c(\bullet)$ is differentiable at this point (otherwise, we arbitrarily set $\theta(s) = 1/2$) and then define $p_0(s)$ from (5.4). If $\beta(s) < 1$, then (5.7) evaluated at a (or equivalently, at c(a)), yields a solution satisfying $g(a) \leq p_0(a) \leq g(c(a))$. $\theta(a)$ is then defined from (5.9), implying that (5.4) holds at $\delta = 1$. From (5.8) and (5.6), it then follows that at every a where $c(\bullet)$ is differentiable, (5.3) holds. []

Several remarks concerning Theorem 5.3 are in order. First, if we strengthen (5.2) and (5.3) to $\beta(s) < 1$ and $\theta(s) \in (0,1)$ for all $s \in (0,s^*) \cup (s^*,\hat{s})$, then the conclusion of the first part of the theorem can be strengthened to "$B_1$ is strictly quasiconcave." Second, if the pairing function $c(a)$ is <u>monotone</u>, (5.6) implies that it satisfies the equation:

$$(5.12) \qquad B_1(a) + B_1(c(a)) = B_1(s^*), \text{ for all } a \in [0,s^*).$$

Third, an alternative way to write the implication of (5.7) and (5.8) is:

$$(5.13) \qquad \theta(a)[\bar{x}_1(a) - g(a)\bar{p}_1(a)] + [1 - \theta(a)][\bar{x}_1(c(a)) - g(c(a))\bar{p}_1(c(a))] = 0.$$

Equation (5.13) has the economic interpretation that <u>revenues are conserved between paired seller types.</u> Since $B_1(\cdot)$ is quasiconcave, $\bar{x}_1(a) - g(a)\bar{p}_1(a)$ is nonnegative while $\bar{x}_1(c(a)) - g(c(a))\bar{p}_1(c(a))$ is nonpositive. Observe that $g(a)\bar{p}_1(a)$ represents the revenue that type a would earn if there were no pooling stage--then, all sales would occur at a price of $g(a)$--while $\bar{x}_1(a)$ equals the revenue type a actually receives. Hence, $\bar{x}_1(a) - g(a)\bar{p}_1(a)$ represents the "cross-subsidy" which seller types below $s^*$ receive, and analogously, $\bar{x}_1(c(a)) - g(c(a))\bar{p}_1(c(a))$ represents the "cross-subsidy" which seller types above $s^*$ pay. For each paired a and $c(a)$, the transfer paid must equal the transfer received--with the appropriate weight placed on each type. As equation (5.13) indicates, the proper weights are the probabilities that $s = a$ and $s = c(a)$, respectively, conditioned on the event $s \in \{a, c(a)\}$. It is also worth noting that, for sellers in $[s^*, \hat{s})$, pricing behavior is <u>not</u> monotone decreasing over time since $p_0(s) < g(s)$ if $\theta(s) \neq 0$. The reason for this price increase is that, after the initial revelation, the seller is relieved of having to cross-subsidize the lower types with which she was pooled.

## 6. <u>Ex Ante Efficiency and Two-Price Mechanisms</u>

Consider any 0-1 mechanism with strictly increasing continuous boundary $g(\cdot)$, and strictly quasiconcave $B_1$. From here on, we will (for the sake of simplicity) assume that the pairing function is <u>monotone</u>. Theorem 5.3 then implies that $g(\cdot)$ induces a two-price mechanism, and that the pairing function satisfies (5.12), provided that the implied $\beta(\cdot)$ satisfies:

$$(6.1) \qquad g(s) \leq \beta(s), \text{ for all } s \in [s^*, \hat{s}).$$

This constraint is somewhat troublesome, and is indeed not always satisfied. In this section we present some results concerning (6.1).

<u>Lemma 6.1</u>: Constraint (6.1) is satisfied if and only if:

$$(6.2) \qquad \bar{x}_1(s)/\bar{p}_1(s) \geq p_0(s), \text{ for all } s \in [s^*, \hat{s}) \text{ s.t. } \theta(s) \neq 0.$$

<u>Proof</u> (informally): In a 0-1 mechanism, $\bar{x}_1(s)/\bar{p}_1(s)$ represents the average price paid to seller s. In a two-price mechanism, $p_0(s)$ and $g(s)$ are the

only prices ever paid, and for $s \in [s^*, \hat{s})$ s.t. $\theta(s) \neq 0$, $g(s) > p_0(s)$. Hence, (6.2) is satisfied if and only if sales are nonnegative at $g(s)$. []

In our proof of implementability of two-price mechanisms, we will require that at least one of the two inequalities (6.1) and (6.2) is satisfied with discrete slack. Observe that (6.1) cannot have slack at $\hat{s}$, since $g(\hat{s}) = \beta(\hat{s}) = 1$. Nor can constraint (6.2) generally have slack at $s^*$.[8] However, it is often possible for (6.1) to have slack at $s^*$, and for (6.2) to have slack at $\hat{s}$:

<u>Theorem 6.2</u>: Suppose $F_1(s)/f_1(s)$ and $[F_2(b) - 1]/f_2(b)$ are strictly increasing functions, and that $\lim_{b \uparrow 1} f_2(b) > 0$. Then there exists $\lambda_s \in (0,1)$ such that, for every $\lambda \in (\lambda_s, 1)$ there exists $\varepsilon^\lambda > 0$ such that the ex ante efficient mechanism with weight $\lambda$ on the seller induces a boundary $g^\lambda$ satisfying at least one of the constraints (6.1) and (6.2) with slack $\varepsilon^\lambda$.

<u>Proof</u>: By Theorem 4.5 the boundary $g^\lambda(\bullet)$ associated with a seller weight $\lambda < 1$ induces a strictly quasiconcave $B_1^\lambda(\bullet)$. Let $\beta^\lambda(\bullet), p_0^\lambda(\bullet), \theta^\lambda(\bullet), \bar{x}_1^\lambda(\bullet), \bar{p}_1^\lambda(\bullet)$ and $\hat{s}^\lambda$ be associated with $g^\lambda(\bullet)$. The proof requires two observations:

a. As $\lambda \uparrow 1$, the function $\beta^\lambda(\bullet)$ converges uniformly to 1. Hence, for sufficiently small $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that $\lim_{\varepsilon \downarrow 0} \eta(\varepsilon) = 0$ and $\beta^\lambda(s) \geq g^\lambda(s) + \varepsilon$ for $s \in [0, \hat{s}^\lambda - \eta(\varepsilon))$ and $\lambda$ in a neighborhood of 1.

b. If $\lim_{b \uparrow 1} f_2(b) > 0$, it can be shown that $\lim_{\lambda \uparrow 1} \theta^\lambda(\hat{s}^\lambda) > 0$.

Consequently, $\lim_{\lambda \uparrow 1} p_0^{\lambda}(\hat{s}^{\lambda}) < 1$, whereas $\lim_{\lambda \uparrow 1} \bar{x}_1^{\lambda}(\hat{s}^{\lambda})/\bar{p}_1^{\lambda}(\hat{s}^{\lambda}) = 1$. Again, for sufficiently small $\varepsilon > 0$, $\bar{x}_1^{\lambda}(s)/\bar{p}_1^{\lambda}(s) \geq p_0^{\lambda}(s) + \varepsilon$ for $s \in [\hat{s}^{\lambda} - \eta(\varepsilon), \hat{s}^{\lambda}]$, and $\lambda$ in a neighborhood of 1.

The existence of $\lambda_s < 1$ is now immediate. []

Theorem 6.2 establishes that, subject to some distributional assumptions, ex ante efficient mechanisms induce two-price (seller) mechanisms, for weights $\lambda \in [\lambda_s, 1)$. Analogously, ex ante efficient mechanisms with weights $\lambda \in (0, \lambda_b]$ induce two-price (buyer) mechanisms, for some $\lambda_b \in (0,1)$. The intervals $[\lambda_s, 1)$ and $(0, \lambda_b]$ are potentially quite large--in fact often $\lambda_s \leq \lambda_b$ so that two-price mechanisms span the entire Pareto frontier.

