A Note on

Returns to Group Size and the Core with Public Goods*

by

Donald John Roberts

Department of Managerial Economics and Decision Sciences
and
Center for Mathematical Studies in Economics and Management Science

Northwestern University
Evanston, Illinois, 60201 U.S.A.

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Address for Proofs:

Until September 1, 1974:

D. J. Roberts
Department of Managerial Economics and Decision Sciences
Graduate School of Management
Northwestern University
Evanston, Illinois, 60201 USA

After September 1, 1974:

D. J. Roberts
CORE
de Croylaan 54
3030 Heverlee
Belgium

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Two recent papers ([2],[4]) have presented counter-examples to possible extensions of the core-competitive equilibrium equivalence theorem. This note points out a close relationship between crucial aspects of these examples and investigates some resulting questions relating to the core and the Lindahl equilibrium of economies with semi-public goods.

In [4], T. Muench posed the question of whether the core and the Lindahl allocations might coincide in economies with pure public goods and a continuum of infinitesimal consumers. He provided a very convincing counter-example to such a modified Edgeworth conjecture. In his example, the Lindahl equilibrium is unique, while the core is very large. More recently, V. Boehm [1] considered another possible extension of the equivalence theorem to the case of private goods economies where the correspondence between coalitions and the production sets available to them is super-additive, i.e., where there exist disjoint coalitions S_1 and S_2 such that $Y(S_1) + Y(S_2)$ is a proper subset of $Y(S_1 \cup S_2)$. He showed that such super-additivity, which we might call "increasing returns to group size," can prevent the equivalence theorem from holding.

In Boehm's example there are two private goods, with an endowment of only the first one. The production set correspondence is defined for each coalition S by

$$Y_B(S) = \{(z_1, z_2) \in R^2 \mid a\mu(S \cap S^*)\, z_1 + z_2 \leqq 0,\ z_2 \geqq 0\}$$

where a is a scalar, μ denotes the Lebesgue measure on $T = [0,1]$, the set of agents, and S^* is a distinguished coalition [1] in T. In Muench's example, on the other hand, all coalitions have access to the same constant returns to scale production set, so that the correspondence is a constant

$$Y_M(S) = \{(z_1, z_2) \in R^2 \mid az_1 + z_2 \leqq 0,\ z_2 \geqq 0\},$$

where the first coordinate of a commodity space pair refers to the private good and the second to the pure public good. Yet, despite the constancy of the production correspondence, the Muench example actually involves a form of super-additivity very closely related to that in Boehm's example.

To see this, consider more explicitly the distribution of the inputs to and the outputs of production over the members of the coalitions in each model. Then, letting $\underset{\sim}{z}_1$ and $\underset{\sim}{z}_2$ denote such functions, we can describe the technology available to a coalition S in the Boehm example by [2]

$$V_B(S) = \{(z_1, z_2) \in R^2 \mid z_1 = \int_S \underset{\sim}{z}_1,\ z_2 = \int_S \underset{\sim}{z}_2 \geqq 0$$

$$a\mu(S \cap S^*) \int_S \underset{\sim}{z}_1 + \int_S \underset{\sim}{z}_2 \leqq 0,\}. \tag{1}$$

Thus, $V_B(S)$ expresses, in per capita terms, the input - output vectors that are technologically available to the coalition S. In applying this procedure to Muench's example, however, we must recognize that quantities of the pure public good are "macro" quantities, with each coalition's consumption of the good being equal to that of the whole economy (see [4], pp. 242-243), and that a coalition producing the public good must produce for the whole economy. Thus, the corresponding expression in the Muench model is

$$V_M(S) = \{(z_1, z_2) \in R^2 \mid z_1 = \int_S \underset{\sim}{z}_1,\ z_2 = \int_S \underset{\sim}{z}_2 \geqq 0,$$

$$\underset{\sim}{z}_2 \text{ constant},\ a\int_S \underset{\sim}{z}_1 + (1/\mu(S)) \int_S \underset{\sim}{z}_2 \leqq 0\}. \tag{2}$$

The super-additivity of the V correspondences is clear. Indeed, cross-multiplying in the definition of V_M yields $a\mu(S) \int_S \tilde{z}_1 + \int_S \tilde{z}_2 \leqq 0$, which essentially duplicates the corresponding expression in the definition of V_B.

