Discussion Paper No. 803

SMALL TALK AND COOPERATION: A NOTE ON BOUNDED RATIONALITY

by

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November 1988

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Acknowledgment: I wish to acknowledge Ehud Kalai, the two referees and the associate editor for many helpful remarks and suggestions.

Abstract

Neyman has shown that bounded rationality can lead to cooperation in the finitely repeated prisoner's dilemma game, if the game is conducted by finite automata of fixed size. We obtain similar results utilizing finite automata which can send messages back and forth over a given communication channel. The communication protocol utilized does not involve the transfer of any relevant information. Rather, it saturates the computational resources of the players thus preventing them from engaging in complex strategies which could potentially undermine cooperation.

1. Introduction

The effect of bounded computational power on the behavior of players in a competitive situation has been recently analyzed in a variety of settings, e.g., Aumann [2], Rubinstein [9], Abreu and Rubinstein [3], Ben Porath [5], Kalai and Stanford [6], Megiddo and Wigderson [7], etc. In particular, Neyman [8] has shown that bounded rationality can lead to cooperation in the finitely repeated prisoner's dilemma game where the play is conducted by finite automata of fixed size. In this note we view Neyman's results from the perspective of communication. Specifically, we demonstrate that the ability to communicate, in the context of bounded rationality, enables players to cooperate, i.e., to reach outcomes which are not attainable otherwise.

We refer to the mode of communication which is utilized in our analysis as "small talk." It is characterized by the fact that no relevant information, whether true or misleading, is being exchanged. (The setup is of complete information.) Similarly, messages are not used for coordination of moves. Rather, communication is used to distract the players for the purpose of preventing them from engaging in complex strategies which could potentially undermine cooperation ("cheating"). To that end, meaningless messages are sent back and forth according to a rigid protocol which is designed in such a way that a player cannot "cheat" and send the right message at the same time. For a rival, who is considering whether cooperation is "safe," a successful execution of the protocol can serve as a "guarantee" that no "cheating" is being contemplated. The following example demonstrates the general principle.

Consider a card game such as "Black Jack." It is well-known that

We begin by introducing the results of [8]. For a given n player game G, let G^N denote the N stage repeated version of G, with the average payoff as evaluation criteria. Denote the set of actions available to player i at each stage by A^i, to his opponents by A^{-i}, and to the entire set of players by A. A finite automaton for player i is a four-tuple $FA^i = \langle S^i, s^i, f^i, g^i \rangle$ where S^i is a finite set, $s^i \in S^i$, $F^i: S^i \to A^i$ and $g^i: S^i \times A^{-i} \to S^i$. In words, S^i is the set of states of the automaton, s^i is the initial state, $f^i(s)$ is the action taken by player i when in state s, and g^i describes the transition from state to state: if at state s the other players choose the action tuple a^{-i}, the automaton's next state is $g^i(s, a^{-i})$. The size of a finite automaton is the number of states. For any N = 1,2,... and positive integers $s_1, \ldots, s_n$, define the n person game $G^N(s_1, s_2, \ldots, s_n)$ as follows: the pure strategies of player i, i = 1,...,n, are all the finite automata of size s_i.

Neyman analyzes in particular the prisoner's dilemma game PD with the action set $A^i = (F,D)$ (friendly and deviating) and the payoff matrix:

	D_2	F_2
D_1	1,1	4,0
F_1	0,4	3,3

Let $s_{max} = \max\{s_1, s_2\}$ and $s_{min} = \min\{s_1, s_2\}$. It is well-known that the only

A.4 For any integer k there is N_0 such that if $N \geq N_0$, and $N^{1/k} \leq s_1, s_2 \leq N^k$, there is a mixed strategy equilibrium for $PD^N(s_1, s_2)$ in which the expected average payoff to each player is at least 3 - 1/k.