<u>Example 6.3</u>: Consider the Chatterjee-Samuelson mechanism (3.2) associated with equal weighting ($\lambda = 1/2$) and uniform distributions. The pairing c(a) is derived from (5.12), using $B_1(s) = (s/6)(3/4 - s)^2$. It can be demonstrated that dc/da is a strictly decreasing function on (0,1/4), with range $(1,\infty)$. In addition, $\beta(a) = (1/2)[1 + a + c(a)]$ (see Figure 2). Constraint (6.1) holds everywhere on $[s^*, \hat{s})$, but $\beta(\cdot)$ and $g(\cdot)$ are tangent at $\hat{s}$, implying that both (6.1) and (6.2) hold with equality as s approaches $\hat{s}$. As we increase $\lambda$ from 1/2, constraint (6.2) is monotonically relaxed. Meanwhile, (6.2) is violated at $\hat{s}^{\lambda}$ for every $\lambda < 1/2$. A symmetric argument establishes that $\lambda_b$ also equals 1/2.

(Insert Figure 2 about here)

We conclude that, in the double-uniform case, two-price (seller) mechanisms span the half of the efficiency frontier favoring the seller, while two-price (buyer) mechanisms span the remaining half, which favors the buyer.

Example 6.4: Let $F_1(s) = s^{\alpha}$ and $F_2(b) = 1 - [1 - b]^{\gamma}$, for $\alpha,\gamma > 0$. The set of ex ante efficient boundaries is then parametrically defined by $g(s) = \rho + (1 - \rho)s/\hat{s}$, where $\hat{s} = [\alpha + \rho(1 + \gamma)]/(1 + \alpha)$ and $\rho$ $(0 \leq \rho \leq 1/(\alpha + \gamma))$ is a parameter reflecting the relative weight to the seller. In particular, if $\rho = 0$, we obtain the monopsony mechanism, if $\rho = 1/(\alpha + \gamma)$ we obtain the monopoly mechanism, and if $\rho = (\alpha - D^{1/2})/(\alpha - \gamma)$ with $D = \alpha\gamma[(1 + \alpha)/(1 + \gamma)]$ we obtain equal weighting. Some tedious algebra establishes that $B_1(s) = [(1 - \rho)/\hat{s}]^{\gamma}[1/(1 + \gamma) - \rho/\hat{s}][\hat{s} - s]^{1+\gamma}s^{\alpha}$, and that $s^* = [\alpha/(1 + \alpha + \gamma)]\hat{s}$. Observe that $F_2(\cdot)$ satisfies the hypotheses of Theorem 6.2 if and only if $\gamma \leq 1$, and analogously for the buyer mechanisms Theorem 6.2 requires $\alpha \leq 1$. Numerical computations indicate that (6.2) is satisfied for the entire rectangle $0 < \alpha,\gamma \leq 1$, $\lambda_s \leq 1/2$ and $\lambda_b \geq 1/2$.

## 7. The Implementability of Efficient Mechanisms

At last, we have developed enough machinery to implement 0-1 mechanisms which require nonconvex splitting. As above, we will cast our results in terms of seller mechanisms and the seller-offer game; entirely analogous results hold for buyer mechanisms and the buyer-offer game. First, let us briefly define our notion of implementation (Ausubel and Deneckere, 1988b).

Definition 7.1: Let $p(\cdot,\cdot)$ be associated with an ICBM. We will say that p is <u>implementable by sequential equilibria of the seller-offer game</u> if there exists a sequence $\{\sigma^n, z^n\}_{n=1}^{\infty}$ such that:

(i) $z^n \downarrow 0$ and, for every $n \geq 1$, $\sigma^n$ is a sequential equilibrium of the seller-offer game where the time between offers is $z^n$; and

(ii) If $m(\cdot)$ denotes the $F_1 \times F_2$-measure on $[0,1] \times [0,1]$, and if $p^n(\cdot,\cdot)$ denotes the probability of trade function induced by $\sigma^n$, then for all $\varepsilon > 0$: $m\{(s,b)\colon |p^n(s,b) - p(s,b)| > \varepsilon\} \to 0$, as $n \to \infty$.

If, furthermore, every $\sigma^n$ is stationary in the sense that history only matters insofar as it is reflected in current beliefs, then we will say that p is <u>implementable by stationary sequential equilibria</u>. In order to prove implementability, we make the following assumptions:

<u>Assumption 7.2</u>: There exists $\varepsilon > 0$ such that $\beta(s) \geq g(s) + \varepsilon$ or $\bar{x}_1(s)/\bar{p}_1(s) \geq p_0(s) + \varepsilon$ for all $s \in [s^*, \hat{s})$.

<u>Assumption 7.3</u>: For every $s \in [0,s^*]$, $g^*(s) > g(s)$. Furthermore, $g^*(s) \neq g(s)$, except for at most finitely many $s \in (s^*,\hat{s}]$.

We may now state our main result, which is proven in the Appendix.

<u>Theorem 7.4</u>: Consider any 0-1 mechanism, p, with a boundary $g(\cdot)$ which is strictly increasing and continuous. Suppose that the implied $B_1(\cdot)$ function is strictly quasiconcave on $[0,\hat{s}]$ with $B_1(\hat{s}) = 0$, and suppose Assumptions

4.1, 7.2 and 7.3 are satisfied. Then p is implementable by stationary sequential equilibria in the seller-offer game.

Corollary 7.5: Suppose $F_1(s)/f_1(s)$ and $[F_2(b) - 1]/f_2(b)$ are strictly increasing functions, that $\lim_{b\uparrow 1} f_2(b) > 0$, and that $\lim_{s\downarrow 0} f_1(s) > 0$. Then there exists $\lambda_s \in (0,1)$ ($\lambda_b \in (0,1)$) such that for every $\lambda \in [\lambda_s, 1]$ ($\lambda \in [0, \lambda_b]$), the ex ante efficient mechanism which places weight $\lambda$ on the seller is implementable in the seller- (buyer-) offer game.

Proof: For every $\lambda \in (\lambda_s, 1)$, the seller result follows directly from Theorems 5.2 and 7.4. To implement the mechanism with weight $\lambda_s$, consider any sequence $\{\lambda^k\}_{k=1}^{\infty} \subset (\lambda_s, 1)$ converging to $\lambda_s$. Each mechanism with weight $\lambda^k$ is implementable; a diagonal argument then shows implementability for $\lambda_s$ (and similarly for 1). The buyer result is proven analogously. []

It is perhaps worth remarking that Corollary 7.5 implies implementability of the entire Pareto frontier for the parametric examples of Section 6. Let us conclude this section by giving a brief description of the sequential equilibria used in the proof of Theorem 7.4. Let $s^* = c_N < c_{N-1} < \ldots < c_1 < c_0 = \hat{s}$ be a grid of (N + 1) seller types partitioning the interval $[s^*, \hat{s}]$, and let $0 = a_N < a_{N-1} < \ldots < a_1 < a_0 = s^*$ be the corresponding grid of seller types in $[0, s^*]$ defined through (5.12). In period zero, seller types belonging to the paired intervals $[a_k, a_{k-1})$ and $[c_k, c_{k-1})$ pool (nonconvexly) by charging the same initial price $p_0(a_k)$. This initial offer reveals to the buyer that $s \in [a_k, a_{k-1}) \cup [c_k, c_{k-1})$, but not whether $s \in [a_k, a_{k-1})$ or $s \in [c_k, c_{k-1})$. In period one, seller types further separate: a seller with $s \in [a_k, a_{k-1})$ charges $g(a_k)$ and a seller with $s \in [c_k, c_{k-1})$ charges $g(c_k)$. With the exception of seller types in the lowest pool, the seller continues to charge the same price, $g(a_k)$ or $g(c_k)$, in all future periods. However, a seller whose valuation belongs to $[a_N, a_{N-1})$ charges a price of $e^{-\lambda(m-1)z}g(a_N)$ in all periods $m \geq 1$ when this price exceeds her valuation and makes nonserious offers thereafter. Finally, seller types belonging to $[\hat{s}, 1]$ always make nonserious offers.