That V_M should show this super-additivity is not surprising. Pure public goods have long been recognized as involving some form of increasing returns phenomena. The cost of providing a given level of a pure public good is independent of the size of the group to whom it is provided. Thus, the per capita cost of provision decreases as group size increases. The analysis associated with expression (2) simply makes explicit the nature of these increasing returns in this example.

Thus, we know that equivalence does hold in large private goods production economies if the production correspondence is additive (see Hildenbrand [3]), and that the introduction of super-additivity may prevent equivalence in this case. Further, we see that the identical phenomenon of increasing returns to group size that makes the Boehm example work is also present in Muench's example with public goods. An obvious question is then that of whether it is this super-additivity alone that prevents equivalence in the Muench example or whether some other aspect of public goods (in particular, the requirement that all the agents for whom the public good is produced must receive the same amount) is also important in this context.

A natural approach to this question is to examine the case of a "semi-public" good. Such a good has the properties that costless exclusion is possible and that it is subject to crowding in its provision, so that per capita costs of providing the good do not fall in strict proportion to the inverse of the number receiving the good. However, as long as per capita costs do fall with the size of the consuming group, so that some element of increasing returns to group size does still obtain, we must expect small coalitions to be too weak for an equivalence theorem to hold. Moreover, should the posited crowding lead to per capita costs which increase with group size (decreasing returns to group size), we would expect the core would be empty, since small coalitions would be able to block all the Pareto optima. Indeed, in examining the core with semi-public goods, Ellickson [2] has presented an example of exactly such an occurrence in the presence of such strict sub-additivity.

These considerations, plus the analogy with the private goods case in which strict additivity does hold, suggest that the one situation where we might hope for an equivalence theorem is that of constant returns to group size. Thus, we will consider economies with semi-public goods with the exclusion property in which all coalitions have access to the same cone of technology <u>in per capita terms</u>. In such economies, the provision of the public good may be restricted to any specified group, and the total cost of providing the good to a group of size 2s is just twice that of supplying it to a group of size s, so per capita costs are constant. Thus, the only remaining element of "publicness" is that all agents for whom the good is provided must receive the same amount. In terms of the Muench example, this means replacing (2) by

$$V_M^*(S) = \{(z_1,z_2) \in R^2 \mid z_1 = \int_S \underset{\sim}{z}_1, z_2 = \int_S \underset{\sim}{z}_2 \geqq 0 ,$$

$$\underset{\sim}{z}_2 \text{ constant}, \ a\int_S \underset{\sim}{z}_1 + \int_S \underset{\sim}{z}_2 \leqq 0\}.$$

We may now ask whether the modified Edgeworth conjecture of equivalence between the core and Lindahl allocations is true in this context. However, the answer is still in general "no". An example, adapted from Ellickson [2], reveals that even in this case the core may be empty while the Lindahl equilibrium exists. [Take $\omega_1 = \omega_2 = 2$, $\omega_3 = 1$, $a = 1$, $b = 2$, $c = 3$ in the example in Section II of Ellickson]. A further simple example also indicates the possibility that the core may be a non-empty, proper subset of the Lindahl equilibrium allocations. Specifically, consider a two person economy with one private good and one public good. Let the two agents be identical, with each having an endowment ω_i of one unit of the private good and preferences over R^2_+ given by a utility function

$$U(x,y) = \begin{cases} x + 2y & 0 \leqq y \leqq 1/2 \\ x + 1 & 1/2 \leqq y. \end{cases}$$