The equilibrium of A.4 works roughly as follows. The simple trigger strategies utilized in A.1 are modified so that a player with more than N states is expected to play according to a complex sequence of D and F moves (rather than the simple sequence of F moves as in A.1). This sequence of moves is designed so that its execution "wastes" an appropriate number of simpler states, leaving a player following it correctly with less than N free states. Thus, the arguments of A.1 can be made and the (complex) set of trigger strategies is in equilibrium. The factor of -1/k present in the payoff achieved in A.4 is due to need to utilize some undesirable D moves in order to achieve the right amount of "complexity" in the sequence. Below we demonstrate how the same effect can be achieved in a simpler way by utilizing finite automata which can communicate. Specifically, we now depart from the previously mentioned models by allowing our finite automata to communicate with each other: at each stage t player i chooses an action a_t^i and sends a message $m_t^i \in M$, where M is a finite set. To accommodate messages, we redefine the transition and actions functions, g and f, as follows. The domain of g^i is enlarged so that g^i: $S^i \times A^{-i} \times M \rightarrow S^i$ (i.e., a transition depends on i's state, and on -i's action and message.) Similarly, the range of f is modified so that f^i: $S^i \rightarrow A^i \times M$, i.e., at each state, both an action and a message are chosen. We let $G^N(s_1, \ldots, s_n: M)$ be the game where the pure strategies to player i are finite automata of size s_i, which can communicate over the message space M.

Q^2 can be chosen in such a way that a player with a given number of states cannot expect to be able to "count to N" and echo correctly at the same time (although he can perform each one of the tasks separately). It follows that the best response of each player is to cooperate throughout the game.

We now spell out the proof of the theorem in detail. For each message $q \in M$ and subset $Q \subseteq M$ consider the (pure) strategy for player i, $S^i(q,Q)$, as follows:

Strategy $S^2(q,Q)$:

1) At stage 1:

1a) Send the message q: $m_1^i = q$

1b) Act friendly: $a_1^i = F$

2) At stages $2 \leq t \leq N$, if player $-i$ is "conforming":

2a) Echo $-i$'s original message: $m_t^i = m_1^{-i}$

2b) Act friendly: $a_t^i = F$

3) At stage $2 \leq t \leq N$, if player $-i$ is "not conforming":

3a) Send the original message: $m_t^i = q$

3b) Deviate: $a_t^i = D$

Player $-i$ is considered <u>conforming</u> as long as the following conditions hold:

(i) At stage 1, player's $-i$ message is chosen from the right set Q, i.e. $m_1^{-i} \in Q$.

(ii) At stage $2 \leq t \leq N - 1$, player $-i$ echoes correctly, i.e., each player is 3. A player with unlimited computational power can do better than that against an opponent playing T. However, the number of states needed can be rather large:

Lemma 2: Let A be any automaton for player $-i$ which achieves an expected average payoff of more than 3 against $T^i(R,Q)$. Then, for $N \geq 4$, A must have at least $|R| + N - 2$ states.

The proof of Lemma 2 is given in the next section.

We are now in a position to complete the proof of the theorem. Recall that player i, $i = 1,2$, is restricted to strategies which can be implemented by automata of size at most s_i. Let x_i, $i = 1,2$, be an integer satisfying

$$s_i - N + 2 < x_i \leq s_i - 2.$$

Such numbers x_i clearly exist, say $x_i = s_i - N + 3$. Let $Q_i \subseteq M$ be specified subsets of messages which satisfy $|Q_i| = x_i$, $i = 1,2$. Consider the pair of strategies $T^1(Q_2,Q_1)$ and $T^2(Q_1,Q_2)$. By Lemma 1, each strategy $T^i(Q_{-i},Q_i)$ can be implemented by an automaton requiring $x_i + 2 \leq s_i$ states. Also, strategy $T^{-i}(Q_i,Q_{-i})$ is the best response to $T^i(Q_{-i},Q_i)$ since by Lemma 2, player $-i$ requires at least $x_{-i} + N - 2 > s_{-i}$ states to improve. Thus, the two strategies constitute an equilibrium for $G_N(s_1,s_2; M)$ as asserted.

3. Proofs of Lemmas 1 and 2

Proof of Lemma 1: Lable the messages of Q as $\{q_1,q_2,\ldots,q_{|Q|}\}$ and consider the following automaton FA(q,Q) with $|Q| + 2$ states labeled $0,\ldots,|Q| + 1$, states of A into the following disjoint subsets

$S = \{s\}$ = starting state.