The buyer forms expectations and optimizes, subject to this seller behavior. After period zero, the buyer assigns probability $\theta$ to $s \in [a_k, a_{k-1})$ and $(1 - \theta)$ to $s \in [c_k, c_{k-1})$, where $\theta \equiv [F_1(a_{k-1}) - F_1(a_k)]/[F_1(a_{k-1}) - F_1(a_k) + F_1(c_{k-1}) - F_1(c_k)]$. Thus, equation (5.4) requires any buyer with valuation exceeding $\beta(a_k)$ to accept the seller's initial offer, and any buyer with lower valuation to reject. Meanwhile, except for the case $s \in [a_N, a_{N-1})$, any buyer with valuation b exceeding $g(a_k)$ (or $g(c_k)$) must accept the seller's offer in period one (since price will never drop again) and any buyer with lower valuation rejects that and all subsequent offers. Finally, for $s \in [a_N, a_{N-1})$, the buyer selects a period in which to purchase, taking account of both time impatience and the probability that the seller will cease to make serious offers in subsequent periods. Meanwhile, the seller is deterred from reducing her price, in an effort to generate additional sales, by the prospect of adverse inferences and hence expectations of still lower prices.

The mechanisms induced by these sequential equilibria differ from the two-price mechanisms considered in Sections 5 and 6 in three major ways. First, the mechanisms are discrete: $\beta(\bullet)$ and $g(\bullet)$ are step functions, piecewise constant on the 2N intervals $[a_k, a_{k-1})$ and $[c_k, c_{k-1})$. Second, the mechanisms involve discounting, and hence generate probability of trade functions taking on one of the three values: 0, $\delta$, and 1 (with the exception of the bottom pool). Finally, sellers in the bottom pool charge a sequence of prices descending towards zero (rather than just two prices). This is necessitated by the fact that the lowest valuation seller cannot be deterred from cutting her price by adverse inferences, and hence must expect sales in all future periods.

## 8. Conclusion

In earlier work (Ausubel and Deneckere, 1986, 1988a), we established two qualitative propositions concerning sequential bargaining with one-sided incomplete information. First, two extensive forms (the seller-offer and buyer-offer games) are sufficient to implement the entire (ex ante) Pareto frontier. Second, the ability to make offers confers bargaining power.[9]

In the current article, we demonstrated similar results for two-sided incomplete information. For fairly general distributions, one segment of the efficient frontier is implementable in the seller-offer game and another segment is implementable in the buyer-offer game. Often, the union of these two segments equals the entire Pareto frontier (see, again, Figure 1). At the same time, the segment we construct from seller-offer equilibria necessarily includes those static mechanisms most favorable to the seller, while usually excluding those mechanisms most favorable to the buyer (and, analogously, for the buyer-offer game). This again draws a connection between bargaining strength and the exclusive ability to make offers. Indeed, the fact that a firm posts prices and refuses to accept counteroffers from consumers may be viewed as a sign of strength: this institutional arrangement confines efficient sequential equilibria to those relatively favorable to the firm.

In one-sided incomplete information, some equilibria of the seller-offer and buyer-offer games can be "embedded" in extensive forms (e.g., alternating offer) which permit both parties to make offers (Fudenberg, Levine and Tirole, 1985; Ausubel and Deneckere, 1988a). It may also be possible to do this in the case of two-sided incomplete information. We plan to pursue this issue in future work.

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## Notes

1. This limiting property of the equilibria is somewhat reminiscent of results by Gul, Sonnenschein and Wilson (1986) and Gul and Sonnenschein (1988). These authors demonstrated that stationarity in bargaining with one-sided incomplete information implies that all information is revealed arbitrarily quickly and that trade occurs instantaneously. This gives efficiency for every weight favoring the informed party.

   Removing the stationarity assumption eliminates the predisposition for instantaneous trade. At the same time, it permits efficiency for weights favoring the uninformed party (Ausubel and Deneckere, 1986, 1988a).

2. In a previous paper (Ausubel and Deneckere, 1988b), we demonstrated the existence of a large class of equilibria. Moreover, our no-trade theorem (Theorem 2) there proves that equilibria with the Coase Conjecture property converge to maximal inefficiency as the interval between offers approaches zero.

3. In Theorem 3.3 below, we demonstrate that $p(\cdot,\cdot)$ is not implementable in any offer-counteroffer game if $p(\cdot,\cdot)$ is not "balanced." Furthermore, $p^4(\cdot,\cdot)$ in Ausubel and Deneckere (1988b) provides an example of a balanced mechanism which is not implementable in the seller-offer game. Counterexamples to general feasibility theorems will be the subject of a future paper.

   The absence of folk theorems in two-sided incomplete information contrasts with our previous (1988a) results on one-sided incomplete information. The seller-offer game there implements the entire set of ex post individually rational ICBM's.

4. We specifically rule out extensive forms which permit simultaneous moves since, even in the one-shot complete information case, these permit any division of the surplus.

5. The terminology here is from Cramton (1985). Note that we are assuming the typical case in which randomization of outcomes over time does not occur.

6. In fact, interim efficient mechanisms often have this characterization as well (see, for example, Myerson, 1985).

7. The prose used in this paragraph does not purport to directly describe either a game or an equilibrium. We have chosen it merely to illuminate the successive splitting of two-price mechanisms. The relationship between these mechanisms and equilibria of the seller-offer game will be fully explored in Section 7.

8. For continuous boundaries $g(\bullet)$, note that $\lim_{s \downarrow s^*} B_1'(s) = 0$, since $s^*$ is the peak of $B_1$, but generally $\lim_{s \downarrow 0} B_1'(s) > 0$. Hence, $\lim_{s \downarrow s^*} \theta(s) = 0$, and so $\bar{x}_1(s^*)/\bar{p}_1(s^*) = p_0(s^*)$.

9. In one-sided incomplete information, if the seller's (buyer's) valuation is commonly known, <u>all</u> ICBM's are implementable in the seller- (buyer-) offer game. Modifying the extensive form, by permitting the silent party to make offers, eliminates those ICBM's most unfavorable to the silent party. These propositions are proven for the case of "no gap," i.e., the uninformed party's valuation is contained in the support of the informed party's valuation.

## Appendix

### Proof of Theorem 7.4

Part I: Construction of a discrete mechanism, $\tilde{p}$, approximating p for $\delta = 1$.

For any $N > 1$, we set $c_0 = \hat{s}$, $c_N = s^*$, and arbitrarily select a (decreasing) grid $\{c_k\}_{k=1}^{N-1}$ of $N - 1$ seller types on the interval $(s^*,\hat{s})$, with the property that the maximum distance between successive sellers on the grid approaches zero as $N \to \infty$. For convenience, let us define:

$$(A.1) \qquad c_k = [(N - k)s^* + k\hat{s}]/N, \qquad k = 0,\dots,N.$$

Given $\{c_k\}_{k=0}^{N}$, we define a second grid $\{a_k\}_{k=0}^{N}$ on the interval $[0,s^*]$ by:

$$(A.2) \qquad B_1(a_k) + B_1(c_k) = B_1(s^*), \qquad k = 0,\dots,N.$$

Observe that $a_0 = c_N = s^*$ and $a_N = 0$. We will now construct a discrete 0-1 mechanism $\tilde{p}$, with associated $\tilde{B}_1(\bullet)$, boundary $\tilde{g}(\bullet)$ and utility function $\tilde{U}_1(\bullet)$, having the property that the utilities from the original mechanism are preserved for seller type $\{c_k\}_{k=0}^{N}$, i.e., $\tilde{U}_1(c_k) = U_1(c_k)$ for $k = 0,\dots,N$. Since $U_1(s) = \int_s^{\hat{s}} [1 - F_2(g(v_1))]dv_1$, this requires:

$$(A.3) \qquad 1 - F_2(\tilde{g}(s)) = \{\int_{c_k}^{c_{k-1}} [1 - F_2(g(v_1))]dv_1\}/(c_{k-1} - c_k), \quad \text{for } s \in [c_k,c_{k-1}) \text{ and } k = 1,\dots,N,$$

and $\tilde{g}(s) = 1$ for $s \in [\hat{s},1]$. Observe, by (4.1), that $\tilde{B}_1'(s)/f_1(s) = \tilde{x}_1(c_k) - [1 - F_2(\tilde{g}(c_k))]\tilde{g}(c_k)$, for all $s \in [c_k,c_{k-1})$. Consequently, $\tilde{B}_1(c_{k-1}) -$

$\tilde{B}_1(c_k) = \int_{c_{k-1}}^{c_k} \tilde{B}_1'(s)ds$ is completely determined by (A.1) and (A.3), for k = 1,...,N.

