Discussion Paper No. 793

THE NATURE OF EQUILIBRIA IN THE BUYER'S BID DOUBLE AUCTION*

Steven R. Williams
Dept. of Economics
Northwestern University
Evanston, IL 60208

August, 1988

If a trader privately knows his own preferences, then he may choose to misrepresent the value that he places upon the good being traded in order to influence the market price in his favor. This misrepresentation may make the outcome of trade inefficient. As a market grows larger, a trader's ability to influence the market price diminishes; he loses his incentive to misrepresent and the market becomes more efficient. This intuitive argument is analyzed here by investigating a Bayesian game model of the buyer's bid double auction, which is a particular procedure for selecting a market-clearing price from a list of offers/bids. The existence of equilibria is proven in a generic instance of the model, and the nature of these equilibria is analyzed in markets of different sizes.

*This research was supported by National Science Foundation Grant No. SES-8705649, and by Northwestern University's Research Grants Committee.

1. Introduction.

This paper concerns a market in which each trader privately knows his own preferences. A trader in such a market may have an incentive to publicly misrepresent his preferences in order to influence the market price in his favor. A prospective buyer, for instance, may bid less than his true evaluation of the good for sale in an attempt to drive the price down; similarly, a seller may have an incentive to set his offer above what he is willing to accept. With behavior of this kind, a seller's offer could exceed a buyer's bid even though a mutually profitable trade between them exists. In the market as a whole, a market price may be determined at which all potential gains from trade are not realized. Misrepresentation of preferences may therefore cause inefficiency.

Our faith in markets rests upon the belief that this inefficiency is insignificant when there are a large number of traders, each of whom is small. A small trader in a large market is unlikely to have much influence on the market price; as a consequence, he has little incentive to misrepresent his preferences. The market would therefore be almost fully efficient. This argument, however, is vague about a very important issue: when is a trader small relative to the rest of the market, and how many traders constitute a large market? The meaning of "small" and "large" in this context is critical to understanding when this intuitive argument applies. Clarifying these words is part of making this argument meaningful.

This argument is investigated here by thoroughly analyzing a particular model of trade. A market is considered in which money may be exchanged for discrete, indivisible units of a good. For $n, m \geq 1$, there are n sellers, each of whom has one unit of the good to sell, and m buyers, each of whom wishes to buy at most unit. Each trader is therefore small in that he trades at most one unit; we shall consider arbitrary values of n and m, however. Trade is organized according to the following rules. Each seller submits an offer, while each buyer submits a bid. The n+m offers/bids are ordered in a list $s_1 \leq s_2 \leq \ldots \leq s_{n+m}$. Any number in $[s_m, s_{m+1}]$ is a market-clearing price; the procedure considered here selects s_{m+1} as the price, for this choice simplifies the subsequent analysis. Trade then occurs between those buyers whose bids are at least as large as s_{m+1} and those sellers whose offers are strictly less than s_{m+1}. This procedure is called the buyer's bid double auction (the BBDA) because in the bilateral case ($n = m = 1$) the buyer's bid is the price whenever trade occurs; in the multilateral case, however, the price may be set by either a buyer or a seller. It is illustrated in the $n = m = 4$ case in Figure 1.1.

Our objective is to investigate how traders strategically use their private information in the marketplace. For this purpose, we model the BBDA as a Bayesian game, as formulated by Harsanyi (1967-68). Each seller's reservation value for his unit of the good is independently drawn from a distribution F_1 on [0,1] and each buyer's reservation value is independently drawn from a distribution F_2 on [0,1]. Though the distributions F_1 and F_2 are common knowledge, a trader privately observes his own reservation value. A seller's utility when he trades equals the price p minus his reservation value, and it is zero when fails to trade; similarly, a buyer's utility is his reservation value minus p when he trades, and zero when he doesn't. Each trader is therefore risk neutral. Our objective is to examine the Bayesian-Nash equilibria of the BBDA, and to understand how they depend upon n and m.

A trader determines the market price when his offer/bid is the (m+1)st largest in the list. A seller only trades when his offer is strictly less than the (m+1)st largest, so he cannot influence the price that he receives. A seller therefore has no incentive to act strategically; it is his unique dominant strategy to set his offer equal to his reservation value.[1] We choose to study the BBDA because its rules permit us to completely characterize the behavior of one side of the market in this way. Let $\tilde{S}$ denote the truthful strategy. We restrict our attention to equilibria in which each seller uses $\tilde{S}$ as his strategy and each buyer uses the same function B as his strategy. Let $\langle\tilde{S},B\rangle$ denote an equilibrium of this kind. The objective is now to study in markets of different sizes the functions B that define such equilibria. We now discuss the results.

A large market might be defined as one in which the difference between each trader's reservation value and his offer/bid is small, for this insures that nearly all gains from trade are realized. To make this definition of the size of a market meaningful, the relationship between the number of traders and the amount by which they misrepresent their reservation values must be clarified. Satterthwaite and Williams (1988) proved the first result of this kind, which we now state. Consider the BBDA when $n = m$, and let $v \in [0,1]$ be any reservation value. There exists a continuous function $\kappa(v)$ such that, for any equilibrium $\langle\tilde{S},B\rangle$ in this market,

$$(1.01) \qquad v - B(v) \le \frac{\kappa(v)}{m}.$$

In words, the amount of misrepresentation $v - B(v)$ at any reservation value v is $O(1/m)$, no matter which equilibrium is considered. The function κ depends upon the distributions F_1 and F_2. This result is extended here to markets in

---

[1] This was observed by Wilson (1983), and proven in the $n = m$ case by Satterthwaite and Williams (1988).

which the number of buyers and the number of sellers may be unequal. We show that for any equilibrium $\langle\tilde{S},B\rangle$ in the market with n sellers and m buyers,

$$(1.02) \qquad v - B(v) \le \frac{\kappa(v)}{\min(n,m)}$$

at every reservation value v, where κ is the same function as in (1.01). To illustrate this result, fix n, m and the distributions F_1, F_2 and consider the market with tn sellers and tm buyers, where t is any natural number; using (1.02), it is easy to show that for any equilibrium $\langle\tilde{S},B\rangle$ in the market indexed by t and any reservation value v, v - B(v) is $O(1/t)$. In any sequence of markets in which the number of sellers and the number of buyers both increase at some constant rate, the amount of misrepresentation at any reservation value quickly converges to zero, no matter which sequence of equilibria is considered.