D = all states other than s in which the action is D.

F_q = All states other than s in which the action is F and the message is q.

Let $f_q = |F_q|$. Note that if A is to achieve an average payoff of more than 3 against T then $|D| \geq 1$. Also, by definition, $|S| = 1$. Thus, $\Sigma_{q \in R} f_q \leq \Sigma_{q \in M} f_q \leq p - 2$.

Let the message sent by A in stage 1 be denoted $q^* \in M$. Partition the set of possible messages of player i, R, into the following four disjoint (possibly empty) sets:

X_1: $\{q \in R: \quad f_q \geq N - 2\}$

X_2: $\{q \in R: \quad N - 2 > f_q \geq 1\}$

X_3: $\{q \in R: \quad f_q = 0, \ q = q^*\}$

X_4: $\{q \in R: \quad f_q = 0, \ q \neq q^*\}$.

We note the following maximal (average) payoffs to player -i when A plays against FA(q,Q), j ∈ R:

1. The maximal attainable payoff is 3 + 1/N. This is possible only if $q \in X_1$.
2. The maximal payoff is 3 if $q \in X_2 \cup X_3$.
3. The maximal payoff is 1 + 5/N if $q \in X_4$.

(1') $3 + (1/N|R|)(x_1 - (2N - 5)x_4)$.

Thus, we can prove the lemma by showing that the maximal value of $(x_1 - (2N - 5)x_4)$ subject to (2)-(5) is zero. This can be done by straightforward substitutions. []

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Consider a card game such as "Black Jack." It is well-known that players who "count cards" in such games can achieve higher payoffs than could be sustained in a long term equilibrium. One could prevent counting by frequent reshuffling of the deck or by expelling apparent offenders. but there is also another way. Specifically, one can require players to perform, from time to time, some simple memory or arithmetic tasks such as repeating long sequences of digits, adding or subtracting large integers, etc. Such tasks, while easily done on their own, can be designed such that they are excessively difficult if one is also concentrating on counting cards. Thus, a successful performance of the task can serve as a proof that a player is not counting. On the whole, the existence of such a proof can benefit all parties involved.

We follow in this note the basic framework and notation of Neyman [8]. The only new feature we add here is equipping the players with a formal channel of communication which allows them to send messages to each other, concurrently with the actual moves of the game. As will be revealed shortly, the ability to communicate offers several advantages. First, it allows the players to achieve complete cooperation, avoiding the waste which is inherent in the scheme of [8]. Also, the approach utilizes a simple communication protocol, which is independent of the game being played and which can be used universally in situations in which it is desirable to waste a certain fraction of one's opponent's computational power. Finally, the analysis itself is quite simple and one can get exact (as opposed to asymptotic) results.

An earlier version of this paper was circulated as [10], which also addresses the case of a general n-person game ($n \neq 3$) and considers communication protocols which utilize the message space more efficiently.

We begin by introducing the results of [8]. For a given n player game G, let G^N denote the N stage repeated version of G, with the average payoff as evaluation criteria. Denote the set of actions available to player i at each stage by A^i, to his opponents by A^{-i}, and to the entire set of players by A. A finite automaton for player i is a four-tuple $FA^i = \langle S^i, s^i, f^i, g^i \rangle$ where S^i is a finite set, $s^i \in S^i$, $F^i: S^i \to A^i$ and $g^i: S^i \times A^{-i} \to S^i$. In words, S^i is the set of states of the automaton, s^i is the initial state, $f^i(s)$ is the action taken by player i when in state s, and g^i describes the transition from state to state: if at state s the other players choose the action tuple a^{-i}, the automaton's next state is $g^i(s, a^{-i})$. The size of a finite automaton is the number of states. For any N = 1,2,... and positive integers $s_1, \ldots, s_n$, define the n person game $G^N(s_1, s_2, \ldots, s_n)$ as follows: the pure strategies of player i, i = 1,...,n, are all the finite automata of size s_i.