We will now determine values for $\tilde{g}(s)$, $s \in [0,s^*)$, so that $\tilde{B}_1(a_{k-1}) - \tilde{B}_1(a_k) = \tilde{B}_1(c_k) - \tilde{B}_1(c_{k-1})$ for k = 1,...,N and so that $\tilde{g}(\bullet)$ is constant on each $[a_k,a_{k-1})$. Since $\tilde{B}_1'(s)/f_1(s) = \tilde{x}_1(a_k) - [1 - F_2(\tilde{g}(a_k))]\tilde{g}(a_k)$, for all $s \in [a_k,a_{k-1})$, we need:

$$\text{(A.4)} \qquad \begin{aligned} \tilde{x}_1(a_k) &- [1 - F_2(\tilde{g}(a_k))]\tilde{g}(a_k) \\ &= [\tilde{B}_1(a_{k-1}) - \tilde{B}_1(a_k)]/[F_1(a_{k-1}) - F_1(a_k)] \\ &= -[\tilde{B}_1(c_{k-1}) - \tilde{B}_1(c_k)]/[F_1(a_{k-1}) - F_1(a_k)] \equiv d_k. \end{aligned}$$

Incentive compatibility of a mechanism necessitates that $d\bar{x}_1(s) = sd\bar{p}_1(s)$ (see Myerson and Satterthwaite, 1983, and Ausubel and Deneckere, 1988a, Theorem 1), implying:

$$\text{(A.5)} \qquad \begin{aligned} \tilde{x}_1(a_k) - \tilde{x}_1(a_{k-1}) &= a_{k-1}[\tilde{p}_1(a_k) - \tilde{p}_1(a_{k-1})] \\ &= a_{k-1}\{[1 - F_2(\tilde{g}(a_k))] - [1 - F_2(\tilde{g}(a_{k-1}))]\}. \end{aligned}$$

First-differencing (A.4) and substituting (A.5) into the resulting equation yields:

$$\text{(A.6)} \qquad \begin{aligned} [\tilde{g}(a_k) - a_{k-1}][1 - F_2(\tilde{g}(a_k))] &- [\tilde{g}(a_{k-1}) - a_{k-1}][1 - F_2(\tilde{g}(a_{k-1}))] \\ &= d_{k-1} - d_k, \text{ for } k = 1,\ldots,N. \end{aligned}$$

Let $\pi^*(a) = \max_p \pi(p,a) = \pi(g^*(a),a)$. Choose small positive $\Delta$. Observe that for sufficiently large $N_\Delta$, $|d_{k-1} - d_k| < \Delta$ for all $N > N_\Delta$ and $k = 1,\ldots,N$. Suppose that, also, $\inf_k|\pi(\tilde{g}(a_{k-1}),a_{k-1}) - \pi^*(a_{k-1})| > \Delta$. Then, equation (A.6) has a solution, $\tilde{g}(a_k) \in [0, g^*(a_{k-1})]$, for $k = 1,\ldots,N$. However, observe that as $N \to \infty$, the solution to the difference equation (A.6) converges uniformly to the solution of the differential equation:

$$\text{(A.7)} \qquad (d/ds)\{[\tilde{g}(s) - s][1 - F_2(\tilde{g}(s))]\} = (d/ds)\{B_1'(s)/f_1(s)\}, \quad \text{for } s \in [0,s^*].$$

Note that (A.7) is uniquely solved by $\tilde{g}(\bullet) \equiv g(\bullet)$ and that $\inf_{s\in[0,s^*]}|\pi(g(s),s) - \pi^*(s)| \equiv \Delta' > 0$. Consequently, there exists $\bar{N} > N_{\Delta'/2} > 0$ such that, for all $N \geq \bar{N}$ and $k = 1,\ldots,N$, the iterative solution to (A.6) satisfies $\inf_k|\pi(\tilde{g}(a_{k-1}),a_{k-1}) - \pi^*(a_{k-1})| > \Delta'/2$. Thus, for all $N \geq \bar{N}$, we have completely defined a discrete boundary, $\tilde{g}(\bullet)$, such that the associated 0-1 mechanism approximates $p(\bullet)$. Since $\tilde{B}_1(0) = 0$, we have also assured $\tilde{B}_1(a_k) + \tilde{B}_1(c_k) = \tilde{B}_1(s^*)$ for all $k = 0,\ldots,N$.

In order to argue that the mechanism with boundary $\tilde{g}(\bullet)$ has the two-price interpretation, all that remains to be shown is that $\tilde{g}(c_k) \leq \tilde{\beta}(c_k) \leq 1$ for $k = 1,\ldots,N$. The second inequality holds strictly by Theorem 5.3, since $\tilde{B}_1(\bullet)$ was constructed to be strictly quasiconcave with peak at $s^*$. We will now establish a result somewhat stronger than the first inequality: there exists $\tilde{N} > \bar{N}$ such that for every $N \geq \tilde{N}$ and for every $k = 1,\ldots,N$, at least one of $\tilde{\beta}(c_k) > \tilde{g}(c_k) + \varepsilon/2$ and $\tilde{x}_1(c_k)/\tilde{p}_1(c_k) > \tilde{p}_0(c_k) + \varepsilon/2$ holds. We will demonstrate this fact using Assumption 7.2 and the uniform convergence of $\tilde{\beta}(\bullet)$, $\tilde{g}(\bullet)$, $\tilde{x}_1(\bullet)/\tilde{p}_1(\bullet)$ and $\tilde{p}_0(\bullet)$ to $\beta(\bullet)$, $g(\bullet)$, $\bar{x}_1(\bullet)/\bar{p}_1(\bullet)$ and $p_0(\bullet)$. Observe that $\tilde{g} \to g$ uniformly, as $N \to \infty$, because the grid width approaches zero. Define $\tilde{\theta}(s) = [F_1(a_{k-1}) - F_1(a_k)]/\{[F_1(a_{k-1}) - F_1(a_k)] + [F_1(c_{k-1}) - F_1(c_k)]\}$ and $\tilde{p}_0(s) \equiv \tilde{\theta}(a_k)\tilde{g}(a_k) + [1 - \tilde{\theta}(c_k)]\tilde{g}(c_k)$, for $s \in [a_k, a_{k-1}) \cup [c_k, c_{k-1})$. Note that $\tilde{\theta} \to \theta$ uniformly and so $\tilde{p}_0 \to p_0$ uniformly. Now $\tilde{x}_1(a_k) = [1 - F_2(\tilde{\beta}(a_k))]\tilde{p}_0(a_k) + [F_2(\tilde{\beta}(a_k)) - F_2(\tilde{g}(a_k))]\tilde{g}(a_k)$, and similarly for $\tilde{x}_1(c_k)$. Recall that $\tilde{\beta}(a_k) = \tilde{\beta}(c_k)$ and $\tilde{p}_0(a_k) = \tilde{p}_0(c_k)$; subtracting yields:

$$\text{(A.8)} \quad F_2(\tilde{\beta}(c_k)) = \frac{\tilde{x}_1(c_k) - \tilde{x}_1(a_k) + \tilde{g}(c_k)F_2(\tilde{g}(c_k)) - \tilde{g}(a_k)F_2(\tilde{g}(a_k))}{\tilde{g}(c_k) - \tilde{g}(a_k)},$$

and an analogous expression for $F_2(\beta(\bullet))$. Since $\tilde{g} \to g$ uniformly and $g(c) - g(a)$ is bounded away from zero (for all paired a and c), (A.8) shows that $F_2(\tilde{\beta}(\bullet)) \to F_2(\beta(\bullet))$ uniformly and hence $\tilde{\beta} \to \beta$ uniformly.