This convergence result does not describe how the equilibria in the BBDA change as only one side of the market grows larger. This issue is not yet completely understood; an example in the Appendix, however, suggests that it is primarily competition between buyers that drives this market toward efficiency. The example concerns the case in which both F_1 and F_2 are uniform. It is shown for this choice of F_1 and F_2 that $B(v) = mv/(m+1)$ is the unique smooth equilibrium in the market with n sellers and m buyers. The amount of misrepresentation at any reservation value is therefore $O(1/m)$, regardless of the number n of sellers. For other distributions the equilibria of the BBDA depend upon both the number of sellers and the number of buyers; the example is therefore not perfectly representative of the general case. It does suggest, however, that in the BBDA the size of a buyer is relative to the total number of buyers, not the total number of traders. This is also suggested by the analysis of the monopsonist (m = 1) and the monopolist ($n = 1$) cases in Section 7, which is carried out in terms of arbitrary distributions F_1 and F_2. It is shown that a monopsonist's strategy in the BBDA is independent of the number of sellers present, while the amount of misrepresentation by a buyer facing a monopolist converges to zero as the number of his fellow buyers increases to infinity. The intuitive argument about convergence at the beginning of this paper thus depends both upon how a market increases in size and upon which procedure is used to organize trade. The convergence result in this paper is an extension of the result of Satterthwaite and Williams (1988) to cases in which $n \neq m$; the proof in fact involves relating the $n \neq m$ case to the $n = m$ case and then applying the result from this earlier paper. What is truly new here is a proof of existence of equilibria in the BBDA. It is shown that for each m, n and for a generic pair of distributions F_1, F_2 there exists a piecewise smooth function B that defines an equilibrium $\langle\tilde{S},B\rangle$ in the BBDA. While a proof of existence of equilibria may not need to be justified, there are aspects of this result that are noteworthy. These are now discussed.

In Bayesian games in which the space of types is finite, it is typically easy to prove existence of equilibria with standard fixed point theorems. It is often advantageous to consider continuum type spaces, for this permits the use of calculus in the analysis of the game; a major disadvantage of this approach, however, is that there are at present few existence theorems for equilibria in such games. While there is currently some research being conducted on this topic, there are few examples to guide this research. One aim of this paper is to provide some insight into when equilibria exist in Bayesian games with continuum type spaces and what kinds of functions must be considered as strategies in order to prove existence.

As a special case of this problem, the difficulty in computing equilibria in double auctions has been a major obstacle to their analysis. For the general discussion that follows, a double auction is any procedure for organizing trade when each trader's preferences are imperfectly known by the other traders. Double auctions are interesting because they can be used to model many real-world trading situations and because they may serve as the foundation of a general noncooperative theory of markets. The theory of double auctions is a natural extension of auction theory, which has been widely applied. Some progress in their analysis has been made by Wilson (1982-85), and in the bilateral case by Leininger et al. (1986) and Satterthwaite and Williams (1987); in general, however, progress has been slowed by the failure to thoroughly understand their equilibria. To illustrate this point, consider a notable result of Wilson (1986). He proved that interim incentive efficiency is achievable in a very simple double auction when the number of traders is sufficiently large; the result depends, however, on the assumption that there exists an equilibrium with certain properties in each of the markets of different size. This assumption considerably weakens the conclusion about double auctions.

The existence proof in this paper is essentially constructive, which provides a great deal of insight into the nature of the equilibria. The approach is to rewrite the first order conditions for a buyer's utility maximization as a differential equation in the strategy B; a solution of this differential equation that satisfies certain additional properties defines an equilibrium in the BBDA. The study of equilibria is in this way grounded in a rich field of mathematics.

The approach has been used before. The BBDA is one example of a general family of double auctions. While it selects the (m+1)st-largest offer/bid s_{m+1} as the market price, any number in the interval $[s_m, s_{m+1}]$ could be selected instead. For $k \in [0,1]$, call the procedure that selects as the market price $ks_{m-1} + (1-k)s_m$ the k-double auction. Wilson (1986) rewrote the first order conditions for utility maximization in a k-double auction as a differential equation; while some properties of equilibria are inferred from this equation, the nature of its solutions (and hence the question of existence of equilibria) was not considered. Satterthwaite and Williams (1987) used this equation to characterize all smooth equilibria in the bilateral k-double auction with $k \in (0,1)$ as a two parameter family; the bilateral case, however, is much simpler than the multilateral case. Using only elementary techniques, the analysis of this differential equation is pushed further in this paper than in these earlier papers. While this paper considers only one particular double auction that is simpler to analyze, the usefulness of this approach in these earlier papers suggests that the methods developed here should be applicable to the study of any k-double auction.

In addition to the papers cited above, several other sources of this work should be mentioned. Myerson and Satterthwaite (1983) revealed the limitations on the gains that can be achieved through trade when each trader privately knows his own preferences. For each pair of distributions and for a given number of traders on each side of the market Gresik and Satterthwaite (1986) designed the trading mechanism that maximized the total gain from trade subject to incentive and individual rationality constraints. They then computed the rate at which inefficiency vanishes along the sequence of mechanisms indexed by the total number of traders. The primary difference between our convergence result and this earlier result is that the BBDA is a single procedure that is adaptable to a wide variety of trading environments, while optimal mechanisms are designed specifically in terms of the underlying distributions. The BBDA is thus a more realistic procedure. Wilson's survey (1987) helped to inspire and guide this work; it lays out a program for research on double auctions. Finally, McAfee and McMillan's (1987) broad survey of auction theory is a good source of background material; it also explains the significance of a theory of double auctions.

2. Assumptions and Elementary Facts.

Each of the distributions F_1, F_2 is a C^2 function on [0,1]. Let f_i denote the density function determined by F_i. Each f_i is strictly positive on [0,1].

As noted in the Introduction, it is assumed throughout this paper that each seller adopts the truthful strategy $\tilde{S}$. For $v \in [0,1]$ and for any function B, let $\pi(v,b;B)$ denote a buyer's expected payoff when (i) v is his reservation value, (ii) b is his bid, and (iii) each of the other buyers adopts B as his strategy. Let s_k denote the kth largest number in any specified sample of offers/bids.