The cone describing aggregate production opportunities is given by $Z = \{(z_1, z_2) \in R^2 \mid z_1 + z_2 = 0,\ z_2 \geqq 0\}$. Assigning measure 1/2 to each agent, we see that the Lindahl equilibria correspond to those allocations with $y = 1/2$, $0 \leqq x_1, x_2 \leqq 1$, $x_1 + x_2 = 1$ If we normalize so that the price of the private good is set at one, then the corresponding individualized public good prices are $q_1 = 2(1-x_1)$ and $q_2 = 2(1-x_2)$. If we allow each agent to produce the public good for his own use at the same per capita cost as the whole economy achieves when production is for both agents, i.e., individual i can achieve consumptions (x_i, y_i) such that $\frac{1}{2}(x_i - \omega_i) + \frac{1}{2}(y_i) = 0$, then all of these Lindahl allocations except that with $x_1 = x_2 = \frac{1}{2}$ are blocked, and the core consists of this single allocation. (The reader will note that the same phenomenon could occur even if the consumers did not become satiated with the public good.)

Thus, we cannot hope for any general equivalence theorem, even in this particularly nice case. However, if we limit our consideration to those cases where the core is non-empty and contains Lindahl allocations, we might still hope for some more limited "equivalence" results, characterizing the two solutions in terms of one another.

Consider then the case of an economy with a measure space, $(T, \mathcal{J}, \nu)$, where $\nu(T) = 1$, of consumers having identical tastes and endowments of private goods (assume zero endowments of public goods). Let there be M private goods, and N semi-public goods which are subject to crowding and for which exclusion is possible. To specify the possibilities for production, let $Z \subset R^{M+N}$ be a closed, convex cone containing the origin. A coalition S can produce $(x(t),y) \in R^{M+N}$ for each $t \in S$ iff $(\int_S \underset{\sim}{x}-\underset{\sim}{\omega}, \int_S \underset{\sim}{y}) \in Z$, where $\underset{\sim}{\omega}$ is the endowment function, which is constant, and $\underset{\sim}{y}$ is the constant function with value y. Thus, every coalition has access to the same technology in per capita terms. Suppose the common preferences are continuous, and convex, and non-satiated. Then we can prove that any Lindahl equilibrium treating all consumers equally [i.e., $(x(t),y)$ is indifferent to $(x(t'),y)$ for almost all t and t'] belongs to the core, and that all the core allocations treat equals equally and, as long as no consumer is at a wealth level which is minimal on his consumption set, can be supported as Lindahl equilibria.

To establish this claim, first suppose $(\bar{\underset{\sim}{x}},\bar{y})$ is a Lindahl allocation which treats equals equally. If this allocation is not in the core, there is a coalition S and a bundle $(x(t),y)$ for each t in S such that $(x(t),y)$ is preferred by t to $(\bar{x}(t),\bar{y})$ and

$$(\int_S \underset{\sim}{x}-\underset{\sim}{\omega}, \int_S \underset{\sim}{y}) \in Z$$

where $\underset{\sim}{y}$ is the constant function with value y. We may take $\underset{\sim}{x}$ to be a constant function, given convexity and the assumption of identical consumers. But for any constant function $\underset{\sim}{f}$ on T, $\int_S \underset{\sim}{f} \equiv \nu(S) \int_T \underset{\sim}{f}$. Thus, if we now consider $\underset{\sim}{x}$ and $\underset{\sim}{y}$ as constant functions on T, we have

$$(\int_T \underset{\sim}{x} - \underset{\sim}{\omega}, \int_T \underset{\sim}{y}) \in Z.$$

Thus, (x,y) could be provided to every member of T, and would be preferred by everyone to the Lindahl allocation. But this contradicts the Pareto-optimality of the Lindahl allocations, which is unaffected by crowding or exclusion.