It remains to be demonstrated that $\tilde{x}_1/\tilde{p}_1 \to \bar{x}_1/\bar{p}_1$ uniformly. This convergence can be shown algebraically, but it is more informative to argue it graphically, via Figure 3. Begin with the original mechanism $p(\bullet)$. Since $U_1(s) = \int_s^{\hat{s}} \bar{p}_1(v_1)dv_1$ for $s \in [0, \hat{s}]$, and since by hypothesis $g(\bullet)$ is continuous, it should be observed that $U_1(\bullet)$ is $C^1$ and convex, with slope $-\bar{p}_1(s)$ at s. Let $T_s$ denote the tangent line through s. Since $\bar{x}_1(s) = U_1(s) + s\bar{p}_1(s)$, it follows that $\bar{x}_1(s)$ is the $U_1$-intercept of $T_s$, and hence $\bar{x}_1(s)/\bar{p}_1(s)$ is the $v_1$-intercept of $T_s$. Now consider the discrete mechanism $\tilde{p}(\bullet)$. By construction, $\tilde{U}_1(c_k) = U_1(c_k)$ for $k = 1,\ldots,N$. Let $L_k$ denote the line through $(c_k, \tilde{U}_1(c_k))$ and $(c_{k-1}, \tilde{U}_1(c_{k-1}))$. Analogous to the above, the slope of $L_k$ equals $-\tilde{p}_1(c_k)$ and $\tilde{x}_1(c_k)/\tilde{p}_1(c_k)$ is the $v_1$-intercept of $L_k$. Since $L_k$ is also the secant line of $U_1$ at $c_k$ and $c_{k-1}$, and since

$U_1(\bullet)$ is convex, $\bar{x}_1(c_k)/\bar{p}_1(c_k) \le \tilde{x}_1(c_k)/\tilde{p}_1(c_k) \le \bar{x}_1(c_{k-1})/\bar{p}_1(c_{k-1})$, as illustrated in Figure 3. But $\bar{p}_1(\bullet)$ is continuous and monotone, and hence $\bar{x}_1(\bullet)/\bar{p}_1(\bullet)$ is continuous and monotone, implying that $\max_k|\bar{x}_1(c_{k-1})/\bar{p}_1(c_{k-1}) - \bar{x}_1(c_k)/\bar{p}_1(c_k)| \to 0$ as $N \to \infty$, and hence establishing uniform convergence.

(Insert Figure 3 about here)

Part II: Construction of an approximating mechanism, $\hat{p}$, for $\delta < 1$.

Consider the following system of 8N equations in the 8N unknowns $\{\hat{x}_1(a_k),\ \hat{x}_1(c_k),\ \hat{p}_1(a_k),\ \hat{p}_1(c_k),\ \hat{g}(a_k),\ \hat{g}(c_k),\ \hat{\beta}(c_k),\ \hat{p}_0(c_k)\}_{k=1}^{N}$:

(A.9):

(a) $\hat{x}_1(a_k) - \hat{x}_1(a_{k-1}) - a_{k-1}[\hat{p}_1(a_k) - \hat{p}_1(a_{k-1})] = 0$

(b) $\hat{x}_1(c_k) - \hat{x}_1(c_{k-1}) - c_{k-1}[\hat{p}_1(c_k) - \hat{p}_1(c_{k-1})] = 0$

(c) $-\hat{p}_1(a_k) + 1 - F_2(\hat{\beta}(a_k)) + \delta[F_2(\hat{\beta}(a_k)) - F_2(\hat{g}(a_k))] = 0$

(d) $-\hat{p}_1(c_k) + 1 - F_2(\hat{\beta}(c_k)) + \delta[F_2(\hat{\beta}(c_k)) - F_2(\hat{g}(c_k))] = 0$

(e) $-\hat{x}_1(a_k) + [1 - F_2(\hat{\beta}(a_k))]\hat{p}_0(a_k) + \delta[F_2(\hat{\beta}(a_k)) - F_2(\hat{g}(a_k))]\hat{g}(a_k) = 0$

(f) $-\hat{x}_1(c_k) + [1 - F_2(\hat{\beta}(c_k))]\hat{p}_0(c_k) + \delta[F_2(\hat{\beta}(c_k)) - F_2(\hat{g}(c_k))]\hat{g}(c_k) = 0$

(g) $-\tilde{B}_1(a_{k-1}) + \tilde{B}_1(a_k) + [F_1(a_{k-1}) - F_1(a_k)]\{\hat{x}_1(a_k) - \delta[1 - F_2(\hat{g}(a_k))]\hat{g}(a_k) - (1 - \delta)[1 - F_2(\hat{\beta}(a_k))]\hat{\beta}(a_k)\} = 0$

(h) $-\tilde{B}_1(c_{k-1}) + \tilde{B}_1(c_k) + [F_1(c_{k-1}) - F_1(c_k)]\{\hat{x}_1(c_k) - \delta[1 - F_2(\hat{g}(c_k))]\hat{g}(c_k) - (1 - \delta)[1 - F_2(\hat{\beta}(c_k))]\hat{\beta}(c_k)\} = 0,$

for $k = 1,\ldots,N$. As before, $\hat{\beta}(a_k) \equiv \hat{\beta}(c_k)$ and $\hat{p}_0(a_k) \equiv \hat{p}_0(c_k)$. $\tilde{B}_1(a_k)$ and $\tilde{B}_1(c_k)$ are constants determined in Part I. $\bar{p}_1(c_0)$ and $\bar{x}_1(c_0)$ are assigned the boundary values of zero.

A solution to (A.9) has the two-price interpretation if, in addition, $\hat{g}(c_k) \le \hat{\beta}(c_k) \le 1$ for $k = 1,\ldots,N$. In the initial period, sellers with valuations in $[a_k,a_{k-1}) \cup [c_k,c_{k-1})$ will be required to pool by offering the price $\hat{p}_0(c_k)$. In the following period (which is now discounted by $\delta$), sellers in $[a_k,a_{k-1})$ will be required to offer $\hat{g}(a_k)$ and sellers in $[c_k,c_{k-1})$ will offer $\hat{g}(c_k)$. The probability-of-trade calculations in (A.9c-d) and the revenue calculations in (A.9e-f) will be justified provided that a buyer with valuation $\hat{\beta}(c_k)$ is indifferent between the initial and second offers, i.e.,

$$\text{(A.10)} \qquad \hat{\beta}(c_k) - \hat{p}_0(c_k) = \delta\{\hat{\beta}(c_k) - \tilde{\theta}(a_k)\hat{g}(a_k) - [1 - \tilde{\theta}(c_k)]\hat{g}(c_k)\}.$$

However, substituting (A.9e) and (A.9f) into (A.9g) and (A.9h), respectively, and adding the resulting two equations implies (A.10).

It should be recalled that, in Part I, we constructed a solution to (A.9) for $\delta = 1$ and arbitrary grids. The implicit function theorem will immediately imply the existence of solutions to (A.9) for all $\delta$ contained in a nonempty interval $(\tilde{\delta}_N,1]$, provided that the Jacobian (with respect to the 8N unknowns) is nonzero. Voluminous calculations establish that this determinant, evaluated at $\delta = 1$, equals a (nonzero) scalar multiple of

$$\begin{aligned}\Pi_{k=1}^{N}\ [1 - F_2(\tilde{\beta}(c_k))]f_2(\tilde{\beta}(c_k))[\tilde{g}(c_k) - \tilde{g}(a_k)]& \\ \cdot\ [1 - F_2(\tilde{g}(a_k)) - f_2(\tilde{g}(a_k))(\tilde{g}(a_k) - a_{k-1})]& \\ \cdot\ [1 - F_2(\tilde{g}(c_k)) - f_2(\tilde{g}(c_k))(\tilde{g}(c_k) - c_{k-1})]&.\end{aligned}$$

Since $\tilde{\beta}(c_k) < 1$ for all k, we have $1 - F_2(\tilde{\beta}(c_k)) > 0$ and $f_2(\tilde{\beta}(c_k)) > 0$. Observe that $1 - F_2(\tilde{g}(c_k)) - f_2(\tilde{g}(c_k))(\tilde{g}(c_k) - c_{k-1}) = 0$ if and only if $\tilde{g}(c_k) = g^*(c_{k-1})$ (and analogously for $a_k$). Judicious choice of the grid $\{c_k\}_{k=1}^N$ together with Assumption 7.3 assure that this is not the case for any $c_k$; meanwhile, Assumption 7.3 also assures that there exists $\hat{N} \geq \tilde{N}$ such that this is not the case for any $a_k$, whenever $N \geq \hat{N}$.