The BBDA is now defined more precisely. In the market with n sellers and m buyers, recall that the (m+1)st largest offer/bid s_{m+1} is chosen as the market price p. Table 2.1 is used to explain exactly who trades at this price. As listed, s is the number of offers and t is the number of bids that exceed p, while k is the number of offers and j is the number of bids that equal p. Because $p = s_{m+1}$ the sum t+j+s+k is at least n, which implies that

$$(2.01) \qquad t+j \geq n-s-k.$$

The demand $t+j$ at price p is therefore at least as large as the supply $n-s-k$. If a unique offer/bid determines the price p, then $j+k = 1$ and $t+s = n-1$; in this case, (2.01) is an equality, and p is a market-clearing price. Ties at price p are conceivable and they must be handled to complete the definition of the BBDA. If at least two offers/bids equal s_{m+1}, then $j+k \geq 2$ and $t+s < n-1$; (2.01) therefore is a strict inequality, so the demand $t+j$ at price p strictly exceeds the supply $n-s-k$. The supply in this case is allocated among the buyers who bid at least p by starting with the buyer who bid the most, and then working down the list of bids. If in this process a bid is reached that was made by several buyers and there are not enough units of the good left to supply each of them, then the remaining units are distributed among these buyers using a lottery that assigns each of them an equal probability of winning. This completes the definition of the BBDA.

Table 2.1. Determination of the Market Price.

	Sellers	Buyers
# of offers/bids $> s_{m+1}$	s	t
# of offers/bids $= s_{m+1}$	k	j
# of offers/bids $< s_{m+1}$	$n - s - k$	$m - t - j$

We now begin to describe a strategy B that defines an equilibrium $\langle\tilde{S},B\rangle$. A standard argument that originated in Chatterjee and Samuelson (1983) shows that B must be nondecreasing on $[0,1]$. Theorem 2.1 states that B must in fact be strictly increasing. Ties between offers/bids are therefore a probability zero event; the way that ties are resolved in the BBDA insures that they never occur. For this reason, the possibility of ties is ignored in the remainder of this paper.

Theorem 2.1. If $<\tilde{S},B>$ is an equilibrium, then (i) B satisfies the bounds $0 < B(v) \leq v$ at every $v \in (0,1]$, and (ii) B is an increasing function that is differentiable almost everywhere.

Differentiability, of course, follows from the monotonicity of B. A proof of the other parts of the theorem in the $m = n$ case can be found in Satterthwaite and Williams (1983); a formal proof is omitted here because the proof in this special case is easily generalized. The bounds on B are derived from elementary incentive conditions. The monotonicity of B in the monopsonist case ($m = 1$) is established in Section 7; the result in this special case is not used elsewhere in the paper. An intuitive explanation of why B must be increasing when there are at least two buyers now follows. Consider a nondecreasing strategy B that assumes the constant value of b' over some interval. If all buyers use the strategy B, then ties between buyers at b' that are resolved using random allocation are a positive probability event. A buyer has an incentive to raise his bid above b' in these situations, for he thereby avoids a lottery and insures that he receives a unit of the good. A function of this kind therefore could not define an equilibrium.

In the remainder of this paper a strategy B is presumed to satisfy the conclusions of Theorem 2.1. For such a function, define the function B^{-1} on [0,1] with the formula

(2.02) $$B^{-1}(b) \equiv \sup\{v \in [0,1] \mid B(v) \leq b\}.$$

The function B^{-1} is continuous and nondecreasing. It is the inverse to B at all points in B's range.

The continuity of π in v and b is useful later in the paper. To prove this, represent $\pi(v,b;B)$ as

$$(2.03) \qquad \pi(v,b;B) = P(b;B)v - C(b;B),$$

where $C(b;B)$ is the expected payment by the selected buyer when he bids b and all other buyers use the strategy B, and $P(b;B)$ is the probability that the selected buyer trades in this situation.

<u>Theorem 2.2</u>. Given any strategy B, a buyers' expected payoff $\pi(v,b;B)$ is continuous in both v and b.

<u>Proof</u>. The continuity of $\pi(v,b;B)$ in v is obvious from (2.03). $P(b;B)$ is the probability that at least m offers/bids are less than b in a sample of offers/bids from n sellers using $\tilde{S}$ and m-1 buyers using B; formally,

$$(2.04) \qquad P(b;B) = \sum_{\substack{i+j>m \\ 0\le i\le m-1 \\ 0\le j\le n}} \binom{m-1}{i}\binom{n}{j} F_1(b)^j F_2(v_b)^i (1-F_1(b))^{n-j} (1-F_2(v_b))^{m-1-i},$$

where $v_b \equiv B^{-1}(b)$. Because B^{-1} is continuous, (2.04) implies that $P(b;B)$ is continuous.

The proof is now completed by showing that $C(b;B)$ is continuous. For $b'' > b'$ consider $C(b'';B) - C(b';B)$ as $b''-b'$ approaches zero. A buyer who raises his bid from b' to b'' increases his payment only if either: (i) he trades with the bid b'', but would fail to trade with the bid b'; or (ii) the bid b' would be the market price, and the buyer just drives up the price by raising his bid. In event (i), his payment is no more than b'', and in event (ii) the change in his payment is no more than $b''-b'$. We therefore have the following bound:

$$\text{(2.05)} \qquad C(b'';B) - C(b';B) \le \left[P(b'';B) - P(b';B)\right]b'' + (b'' - b').$$

The continuity of $C(b;B)$ now follows from the continuity of $P(b;B)$. Q.E.D.

It is convenient at this point to note a property of the function $\pi(v,b;B)$ that is needed for the construction of equilibria in Section 6. It is clear from (2.04) that $P(b;B)$ depends upon b and $B^{-1}(b)$, and $B^{-1}(b)$ is defined using the definition of B over $[0,B^{-1}(b)]$; $P(b;B)$ does not depend, however, upon how B is defined over $(B^{-1}(b),1]$. Similarly, because the selected buyer's payment when he bids b and trades is less than or equal to b, his expected payment $C(b;B)$ depends only upon how B is defined over $[0,B^{-1}(b)]$, for this determines the distribution of prices between zero and b. The value of $\pi(v,b;B)$ therefore does not depend upon the definition of B over the interval $(B^{-1}(b),1]$.