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	D_2	F_2
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Let $s_{max} = \max\{s_1, s_2\}$ and $s_{min} = \min\{s_1, s_2\}$. It is well-known that the only equilibrium strategy in PD^N is to deviate continuously. However, for $PD^N(s_1,s_2)$, Neyman has shown:

A.1 If $s_{min} \geq 2$, $s_{max} \leq N - 1$, then there are equilibrium strategies in $PD^N(s_1,s_2)$ which result in the play (F,F) at each stage.

However,

A.2 If $s_{max} \geq N$, there are no equilibrium strategies in $PD^N(s_1,s_2)$ which result in the play (F,F) at each stage.

A.3 [7] If $s_{min} \geq N$ then <u>no fixed</u> trajectory of moves, except for the constant play of (D,D), can be achieved as a result of an equilibrium of $PD^N(s_1,s_2)$.

It is easy to see how the equilibrium specified in A.1 is achieved utilizing a pair of simple, identical trigger strategies. Each player acts friendly as long as his opponent does, but reverts to the constant D play upon the first deviation by the other player. The only improving response to such a trigger strategy is to play friendly for the first N - 1 stages and then deviate at stage N. However, such a response requires a finite automaton with at least N states. A.2 and A.3 can be traced to similar arguments. The main contribution of [8] concerns the case of machines with more than N states. Its effect is, asymptotically, to mitigate A.2 and A.3 considerably using <u>mixed</u> strategies:

A.4 For any integer k there is N_0 such that if $N \geq N_0$, and $N^{1/k} \leq s_1, s_2 \leq N^k$, there is a mixed strategy equilibrium for $PD^N(s_1, s_2)$ in which the expected average payoff to each player is at least 3 - 1/k.

The equilibrium of A.4 works roughly as follows. The simple trigger strategies utilized in A.1 are modified so that a player with more than N states is expected to play according to a complex sequence of D and F moves (rather than the simple sequence of F moves as in A.1). This sequence of moves is designed so that its execution "wastes" an appropriate number of simpler states, leaving a player following it correctly with less than N free states. Thus, the arguments of A.1 can be made and the (complex) set of trigger strategies is in equilibrium. The factor of -1/k present in the payoff achieved in A.4 is due to need to utilize some undesirable D moves in order to achieve the right amount of "complexity" in the sequence. Below we demonstrate how the same effect can be achieved in a simpler way by utilizing finite automata which can communicate. Specifically, we now depart from the previously mentioned models by allowing our finite automata to communicate with each other: at each stage t player i chooses an action a_t^i and sends a message $m_t^i \in M$, where M is a finite set. To accommodate messages, we redefine the transition and actions functions, g and f, as follows. The domain of g^i is enlarged so that g^i: $S^i \times A^{-i} \times M \rightarrow S^i$ (i.e., a transition depends on i's state, and on -i's action and message.) Similarly, the range of f is modified so that f^i: $S^i \rightarrow A^i \times M$, i.e., at each state, both an action and a message are chosen. We let $G^N(s_1, \ldots, s_n: M)$ be the game where the pure strategies to player i are finite automata of size s_i, which can communicate over the message space M.

Theorem: Let $s_{min} \geq 3$, $N \geq 5$. Then for $|M| \geq s_{max} - N + 3$ there exists an equilibrium of $PD^N(s_1, s_2; M)$ in which the average payoff to each player is 3.

Note that the theorem implies that both players act cooperatively in each stage. We devote the next section to the proof of the theorem.

2. Analysis

We start this section with an informal outline of the strategies which supports the equilibrium of the theorem. Basically, the (simple) trigger strategies of A.1 are modified in order to waste the excess states as in A.4. However, in contrast to A.4, the modification involves the communication protocol rather than the sequence of actions. Thus, as in A.1, each player is expected to act F continuously. At the same time players are expected to follow precisely a given communication protocol. Failure to perform either one of these requirements causes the opponent to trigger a constant D action. The communication protocol utilized is rather Simple. First, each player i chooses randomly a message m_1^i from a given subset Q^i of the message space M. Each player then uses an automaton, FA^i, which sends the chosen message m_1^i in stage 1. In subsequent stages, the automaton echoes the opponent message, i.e., sends back the message m_1^{-i}. A player i is considered nonconforming to the requirements of the protocol if his original message is not chosen from the given message subset Q^i, or if in subsequent stages he echoes incorrectly the message m_1^{-i}. The proof of the theorem now reduces to a demonstration that that subsets Q^1 and Q^2 can

Q^2 can be chosen in such a way that a player with a given number of states cannot expect to be able to "count to N" and echo correctly at the same time (although he can perform each one of the tasks separately). It follows that the best response of each player is to cooperate throughout the game.