Finally, recall that for $N \geq \tilde{N}$, $\tilde{\beta}(c_k) > \tilde{g}(c_k) + \varepsilon/2$ or $\tilde{x}_1(c_k)/\tilde{p}_1(c_k) > \tilde{p}_0(c_k) + \varepsilon/2$, for every $k = 1,\ldots,N$. Since each of $\hat{\beta}(\bullet)$, $\hat{g}(\bullet)$, $\hat{x}_1(\bullet)/\hat{p}_1(\bullet)$ and $\hat{p}_0(\bullet)$ are continuous in $\delta$, there exists for each N a value $\hat{\delta}_N \in (\tilde{\delta}_N,1)$ such that $\hat{\beta}(c_k) < 1$, and $\hat{\beta}(c_k) > \hat{g}(c_k)$ or $\hat{x}_1(c_k)/\hat{p}_1(c_k) > \hat{p}_0(c_k)$, for all $\delta \in (\hat{\delta}_N,1]$ and all $k = 1,\ldots,N$.

<u>Part III</u>: Construction of an approximating mechanism for $\delta < 1$ and $\lambda > 0$.

Consider the artificial game in which the seller must select from a menu of 2N price paths: $\hat{p}_0(c_k)$ in the initial period, followed by $\hat{g}(a_k)$ in all subsequent periods; and $\hat{p}_0(c_k)$ followed by $\hat{g}(c_k)$ forever $(k = 1,\ldots,N)$. If $\{\hat{p}_0(c_k),\hat{g}(a_k),\hat{g}(c_k)\}_{k=1}^N$ solved (A.9), it is incentive compatible for every seller in $[a_k,a_{k-1})$ and $[c_k,c_{k-1})$ to select her assigned price path. We will now modify the menu by replacing the price paths assigned to $[a_N,a_{N-1})$ and $[c_N,c_{N-1})$. Each seller in $[c_N,c_{N-1})$ will offer $p_0^*$ in the initial period, followed by a constant price path of $g_c^*$ in all subsequent periods. Each seller in $[a_N,a_{N-1})$ will also offer $p_0^*$ in the initial period, but will then follow an exponentially-descending price path in subsequent periods. In particular, if $s \in [a_N,a_{N-1})$, the seller will charge $e^{-\lambda(m-1)z}g_a^*$ in all periods $m \geq 1$ that this price exceeds s, and will charge

1 in all subsequent periods. Let us define $W(g_a^*,\lambda,\beta^*;s)$ to be the net present value of utility to the seller of valuation s from charging an initial price of $g_a^*$ and cutting price by a factor of $e^{-\lambda z}$ in each subsequent period (until price drops below s), if the initial buyer distribution is $F_2(\bullet)$ truncated at $\beta^*$ and if the buyer purchases optimally (under the beliefs that the seller distribution is $F_1(\bullet)$ truncated at $a_{N-1}$).

In order to preserve the incentive compatibility of the modified menu, it is sufficient to guarantee that the following system of four equations:

$$\text{(A.11a)} \quad \beta^* - p_0^* - \delta\{\beta^* - \tilde{\theta}(a_N)g_a^* - [1 - \tilde{\theta}(a_N)]g_c^*\} = 0$$

$$\text{(A.11b)} \quad [1 - F_2(\beta^*)][p_0^* - a_0] + \delta[F_2(\beta^*) - F_2(g_c^*)][g_c^* - a_0] - [\hat{x}_1(a_1) - a_0\hat{p}_1(a_1)] = 0$$

$$\text{(A.11c)} \quad [1 - F_2(\beta^*)][p_0^* - c_{N-1}] + \delta[F_2(\beta^*) - F_2(g_c^*)][g_c^* - c_{N-1}] - [\hat{x}_1(c_{N-1}) - c_{N-1}\hat{p}_1(c_{N-1})] = 0$$

$$\text{(A.11d)} \quad [1 - F_2(\beta^*)][p_0^* - a_{N-1}] + \delta W(g_a^*,\lambda,\beta^*,a_{N-1}) - [\hat{x}_1(a_{N-1}) - a_{N-1}\hat{p}_1(a_{N-1})] = 0,$$

is solved and that the implied $\bar{p}_1(\bullet)$ function on $[a_N,a_{N-1})$ satisfies $\lim_{s\uparrow a_{N-1}} \bar{p}_1(s) \geq \hat{p}_1(a_{N-1})$. The latter inequality will hold for a rectangle of pairs $(\lambda,\delta)$, since it holds strictly when $\lambda = 0$ and $\delta = 1$ (i.e., $\tilde{p}_1(a_N) > \tilde{p}_1(a_{N-1})$). Equation (A.11a) requires the buyer to respond optimally to $p_0^*$, (A.11b) makes $a_0$ (= $c_N$) indifferent between the price paths for $[a_1,a_0)$ and $[c_N,c_{N-1})$, (A.11c) makes $c_{N-1}$ indifferent between the paths for $[c_N, c_{N-1})$ and $[c_{N-1}, c_{N-2})$, and (A.11d) makes $a_{N-1}$ indifferent between the paths for $[a_N, a_{N-1})$ and $[a_{N-1}, a_{N-2})$.

It should be observed that, in Part I, we constructed a solution to (A.11) for $(\delta,\lambda) = (1,0)$ and for arbitrary grids. (Indeed, the solution was $(\beta^*, p_0^*, g_a^*, g_c^*) = (\tilde{\beta}(c_N), \tilde{p}_0(c_N), \tilde{g}(a_N), \tilde{g}(c_N))$.) Also, we established in Part II that for each N there exists $\hat{\delta}_N < 1$ such that, for every $\delta \in (\hat{\delta}_N, 1]$, the six terms $\hat{p}_1(c_{N-1}), \hat{x}_1(c_{N-1}), \hat{p}_1(a_1), \hat{x}_1(a_1), \hat{p}_1(a_{N-1})$ and $\hat{x}_1(a_{N-1})$ which appear in (A.11) may all be parameterized with respect to $\delta$. Thus, the implicit function theorem will immediately imply the existence of solutions to (A.11) for all pairs $(\delta,\lambda)$ contained in a nonempty rectangle $(\delta_N^*, 1] \times [0, \lambda_N^*)$, where $\delta_N^* \geq \hat{\delta}_N$, provided that the system (A.11) is continuously differentiable in the four unknowns $(\beta^*, p_0^*, g_a^*, g_c^*)$ and that the Jacobian is nonzero.

It is more difficult than one might guess to establish continuous differentiability, on account that it is not immediately obvious that $W(g_a^*, \lambda, \beta^*; s)$ is even differentiable in its first coordinate. This fact is demonstrated via the following lemma (whose proof is omitted here): for every $r > 0$, $\lambda > 0$, $z > 0$, $\beta^* > 0$, $s \geq 0$, and $g_a^* > s$, and under Assumption 4.1(a), if the price path $e^{-\lambda(m-1)z} g_a^*$ induces a positive probability of acceptance in the initial period ($m = 1$), then it also induces a positive probability of acceptance in all subsequent periods that $e^{-\lambda(m-1)z} g_a^*$ exceeds s. It immediately follows from this lemma that the lowest buyer valuation, $v_m$, to purchase in period m is given by indifference between consecutive prices, with proper discounting. Thus, $v_m - e^{-\lambda(m-1)z} g_a^* = \delta\{v_m - e^{-\lambda m z} g_a^*\}$, when $e^{-\lambda m z} g_a^* \geq a_{N-1}$. Meanwhile, $v_m - e^{-\lambda(m-1)z} g_a^* = [\delta F_1(e^{-\lambda m z} g_a^*)/F_1(e^{-\lambda(m-1)z} g_a^*)]\ \{v_m - e^{-\lambda m z} g_a^*\}$, when $e^{-\lambda(m-1)z} g_a^* \leq a_{N-1}$, since $F_1(e^{-\lambda m z} g_a^*)/F_1(e^{-\lambda(m-1)z} g_a^*)$ is the probability that the buyer will hear the offer $e^{-\lambda mz}g_a^*$ conditional on hearing $e^{-\lambda(m-1)z}g_a^*$. Finally, it should be observed that, under the conditions of the lemma, $W(g_a^*,\lambda,\beta^*;s) = [F_2(\beta^*) - F_2(v_1)][g_a^* - s] + \Sigma_{i=1}^{T(s)} \delta^i[F_2(v_i) - F_2(v_{i+1})][e^{-\lambda iz}g_a^* - s]$, where $T(s) \equiv \max\{i: e^{-\lambda iz}g_a^* \geq s\}$. By explicit calculation, $W(g_a^*,\lambda,\beta^*;s)$ is continuously differentiable in $g_a^*$, and it is then easy to conclude that the system (A.11) is continuously differentiable.