The subsequent analysis rests upon the first order conditions for a Bayesian-Nash equilibrium, which involve $d\pi/db(v,b:B)$. Several definitions are a prerequisite to writing out formulas for this marginal expected payoff. We define the following functions of v and b:

$$\text{(2.06)} \quad K_{n,m}(v,b) \equiv \sum_{\substack{i+j=m-1\\ 0\le i\le m-1\\ 0\le j\le n-1}} \binom{n-1}{j}\binom{m-1}{i} F_1(b)^j F_2(v)^i (1-F_1(b))^{n-1-j}(1-F_2(v))^{m-1-i};$$

$$\text{(2.07)} \quad L_{n,m}(v,b) \equiv \sum_{\substack{i+j=m-1\\ 0\le i\le m-2\\ 0\le j\le n}} \binom{n}{j}\binom{m-2}{i} F_1(b)^j F_2(v)^i (1-F_1(b))^{n-j}(1-F_2(v))^{m-2-i};$$

$$\text{(2.08)} \quad M_{n,m}(v,b) \equiv \sum_{\substack{i+j=m\\ 0\le i\le m-1\\ 0\le j\le n}} \binom{n}{j}\binom{m-1}{i} F_1(b)^j F_2(v)^i (1-F_1(b))^{n-j}(1-F_2(v))^{m-1-i}.$$

There are two formulas for $d\pi/db(v,b;B)$. Which is appropriate depends upon whether or not b is in the range of B. As above, let v_b denote $B^{-1}(b)$. If b is in the interior of $B([0,1])$ and $B'(v_b)$ exists, then

$$(2.09) \qquad d\pi/db(v,b;B) =$$

$$\left[nf_1(b)K_{n,m}(v_b,b) + (m-1)\frac{f_2(v_b)}{B'(v_b)} L_{n,m}(v_b,b) \right](v-b) - M_{n,m}(v_b,b).$$

If b is in the interior of the complement of $B([0,1])$, then

$$(2.10) \qquad d\pi/db(v,b;B) = nf_1(b)K_{n,m}(v_b,b)(v-b) - M_{n,m}(v_b,b).$$

These formulas can be carefully derived using the arguments in Satterthwaite and Williams (1988). An intuitive explanation using differentials follows.

Consider a buyer with reservation value v and let $b < v$. The buyer weighs two factors as he incrementally raises his bid from b to $b+\Delta b$: (i) if his bid b is the market price, then he loses by driving up the price that he pays; (ii) if the market price is in $(b,b+\Delta b)$, then he gains a profitable trade that he fails to make with the bid b. The loss corresponds to the term that is subtracted in (2.09-10), while the gain is the positive terms in these expressions. The first event occurs when b lies between s_m and s_{m+1} in the sample of offers/bids from the n+m-1 other traders; the probability of this event is $M_{n,m}(v_b,b)$, and the buyer's expected loss is approximately $-M_{n,m}(v_b,b)\Delta b$. The second event is a bit more complex. For it to occur, there must be an offer/bid between b and $b+\Delta b$, and exactly m-1 offers/bids below b. (Recall that all offers/bids are assumed to be distinct, and Δb can be arbitrarily small.) If b is in the interior of the compliment of $B([0,1])$, then only an offer could lie in $(b,b+\Delta b)$. Select a seller; the probability that his offer lies in $(b,b+\Delta b)$ is approximately $f_1(b)\Delta b$, and the probability that exactly m-1 of the remaining n+m-2 offers/bids are below b is $K_{n,m}(v_b,b)$. There are n sellers, so the expected gain in this case is approximately $nf_1(b)K_{n,m}(v_b,b)(v-b-\Delta b)\Delta b$, which completes the explanation of (2.10). If b is in the interior of $B([0,1])$, then the expected gain to the selected buyer is larger, for he may outbid both sellers and buyers as he raises his bid. The probability that an offer of one of the m-1 other sellers lies in $(b,b+\Delta b)$ is approximately $(m-1)f_2(v_b)\Delta b/B'(v_b)$, and the probability that exactly m-1 of the remaining n+m-2 offers/bids are below b is $L_{n,m}(v_b,b)$. This explains the term in brackets in (2.09) that is missing from (2.10), and it completes the explanation of these formulas.

Now consider an equilibrium $\langle\tilde{S},B\rangle$. If $B'(v)$ exists, then the following first order condition holds at v and $b = B(v)$:

$$(2.11) \qquad 0 = d\pi/db(v,b;B)$$
$$= \left[nf_1(b)K_{n,m}(v,b) + (m-1)\frac{f_2(v)}{B'(v)}L_{n,m}(v,b)\right](v-b) - M_{n,m}(v,b).$$

When there are at least two buyers ($m \geq 2$), (2.11) is a differential equation in B that must be satisfied at almost every reservation value v. Most of this paper concerns a geometric representation of (2.11) that is derived in the next section. Suppose instead that B is discontinuous at v, with left-hand limit of ℓ_1 and right-hand limit of ℓ_2 at this reservation value. For $b \in (\ell_1,\ell_2)$, $B^{-1}(b) = v$; formula (2.10) for $d\pi/db$ applies and reduces to

$$(2.12) \qquad d\pi/db(v,b;B) = nf_1(b)K_{n,m}(v,b)(v-b) - M_{n,m}(v,b).$$

This formula is used extensively in the analysis of discontinuities.

3. A Geometric Representation of the First Order Condition.

Solving for the inverse of a trader's strategy is a standard technique in the auction literature. It is used here to investigate equilibria when there are at least two buyers. The monopsonist case is different, so it is treated separately in Section 7. It is assumed in the rest of the paper that there are at least two buyers.

If $v = B^{-1}(b)$, then $\dot{v} = dv/db = 1/B'(v)$ whenever $B'(v)$ exists. Substitute this into (2.11); solving for $\dot{v}$ and adding the tautology $\dot{b} = db/db = 1$ then defines the vector field

$$(3.01) \quad \dot{v}(v,b) = \frac{M_{n,m}(v,b) - nf_1(b)K_{n,m}(v,b)(v-b)}{(m-1)f_2(v)L_{n,m}(v,b)(v-b)}$$

$$(3.02) \quad \dot{b} = 1.$$

By Theorem 2.1, an equilibrium strategy B satisfies the bounds $0 < B(v) \le v$ at every $v \in (0,1]$, and (3.01-02) are satisfied at almost every point on the graph of B. We therefore investigate the solutions of (3.01-02) in the triangle $0 \le b \le v \le 1$.