We now spell out the proof of the theorem in detail. For each message $q \in M$ and subset $Q \subseteq M$ consider the (pure) strategy for player i, $S^i(q,Q)$, as follows:

Strategy $S^2(q,Q)$:

1) At stage 1:

   1a) Send the message q: $m_1^i = q$

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   3a) Send the original message: $m_t^i = q$

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Player -i is considered <u>conforming</u> as long as the following conditions hold:

(i) At stage 1, player's -i message is chosen from the right set Q, i.e. $m_1^{-i} \in Q$.

(ii) At stage $2 \le t \le N - 1$, player -i echoes correctly, i.e.,

$m_t^{-i} = q$.

(iii) At stage $\ell \le t \le N - 1$ player $-i$ acts friendly, i.e., $a_t^{-i} = F$.

If any of the conditions (i)-(iii) are violated by player -i, then he is considered nonconforming for the rest of the game.

Recall that we have identified pure strategies with finite automata and that the size of a finite automata is the number of states. Lemma 2 specifies the size of the automaton needed to implement S(q,Q):

Lemma 1: S(q,Q) can be implemented by a finite automaton of size $|Q| + 2$.

Lemma 1 is proved in Section 3. We denote the automaton used in this Lemma by FA(q,Q). The equilibrium of Theorem 1 is achieved by automata of this type where the initial messages q are chosen randomly from appropriate sets. Specifically, for two nonempty message sets $\emptyset \neq R,Q \subseteq M$ consider the randomized strategy $T^i(R,Q)$ for player :

Strategy $T^i(R,Q)$:

1. Choose a message q from the set R according to the uniform distribution.
2. Play with the automaton FA(q,Q), i.e., according to the pure strategy $S^i(q,Q)$.

Clearly, when two automata use strategies of the type T against each other, the resulting action in each stage is F and the average payoff to each player is 3. A player with unlimited computational power can do better than that against an opponent playing T. However, the number of states needed can be rather large:

Lemma 2: Let A be any automaton for player -i which achieves an expected average payoff of more than 3 against $T^i(R,Q)$. Then, for $N \geq 4$, A must have at least $|R| + N - 2$ states.

The proof of Lemma 2 is given in the next section.

We are now in a position to complete the proof of the theorem. Recall that player i, i = 1,2, is restricted to strategies which can be implemented by automata of size at most s_i. Let x_i, i = 1,2, be an integer satisfying

$$s_i - N + 2 < x_i \leq s_i - 2.$$

Such numbers x_i clearly exist, say $x_i = s_i - N + 3$. Let $Q_i \subseteq M$ be specified subsets of messages which satisfy $|Q_i| = x_i$, i = 1,2. Consider the pair of strategies $T^1(Q_2,Q_1)$ and $T^2(Q_1,Q_2)$. By Lemma 1, each strategy $T^i(Q_{-i},Q_i)$ can be implemented by an automaton requiring $x_i + 2 \leq s_i$ states. Also, strategy $T^{-i}(Q_i,Q_{-i})$ is the best response to $T^i(Q_{-i},Q_i)$ since by Lemma 2, player -i requires at least $x_{-i} + N - 2 > s_{-i}$ states to improve. Thus, the two strategies constitute an equilibrium for $G_N(s_1,s_2; M)$ as asserted.

3. Proofs of Lemmas 1 and 2

Proof of Lemma 1: Lable the messages of Q as $\{q_1,q_2,\ldots,q_{|Q|}\}$ and consider the following automaton FA(q,Q) with $|Q| + 2$ states labeled $0,\ldots,|Q| + 1$.

with the starting state being 0:

1. Actions and Messages (the Function f):

Actions: In states $0, 1, \ldots, |Q|$, the action is F.