Direct calculation of the Jacobian at $(\delta,\lambda) = (1,0)$ yields the value $-f_2(\tilde{g}(c_N))\tilde{\theta}(c_N)[\tilde{g}(c_N) - \tilde{g}(a_N)][1 - F_2(\tilde{g}(a_N)) - f_2(\tilde{g}(a_N))(\tilde{g}(a_N) - a_{N-1})]$. As argued in Part II, Assumption 7.3 assures that this Jacobian is nonzero for $N \geq \hat{N}$. Consequently, for each $N \geq \hat{N}$, there exists $\delta_N^* < 1$ and $\lambda_N^* > 0$ such that the implicit function theorem is applicable on the rectangle $(\delta_N^*,1] \times [0,\lambda_N^*)$.

Finally, recall that for $N \geq \tilde{N}$, $\tilde{\beta}(c_N) > \tilde{g}(c_N) + \varepsilon/2$ or $\tilde{x}_1(c_N)/\tilde{p}_1(c_N) > \tilde{p}_0(c_N) + \varepsilon/2$ and also $\tilde{\beta}(c_N) < 1$. Since each of $\beta^*$, $p_0^*$, $g_a^*$, and $g_c^*$ are jointly continuous in $\delta$ and $\lambda$, there exist for every $N \geq \hat{N}$ values $\delta_N^{**} \in (\delta_N^*,1)$ and $\lambda_N^{**} \in (0,\lambda_N^*)$ such that $g_c^* < \beta^* < 1$ for all pairs $(\delta,\lambda) \in (\delta_N^{**},1] \times [0,\lambda_N^{**})$. Thus, for every $(\delta,\lambda)$ contained in this rectangle, we have shown the existence of an approximating mechanism, $p^*_{N,\delta,\lambda}$, which has the two-price interpretation.

Part IV: Construction of the stationary sequential equilibrium.

Let us begin the construction of equilibria by assuring that our initial choices for the grid $\{c_k\}_{k=0}^N$ had the property that the N implied values $\tilde{p}_0(c_1),\ldots,\tilde{p}_0(c_N)$ were all different. (Observe that for generic choices of the grid, the N values are different, but in the nongeneric case where $\tilde{p}_0(c_j) = \tilde{p}_0(c_k)$ for some $j \neq k$, it will be necessary to perturb the grid.) Also, if necessary, redefine $\delta_N^{**}$ closer to 1 and $\lambda_N^{**}$ closer to 0, so that for all pairs $(\delta,\lambda) \in (\delta_N^{**},1] \times [0,\lambda_N^{**})$, the N values $\hat{p}_0(c_1),\ldots,\hat{p}_0(c_{N-1})$, $p_0^*$ are all different. This will enable the seller's initial offer, in the equilibrium we construct below, to fully convey the fact that $s \in [a_k,a_{k-1}) \cup [c_k,c_{k-1})$.

For any $\delta < 1$, let $\sigma_\delta$ be a weak-Markov equilibrium in the seller-offer game with discount factor $\delta$ between periods, where the seller's valuation is commonly known to equal zero and the buyer's valuation is distributed according to $F_2(\cdot)$. Existence of weak-Markov equilibria is guaranteed by theorems of Fudenberg, Levine and Tirole (1985, Proposition 2), or Ausubel and Deneckere (1986, Theorem 4.2). Let $\sigma_\delta^b$ denote the buyer's strategy in $\sigma_\delta$. We may now specify the equilibrium strategies:

<u>Seller's strategy</u>:

- If there has been no prior seller deviation, follow the price path specified for seller type s in the artificial game of the first paragraph of Part III. If $s \in [\hat{s},1]$, charge a price of 1 forever.
- If there has been an undetectable seller deviation (but no detectable seller deviation), follow the price path which maximizes utility subject to keeping the deviation undetectable. If that involves pricing below cost, charge a price of 1 instead.
- If there has been a detectable seller deviation, optimize against a buyer strategy of $\sigma_\delta^b$.

<u>Buyer's strategy</u>:

- If there has been no prior detectable seller deviation, optimize for buyer type b against the seller's chosen price path and the induced beliefs about the seller's type.

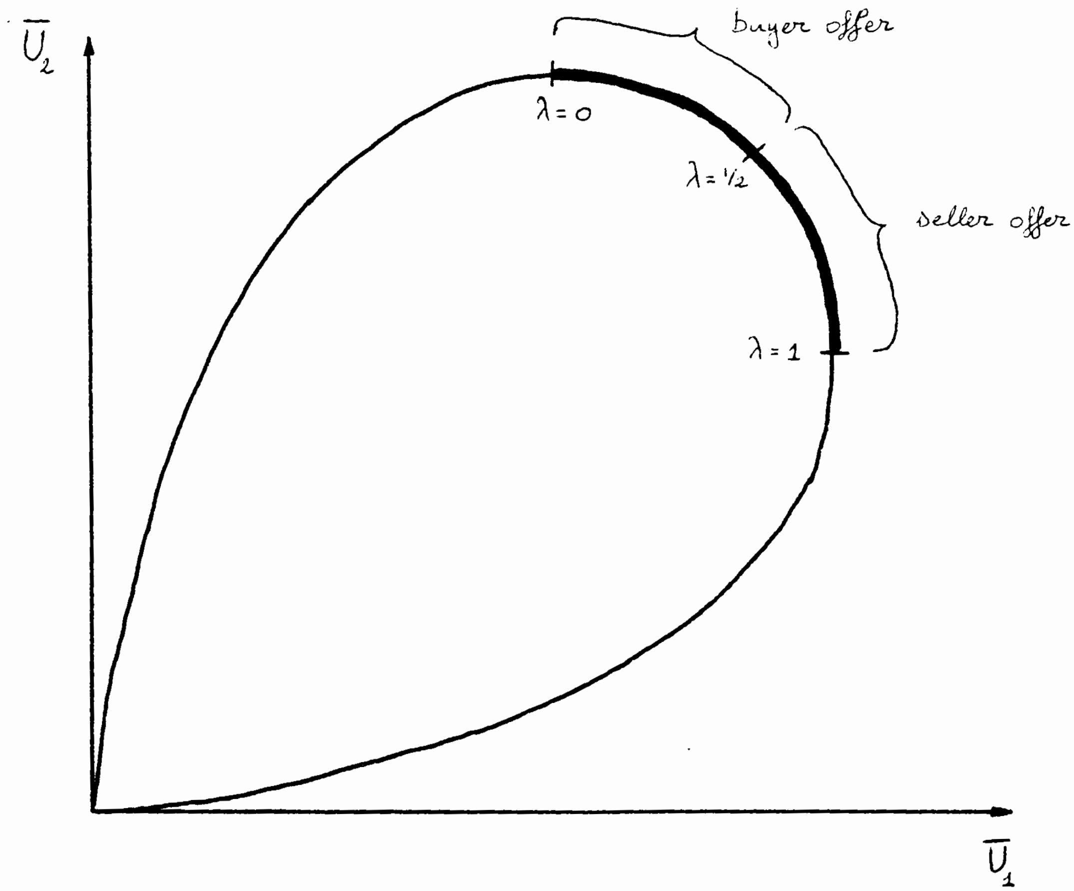
- If there has been a detectable seller deviation, update beliefs to $s = 0$ and maintain these beliefs forever after. Accept or reject using the strategy $\sigma_{\delta}^{b}$.

For sufficiently large N, we constructed in Part III a rectangle $(\delta_N^{**},1] \times [0,\lambda_N^{**})$ where the implicit function theorem was applicable. Select any $\lambda_N \in (0,\lambda_N^{**})$. Using the same argument as in Ausubel and Deneckere (1988b, proof of Theorem 3, Part III), there exists $\delta_N^{***}(\lambda_N) \in (\delta_N^{**},1)$ such that, if $\delta_N$ satisfies $\delta_N^{***}(\lambda_N) < \delta_N < 1$, the weak-Markov strategy $\sigma_{\delta}^{b}$ is sufficiently severe to deter all detectable seller deviations. It is straightforward to verify that our mechanism construction precludes undetectable deviations as well.