On the other hand, it is clear that all core allocations in this set-up must treat (almost) all consumers equally, since each coalition is a microcosm and so any coalition of agents, each of whom received a bundle less preferred than the average bundle, could improve upon the given allocation by producing that average bundle for each of its members[3]. Under our assumptions, such an allocation can be made a Lindahl equilibrium by giving all agents the same prices, which are generated via a standard hyperplane argument applied to Z and the integral of the individual upper contour sets to the given consumption bundles.[4]

Note, however, that the Lindahl equilibria need not treat equals equally, so that the restriction to the equal treatment Lindahl allocations is necessary. This is illustrated by the example presented earlier, in which there are a continuum of Lindahl equilibria which do not treat equals equally.

Thus, given equal treatment, there is some possibility of proving a very limited form of equivalence. But the possibilities for extending this result would seem even more limited. Suppose we now allow the agents to differ, while setting $M = N = 1$ and $Z = \{(z_1, z_2) \in R^2 \mid z_1 + z_2 \leqq 0,\ z_2 \geqq 0\}$. Then the following is true: if $(\underset{\sim}{x}, y)$ is a Lindahl allocation with $y > 0$, if preferences are monotonic [5] and if, for some coalition S, we have $\int_S \underset{\sim}{q} > \nu(S)$, where $\underset{\sim}{q}$ is the function giving public goods prices, then the Lindahl allocation $(\underset{\sim}{x}, y)$ is blocked by S. To see this, note that, from the budget constraints,

$$\int_S \underset{\sim}{x} + y \int_S \underset{\sim}{q} \leqq \int_S \underset{\sim}{\omega},$$

where the private good's price is taken to be normalized at one. Rewrite y as $(1/\nu(S)) \int_S \underset{\sim}{y}$. Then

$$\int_S \underset{\sim}{x} + (1/\nu(S)) \int_S \underset{\sim}{y} \int_S \underset{\sim}{q} \leqq \int_S \underset{\sim}{\omega}.$$

Since $(\int_S \underset{\sim}{q})/\nu(S) > 1$, this implies

$$\int_S \underset{\sim}{x} + \int_S \underset{\sim}{y} < \int_S \underset{\sim}{\omega}.$$

But then S can, using its own resources, produce $(\bar{x}(t)+d, \bar{y}+d)$ for each of its members, where d is some positive number. Then, with monotonicity, $(\underset{\sim}{x}, y)$ is blocked.

Thus, if any coalition pays more for the public good than the share determined strictly by its size relative to the whole economy, the Lindahl equilibrium is blocked. Since it would only be under the most special of circumstances that Lindahl equilibrium would correspond to all agents paying the same price for the public good, we must despair of finding any general relationship between cores and equilibria when we have crowding and exclusion.

In the pure public goods model, the small coalitions (which can do all the blocking in a private goods world) are very weak. Consequently, the core remains large. If we admit crowding and exclusion, the power of the smaller coalitions increases so much that the Lindahl allocations may be blocked and the core may even be empty. This increased power is even more striking when we note that the limited equivalence result given above does not depend on the measure being non-atomic. We are then left with a rather unsatisfactory state: when the public goods are pure, the core is very large (but contains the Lindahl solutions), while if we introduce crowding and exclusion the core may become empty. This may suggest that if some core-like notion is to be a useful solution concept for public goods economies, some reconsideration of the basic definitions (such as has been begun by Rosenthal [6], Starrett [7] and Richter [5]) may be in order.

Footnotes

1. Throughout, "coalition" means a set of agents of positive measure. We usually ignore sets of measure zero, writing "every" and "all" instead of "almost every" or "almost all."

2. Since all integrals are of functions on T, given a measure ν, we will simply write $\int_S f$ for $\int_S f(t)\nu(dt)$.

3. Note that this argument would not be valid if applied to the Muench model, since there a coalition of less than full measure faces higher per capita costs for the public good than does the economy as a whole.

4. The consumers being identical is crucial here, since it means that the correspondence from agents to their upper contour sets is constant-valued and its value for each t is equal to its integral over T.

5. That is, $[x > \bar{x},\ y > \bar{y}]$ implies $[(x,y)$ strictly preferred to $(\bar{x},\bar{y})]$.

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