Figure 3.1 shows the direction of (3.01-02) on the edges of the triangle. At the vertices (0,0) and (1,1), $\dot{v}$ is an indeterminate form. On the edge $v = 1$, $\dot{v}$ is infinite, so the normalized vector is (1,0). On the edge $b = 0$, $\dot{v}$ equals negative infinity, so the normalized vector is (-1,0). On the edge $v = 1$, the vector field points into the triangle when $v >$

$b + F_1(b)/f_1(b)$, and out of the triangle when $v < b + F_1(b)/f_1(b)$; in particular, it points outward on this edge near (1,1), and inward near (1,0). Solution curves to (3.01-02) therefore enter the triangle through the edge $v = b$ and subintervals of $v = 1$ in which $v > b + F_1(b)/f_1(b)$; solution curves exit through subintervals of the edge $v = 1$ in which $v < b + F_1(b)/f_1(b)$. Higher order terms determine whether a curve enters or exits through a point on the edge $v = 1$ at which $v = b + F_1(b)/f_1(b)$. Solution curves neither enter nor exit through the edge $b = 0$.

If an equilibrium $\langle\tilde{S},B\rangle$ exists such that B is smooth, then the graph of B is a solution curve to (3.01-02) that enters the triangle through (0,0). Theorem 3.1 states that there exists a unique solution curve that enters through this vertex; consequently, no more than one smooth function B can define an equilibrium of the form $\langle\tilde{S},B\rangle$. As explained later, the solution curve through (0,0) may fail to define an equilibrium; it will still be useful, however, in the construction of equilibria in Section 6.

Theorem 3.1. There exists a unique solution curve to (3.01-02) that enters the triangle through the vertex (0,0). Its tangent at (0,0) is $b = mv/(m+1)$.

Proof. Topological considerations imply that some solution curves enter through (0,0). To determine the tangent at this vertex of a solution curve, consider the family of curves

$$(3.03)\quad v = \tau_\delta(b) \equiv F_2^{-1}(\delta F_1(b))$$

parameterized by $\delta > 0$. The derivative $\tau'_\delta(b) = \delta f_1(b)/f_2(\tau_\delta(b))$ exists at each point on the curve defined by δ, and $\tau'_\delta(0) = \delta f_1(0)/f_2(0)$. We examine $\dot{v}$ along the curve $v = \tau_\delta(b)$ as b approaches zero. A solution curve to (3.01-02) has the same tangent at (0,0) as the curve $v = \tau_\delta(b)$ if and only if

$$(3.04) \quad \lim_{b \downarrow 0} \dot{v}\,(\tau_\delta(b),b) - \tau_\delta'(b) = 0.$$

In the Appendix, it is shown that $\delta = (m+1)f_2(0)/mf_1(0)$ is the unique solution to (3.04); consequently, any solution curve through (0,0) has $b = mv/(m+1)$ as its tangent at this vertex.

In order to rule out multiple solution curves that share this tangent (e.g., a bifurcation at (0,0)), we consider $\partial\dot{v}/\partial b$ along the curve $v = \tau_\delta(b)$ for the value of δ that solves (3.04). In the Appendix, it is also shown that

$$(3.05) \quad \lim_{b \downarrow 0} \partial\dot{v}/\partial b = \infty$$

along this curve. Consider Figure 3.2, and let $v = \rho(b)$ denote a solution curve to (3.01-02) through (0,0). Near (0,0) in some neighborhood of this solution curve, $\dot{v}$ is increasing in b. As v decreases, a solution curve that is above ρ could not move down to share ρ's tangent at (0,0), and a solution curve below ρ could not move up to share its tangent. In this way the proof of uniqueness is completed. Q.E.D.

4. Convergence of All Equilibrium Strategies to Truthful Revelation.

Inspection of $\dot{v}$ on the edges of the triangle reveals that it is negative in some nonempty open subset. Call this subset $\Gamma_{n,m}$,

$$(4.01) \quad \Gamma_{n,m} = \{(v,b) \mid \dot{v}(v,b) < 0\},$$

and let $\gamma_{n,m}$ denote its boundary. This region contains an open set that borders on both the edge $b = 0$ and on a subinterval of the edge $v = 1$.

Depending upon the distributions F_1 and F_2, it may also contain open regions in the interior of the triangle, or additional regions that border on the edge $v = 1$.

By Theorem 2.1, an equilibrium strategy B has a nonnegative derivative at almost every value in its domain. Recall that $B'(v) = 1/\dot{v}(v,B(v))$. The region $\Gamma_{n,m}$ is significant because the graph of an equilibrium strategy must lie outside $\Gamma_{n,m}$ at almost all reservation values; in fact, it is proven below that such a graph must lie completely outside $\Gamma_{n,m}$. This observation will be used first to bound the buyers' strategies and then to prove convergence.

The reader should note that $\dot{v}$ and formula (2.12) for $d\pi/db$ are opposite in sign. This follows easily from comparing formula (2.12) with equation (2.11), which was solved to define $\dot{v}$. It is a very useful fact that is needed both in the following proof and throughout the analysis of discontinuous strategies.

Theorem 4.1. If $\langle\tilde{S},B\rangle$ is an equilibrium, then the graph of B lies completely outside $\Gamma_{n,m}$.

Proof. The proof is by contradiction. Suppose $(v,B(v)) \in \Gamma_{n,m}$, and let ℓ_1 and ℓ_2 be the left- and right-hand limits (respectively) of B at v. At least one of the following statements is true: (i) $B(v) = \ell_1$; (ii) $B(v) = \ell_2$; (iii) $\ell_1 < B(v) < \ell_2$. If either (i) or (ii) is true, then the continuity of $\dot{v}$ and the differentiability of B almost everywhere imply that B' exists and is negative at some reservation value near v. This contradicts the fact that B is increasing. If (iii) is true, then formula (2.12) specifies a positive value for $d\pi(v,B(v);B)$, which contradicts the hypothesis that $\langle\tilde{S},B\rangle$ is an equilibrium. Q.E.D.

The sign of $\dot{v}$ is determined by the numerator of (3.01); it implies that a pair (v,b) is outside $\Gamma_{n,m}$ if and only if

$$(4.02) \quad v - b \le \frac{1}{f_1(b)} \frac{M_{n,m}(v,b)}{nK_{n,m}(v,b)} .$$

By Theorem 4.1, the inequality (4.02) bounds the amount of misrepresentation v-B(v) in any equilibrium strategy B. The convergence result now follows from examining the right-hand side of this inequality.