In state $|Q| + 1$, the action is D.

Messages: In states $j = 1, \ldots, |Q|$, the message is q_j.

In states 0, $|Q| + 1$, the message is q.

Transitions (the Function g):

Beginning State (0):

a. If $a_1^{-i} = D$ or $m_1^{-i} \notin Q$: move to state $|Q| + 1$.

b. Otherwise, $m_1^{-i} = q_j$ for some $j \leq r$: move to state j.

Punishing State ($|Q| + 1$):

Stay in state $|Q| + 1$.

Play state ($1 \leq j \leq |Q|$):

a. If $a_t^{-i} = D$ or $m_t^{-i} \neq q$: move to state $|Q| + 1$.

b. Otherwise: remain in state j.

It is a simple matter to verify that FA(q,Q) in fact implements $S^i(q,Q)$. []

Proof of Lemma 2: Let $p = |R| + N - 3$ and assume, on the negative, the existence of an automaton A with p states which achieves an expected average value for player $-i$ of more than 3 against $T^i(R,Q)$. Partition the set of states of A into the following disjoint subsets

$S = \{s\}$	=	starting state.
D	=	all states other than s in which the action is D.
F_q	=	All states other than s in which the action is F and the message is q.

Let $f_q = |F_q|$. Note that if A is to achieve an average payoff of more than 3 against T then $|D| \geq 1$. Also, by definition, $|S| = 1$. Thus, $\Sigma_{q\in R} f_q \leq \Sigma_{q\in M} f_q \leq p - 2$.

Let the message sent by A in stage 1 be denoted $q^* \in M$. Partition the set of possible messages of player i, R, into the following four disjoint (possibly empty) sets:

X_1: $\{q \in R: \quad f_q \geq N - 2\}$

X_2: $\{q \in R: \quad N - 2 > f_q \geq 1\}$

X_3: $\{q \in R: \quad f_q = 0,\ q = q^*\}$

X_4: $\{q \in R: \quad f_q = 0,\ q \neq q^*\}$.

We note the following maximal (average) payoffs to player -i when A plays against FA(q,Q), j $\in$ R:

1. The maximal attainable payoff is 3 + 1/N. This is possible only if $q \in X_1$.
2. The maximal payoff is 3 if $q \in X_2 \cup X_3$.
3. The maximal payoff is 1 + 5/N if $q \in X_4$.

3. is due to the fact that $FA^i(q,Q)$ expects A to echo the message q and play F in stages 2,...,N. However, A does not have any state with this action-message combination, since $f_q = 0$ and $q^* \neq q$. Consequently, FA declares A as nonconforming at the latest in stage 2, and therefore will play D against A in stages 3,...,N. Thus, the maximum average payoff to A is $(1/N)(7 + N - 2)$ where 7 represents the maximal amount A can achieve in stages 1 and 2 (play F first, then D).

Denote the cardinalities of X_i by x_i, i = 1,...,4. Then the expected average payoff to player -i is bounded from above by

(1) $$(1/N|R|)[(3N + 1)x_1 + 3Nx_2 + 3Nx_3 + (N + 5)x_4]$$

where the x_i must satisfy

(2) $$x_1 + x_2 + x_3 + x_4 = |R|$$

(3) $$(N - 2)x_1 + x_2 \leq p - 2 = |R| + N - 5$$

(4) $$x_3 \leq 1$$

(5) $$x_1, x_2, x_3, x_4 \geq 0, \text{ integers.}$$

(2) is due to the fact that the X_i induce a partition of $|R|$. (3) reflects the fact that for each message $q \in X_1$ we need at least N - 2 states of A and for each $q \in X_2$ we need at least one state. On the other hand, the total of available states is $\Sigma f_q \leq p - 2$. Finally, (4) is due to the fact that there can be at most one q such that $q = q^*$. Note that (1) can be written in the form:

(1') $3 + (1/N|R|)(x_1 - (2N - 5)x_4)$.

Thus, we can prove the lemma by showing that the maximal value of $(x_1 - (2N - 5)x_4)$ subject to (2)-(5) is zero. This can be done by straightforward substitutions. []

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