Selecting $\lambda_N$ and $\delta_N$ so that $\lambda_N \downarrow 0$ and $\delta_N \uparrow 1$, we see that the sequence of mechanisms $p^*_{N,\delta_N,\lambda_N}$ induced by our constructed sequential equilibria converges in measure to our original p, as $N \to \infty$. Thus, the mechanism p is implementable by sequential equilibria in the seller-offer game.

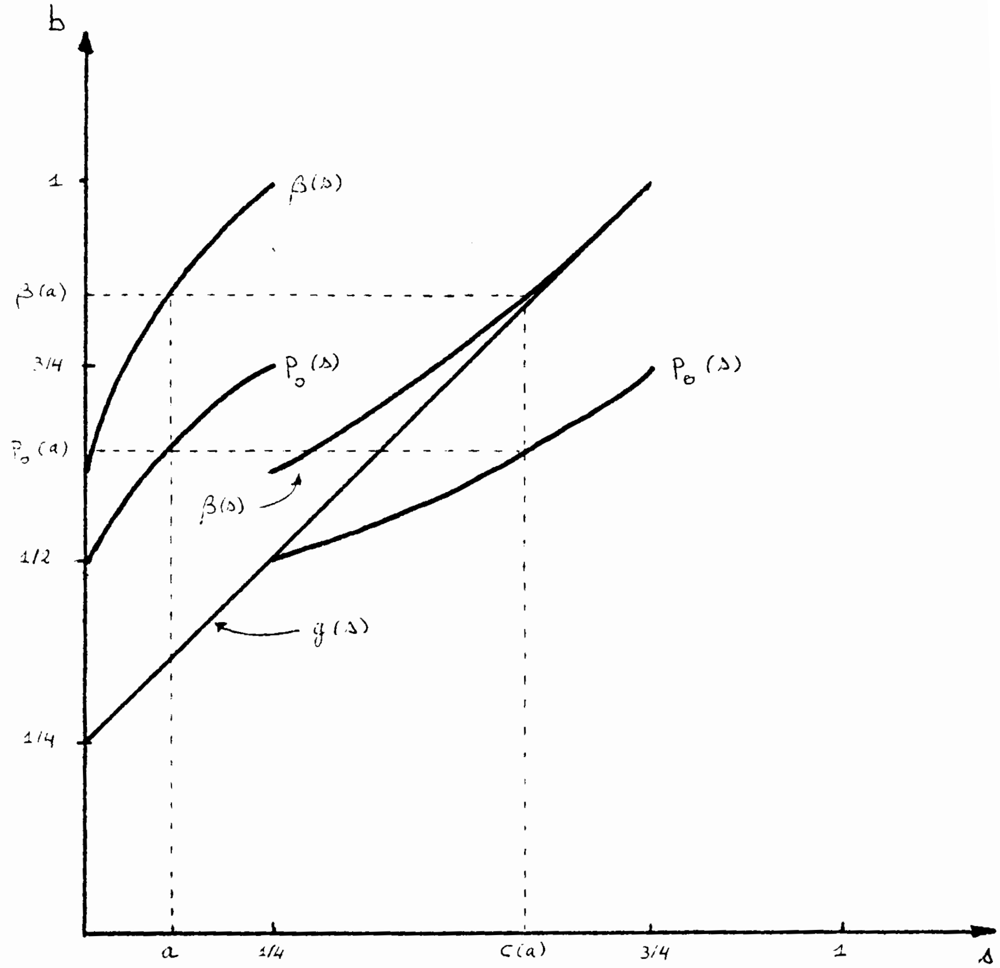
Moreover, in the constructed sequential equilibria, in any period in which the players' beliefs are the same as in the previous period, the players' equilibrium actions are also the same as in the previous period (and will continue to be henceforth). Players' updating rules are also stationary, establishing that the equilibria utilized in showing implementation are also stationary. []

Figure 1



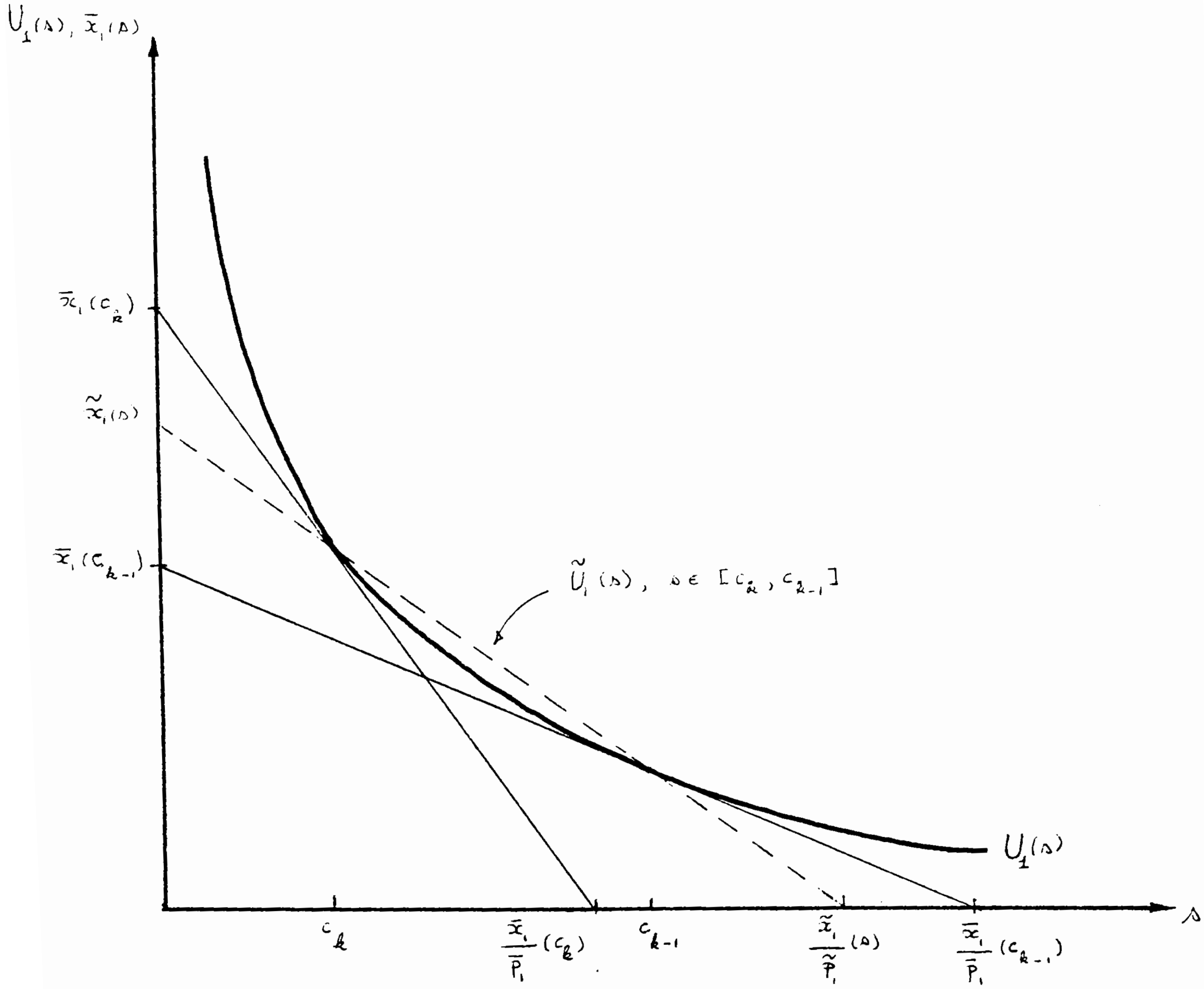

Ex ante feasible utility set and portion of the Pareto frontier implementable in the seller- and buyer-offer game (where $\overline{U}_i = \int_0^1 U_i(z) f_i(z) dz$).

Figure 2




The two-price splitting of the Chatterjee-Samuelson mechanism.

Figure 3




Graphical interpretation of $\bar{x}_1(s)$ and $\bar{x}_1(s)/\bar{p}_1(s)$.