<u>Lemma 4.2</u>. For $0 < b \le v < 1$, $M_{n,m}(v,b)/nK_{n,m}(v,b)$ is:

(4.03) decreasing in n when $m \ge 2$;

(4.04) decreasing in m when $n \ge 2$;

(4.05) constant in m when $n = 1$.

The proof of Lemma 4.2 is a purely formal analysis of the polynomials $K_{n,m}$ and $M_{n,m}$; it can be found in the Appendix. The lemma can be interpreted geometrically by referring back to (4.02). Using this inequality as a definition of $\Gamma_{n,m}$, the lemma states that $\Gamma_{n,m}$ is monotonically increasing in both m and n whenever n, $m \ge 2$. As the number of traders on both sides of the market increases, the graphs of equilibrium strategies of buyers are therefore confined to a smaller and smaller region of the triangle. Theorem 4.3 makes this precise by specifying the rate at which all equilibria are pushed towards the $v = b$ edge as the size of both sides of the market increases. Statement (4.05) in the lemma implies that $\Gamma_{1,m}$ is constant in m; the bound on buyers' strategies that is obtained from $\Gamma_{1,m}$ thus provides little insight. Convergence can be proven in the monopolist case by analyzing the vector field (3.01-02) a bit more carefully. This is done in Section 7.

<u>Theorem 4.3</u>. Consider the BBDA when sellers' reservation values are drawn from F_1 and buyers' reservation values are drawn from F_2. There exists a continuous function $\kappa(v;F_1,F_2)$ such that for any n, m $\geq$ 2 and for any equilibrium $\langle\tilde{S},B\rangle$ in the market with n sellers and m buyers

$$(4.06) \quad v - B(v) \leq \frac{\kappa(v;F_1,F_2)}{\min(n,m)}$$

at every $v \in (0,1)$.

<u>Proof</u>. Let $t = \min(n,m)$. The graph of B lies outside $\Gamma_{m,n}$. By (4.02-04),

$$(4.07) \quad v - B(v) \leq \frac{1}{f_1(B(v))} \frac{M_{n,m}(v,B(v))}{nK_{n,m}(v,B(v))} \leq \frac{1}{f_1(B(v))} \frac{M_{t,t}(v,B(v))}{tK_{t,t}(v,B(v))}$$

at every $v \in (0,1)$. Satterthwaite and Williams (1988, Thm. 5.3) constructed continuous functions $\mu(v;F_1,F_2)$ and $\kappa(v;F_1,F_2)$ of v that have the following properties at each $v \in (0,1)$: (i) $0 < \mu(v;F_1,F_2) < v$; (ii) $(v,b) \in \Gamma_{2,2}$ for every $b \in [0,\mu(v;F_1,F_2)]$; (iii) for $b > \mu(v;F_1,F_2)$,

$$(4.08) \quad \frac{1}{f_1(b)} \frac{M_{t,t}(v,b)}{tK_{t,t}(v,b)} \leq \frac{\kappa(v;F_1,F_2)}{t} .$$

Because $\Gamma_{2,2} \subset \Gamma_{n,m}$, B(v) is greater than $\mu(v;F_1,F_2)$, so (4.08) holds at $b = B(v)$. Combining (4.07-08) then produces the desired bound on $v - B(v)$. Q.E.D.

5. <u>Discontinuities in the Buyers' Strategy</u>.

Besides its role in the convergence result, the shape of $\Gamma_{n,m}$ also plays an important role in determining the nature of equilibrium strategies.

Consider, for instance, Figure 5.1 in which the solution curve ρ through (0,0) enters $\Gamma_{n,m}$ at point q in the interior of the triangle. At point q this curve turns back on itself and fails to define b as a function of v. A smooth equilibrium strategy for buyers therefore does not exist in this case.

It is shown in the next section that a piecewise smooth equilibrium exists both in this example and in a generic problem. A thorough understanding of discontinuities of equilibrium strategies is needed for this result. This is provided by Theorem 5.1, which states necessary and sufficient conditions for a piecewise smooth strategy of buyers to define an equilibrium.

Theorem 5.1.

I. Let $\ell_1 < \ell_2$ be the left- and right-hand limits (respectively) of a strategy B at a point of discontinuity $\bar{v}$. If $\langle\tilde{S},B\rangle$ is an equilibrium, then:

(5.01) $\pi(\bar{v},\ell_1;B) = \pi(\bar{v},B(\bar{v});B) = \pi(\bar{v},\ell_2;B)$;

(5.02) $\pi(\bar{v},b;B)$ is maximized over $b \in [\ell_1,\ell_2]$ at $b = \ell_1$, $B(\bar{v})$, and ℓ_2;

(5.03) $\dot{v}(\bar{v},\ell_2) = 0$;

(5.04) $\pi(1,b;B)$ is maximized over $b \in [B(1),1]$ at $b = B(1)$.

II. Conversely, suppose B is an increasing function whose graph is a finite union of segments of solution curves to (3.01-02), each segment of which has nonempty interior. If B satisfies (5.01-03) at each of its discontinuities and (5.04) at $v = 1$, then $\langle\tilde{S},B\rangle$ is an equilibrium.

In part I, (5.02) follows from the definition of an equilibrium once (5.01) has been proven; (5.04) also follows directly from this definition.

Because an equilibrium strategy is increasing, it can have only jump discontinuities. Consider the graph of such a strategy. By (5.03), the upper endpoint of a jump discontinuity must be in $\gamma_{n,m}$ where $\dot{v} = 0$. This limits where discontinuities may occur. Equation (5.01) has the same purpose. It is interesting to note that an equation similar to (5.01) holds in the bilateral k-double auction; it underlies the construction of step function equilibria in this game by Leininger et. al. (1986).

<u>Proof of part I of Theorem 5.1</u>. For the reasons stated above, only (5.01) and (5.03) are proven here. We begin with (5.01). The definition of an equilibrium implies that

(5.05) $\pi(\bar{v},B(\bar{v});B) \geq \pi(\bar{v},\ell_1;B),\ \pi(\bar{v},\ell_2;B).$

Suppose the inequality for $\pi(\bar{v},\ell_1;B)$ is strict. By the continuity of $\pi(v,b;B)$ in v and b, there exists an $\varepsilon > 0$ such that

(5.06) $\pi(v,B(\bar{v});B) > \pi(v,B(v);B)$

for all $v \in [\bar{v} - \varepsilon, v)$. This contradicts the hypothesis that $<\tilde{S},B>$ is an equilibrium. We therefore conclude that equality holds in (5.05) for $\pi(\bar{v},\ell_1;B)$. A similar argument proves equality in the case of $\pi(\bar{v},\ell_2;B)$.

We now turn to (5.03). The points (v,B(v)) lie outside $\Gamma_{n,m}$; by continuity, it follows that $\dot{v}(\bar{v},\ell_2) \geq 0$. Formula (2.12) specifies $d\pi/db(\bar{v},b;B)$ for $b \in (\ell_1,\ell_2)$. If $\dot{v}(\bar{v},\ell_2)$ were strictly positive, then $d\pi/db(\bar{v},b;B)$ would be negative for b in (ℓ_1,ℓ_2) near ℓ_2. For such b, it would therefore be true that

(5.06) $\pi(\bar{v},b;B) > \pi(\bar{v},\ell_2;B) = \pi(\bar{v},B(\bar{v});B),$

where the equality in (5.06) is from (5.01). This contradicts the hypothesis that $<\tilde{S},B>$ is an equilibrium. We conclude that $\dot{v}(\bar{v},\ell_2)$ must be zero. Q.E.D.

Proof of part II of Theorem 5.1. Consider a buyer with reservation value $v^* > 0$. We must show that $b = B(v^*)$ maximizes $\pi(v^*,b;B)$. It is elementary to show that the buyer need only consider bids in $(0,v^*]$. Each bid b in this interval lies in one of three sets: (i) b may be in the range of B over some subinterval of $[0,1]$ in which B is C^1; (ii) b may be in an interval that the graph of B jumps across; (iii) b may be in $[B(1),v^*]$. Each of these cases shall be discussed in turn.

Let (a,b) denote an interval in which B is C^1. For any $v \in [0,1]$, formula (2.09) specifies $d\pi/db(v,b;B)$ for b in the range of B over (a,b); in particular, if $v \in (a,b)$, then

$$(5.07)\quad d\pi/db(v,B(v);B) = 0.$$

Formula (2.09) for $d\pi/db$ is linear in v. For the specified value v^* and for $v \in (a,b)$, it follows that $d\pi/db(v^*,B(v);B)$ is positive if $v < v^*$, zero if $v = v^*$, and negative if $v > v^*$. We shall use this fact after discussing case (ii).

Now suppose $\ell_1 < \ell_2$ are the left- and right-hand limits (respectively) of B at a point of discontinuity $\bar{v}$, and consider $b \in [\ell_1,\ell_2]$. Assume first that $v^* \le \bar{v}$. Represent $\pi(v,b;B)$ as in (2.03). By (5.02),

$$(5.08)\quad \pi(\bar{v},\ell_1;B) = P(\ell_1;B)\bar{v} - C(\ell_1;B) \ge P(b;B)\bar{v} - C(b;B) = \pi(\bar{v},b;B).$$

$P(\ell_1;B)$ is less than or equal to $P(b;B)$ because $P(b;B)$ is increasing in b, and we have assumed that $v^* \le \bar{v}$; (5.08) therefore implies that

(5.09) $P(\ell_1;B)v^* - C(\ell_1;B) \geq P(b;B)v^* - C(b;B)$,

or equivalently,

(5.10) $\pi(v^*,\ell_1;B) \geq \pi(v^*,b;B)$.

A similar argument shows that if $v^* \geq \bar{v}$ and $b \in [\ell_1,\ell_2]$, then

(5.11) $\pi(v^*,\ell_2;B) \geq \pi(v^*,b;B)$.

If $v^* \leq \bar{v}$, then $\pi(v^*,b;B)$ is maximized over $[\ell_1,\ell_2]$ at its lower endpoint ℓ_1; if $v^* \geq \bar{v}$, then $\pi(v^*,b;B)$ is maximized over this interval at its upper endpoint ℓ_2. Combining this with the conclusion of case (i), it is clear that $\pi(v^*,b;B)$ is maximized over the interval $(0,\min(v^*,B(1))]$ at $b = B(v^*)$.

Finally, assume that $v^* > B(1)$ and consider $b \in [B(1),v^*]$. It has been shown thus far that $\pi(v^*,B(v^*);B) \geq \pi(v^*,B(1);B)$; to complete the proof, it is sufficient to prove that $\pi(v^*,b;B)$ is maximized over $[B(1),v^*]$ at $b = B(1)$. Statement (5.04) implies that $\pi(1,B(1);B) \geq \pi(1,b;B)$ for $b \in [B(1),v^*]$. In (5.08-10), replacing $\bar{v}$ with 1, ℓ_1 with $B(1)$ and $[\ell_1,\ell_2]$ with $[B(1),v^*]$ produces the desired proof that $\pi(v^*,b;B)$ is maximized over $[B(1),v^*]$ at $b = B(1)$. Q.E.D.

6. Existence of Equilibria.

Theorem 3.1 states that there exists a unique solution curve to (3.01-02) that emanates from the vertex (0,0). This curve passes through the triangle and exits along the edge $v = 1$. Suppose that it is the graph of a function $b = B(v)$. This function is necessarily increasing, for $\dot{b} = 1$ and $\dot{v}$ is finite along its graph. If (5.04) holds (i.e., if $\pi(1,b;B)$ is maximized over

$b \in [B(1),1]$ at $b = B(1)$), then by Theorem 5.1 $<\tilde{S},B>$ is an equilibrium.

This discussion highlights two difficulties that arise in constructing an equilibrium strategy B from solution curves to (3.01-02), starting with the curve that emanates from (0,0). First, as noted earlier this curve may enter $\Gamma_{n,m}$ as it passes through the triangle. It would therefore not define b as a function of v. Second, B must be defined so that (5.04) holds. Both of these difficulties will be overcome by introducing jump discontinuities into the definition of B that satisfy the conditions in Theorem 5.1.

The following examples illustrate how the shape of $\gamma_{n,m}$ may cause these difficulties to occur, and how jump discontinuities may overcome them.

Example 6.1. We first consider a case in which these difficulties do not arise because of the simple nature of $\gamma_{n,m}$. As illustrated in Figure 6.1, suppose that $\gamma_{n,m}$ consists of the graph of a continuous function of v plus (perhaps) several isolated points on the $v = 1$ edge. The solution curve through (0,0) enters the interior of the triangle above $\gamma_{n,m}$ into the region in which $\dot{v}$ is positive. Because the vector field (3.01-02) points straight upward at points in $\gamma_{n,m}$, this solution curve cannot enter $\Gamma_{n,m}$ as it continues through the triangle. It therefore defines an increasing function $b = B(v)$. Formula (2.12) specifies $d\pi/db(1,b;B)$ for $b \in [B(1);1]$; because $\dot{v}(1,b)$ is nonnegative for b in this interval, $d\pi/db(1,b;B)$ is nonpositive, so $\pi(1,b;B)$ is maximized over this interval at $b = B(1)$. By Theorem 5.1, $<\tilde{S},B>$ is an equilibrium.

Example 6.2. Suppose $\gamma_{n,m}$ has the shape depicted in Figure 5.1, which is duplicated in Figure 6.2. The solution curve ρ that emanates from (0,0) may pass above p_2; in this case, the discussion in Example 6.1 shows that it defines an equilibrium $<\tilde{S},B>$. The curve ρ could pass below p_2, in which case it enters $\Gamma_{n,m}$ at some point q. As explained earlier, the solution curve in this case fails to define b as a function of v.

A piecewise smooth equilibrium strategy can be constructed in this case. Let x and y be the v-coordinates of p_2 and q, respectively, and let $b = \phi(v)$ and $b = \lambda(v)$ be the functions on [x,y] whose graphs are the indicated segments of $\gamma_{n,m}$. For any v in [x,y], define B_v as the strategy whose graph consists of the segment of ρ from (0,0) to v together with the solution curve through $(v,\lambda(v))$ from v to the right-hand edge of the triangle. Define $B_v(v)$ so that $(v,B_v(v))$ is in ρ. For B_v to define an equilibrium, (5.01-02) must hold at the discontinuity v, i.e.,

(6.01) $\pi(v,B_v(v);B_v) = \pi(v,\lambda(v);B_v)$, and

(6.02) $\pi(v,b;B_v)$ is maximized over $b \in [B_v(v),\lambda(v)]$ at $b = B_v(v)$.

Because $\dot{v}(x,b)$ is positive for $b \in [x,\lambda(x)]$, formula (2.12) for $d\pi/db(x,b;B_x)$ is negative over this interval; consequently,

(6.03) $\pi(x,B_x(x);B_x) > \pi(x,\lambda(x);B_x)$.

A similar argument shows that

(6.04) $\pi(y,B_y(y);B_y) < \pi(y,\lambda(y);B_y)$.

By continuity, there exists a value $v \in [x,y]$ at which (6.01) holds. Note that $d\pi/db(v,b;B_v)$ is negative for $b \in (B_v(v),\phi(v))$ and positive for $b \in (\phi(v),\lambda(v))$. Condition (6.03) therefore also holds at v. An argument from Example 6.1 then shows that (5.04) holds with this choice of B_v, and the construction of an equilibrium is complete.

Example 6.3. Now assume that $\gamma_{n,m}$ has the shape depicted in Figure 6.3. It has two components, one of which is the graph of a continuous function of v, while the other is a curve that lies above this graph and has endpoints on the $v = 1$ edge. The solution curve through (0,0) may pass completely above $\gamma_{n,m}$ as in Example 6.1 and thus define an equilibrium $\langle\tilde{S},B\rangle$, or it may pass below p_3 and intersect $\gamma_{n,m}$ at some point in the interior of the triangle, as in Example 6.2. In this last case, a piecewise smooth equilibrium strategy can be constructed by the procedure outlined above.

A third possibility is that the curve may pass between the components of $\lambda_{n,m}$ and define an increasing function $b = B(v)$. This function may or may not define an equilibrium, however. Let b_1 and b_2 be the b-coordinates of the indicated points in Figure 6.3 at which $\gamma_{n,m}$ intersects the $v = 1$ edge. Formula (2.12) specifies $d\pi/db(1,b;B)$ for $b \in [B(1),1]$; this derivative is positive for $b \in (b_1,b_2)$, and negative for $b \in (B(1),b_1) \cup (b_2,1]$. The function $\pi(1,b;B)$ therefore has a local maximum at $b = b_2$. The strategy B defines an equilibrium if and only if $\pi(1,B(1);B) \geq \pi(1,b_2;B)$.

If $\pi(1,b_2;B) > \pi(1,B(1);B)$, then an equilibrium can be constructed using the procedure developed in Example 6.2. Let x denote the v-coordinate of the point p_3, let $b = \lambda(v)$ denote the function whose graph is the indicated segment of $\gamma_{n,m}$ and set $y = 1$. Consider again strategies of the form B_v for $v \in [x,y]$. It is easy to show that (6.03) holds, and (6.04) holds by hypothesis. The argument in the preceding example then shows that there exists a value of v such that $\langle\tilde{S},B_v\rangle$ is an equilibrium.

Several regularity assumptions on $\gamma_{n,m}$ are needed to prove a general existence result using the construction in Examples 6.2-3. Let G(v,b) denote the numerator of (3.01),

(6.05) $G(v,b) \equiv M_{n,m}(v,b) - nf_1(b)\ K_{n,m}(v,b)(v-b)$,

so that $\gamma_{n,m}$ is the solution set of the equation $G(v,b) = 0$. We shall assume that the following conditions hold at all points (v,b) in $\gamma_{n,m}$:

(6.06) $\nabla G(v,b) \neq 0$;

(6.07) if $\partial G/\partial b(v,b) = 0$, then $\partial^2 G/\partial b^2(v,b) \neq 0$.

These conditions are satisfied in Figures 6.1-3. Condition (6.06) implies that $\gamma_{n,m}$ is a finite union of smooth curves, one of which emanates from (0,0), and perhaps several isolated points on the $v = 1$ edge. Condition (6.07) states that if one of these curves has a vertical tangent at some point, then $\gamma_{n,m}$ does not cross the tangent at that point (i.e., it turns back on itself). A point of this kind is a <u>turning point</u> of $\gamma_{n,m}$. Points $p_1 - p_3$ in Figures 6.2-3 are turning points. The preceding examples show that smooth equilibria may not exist when such points are present. If (6.06-07) hold at all points on $\gamma_{n,m}$, then this boundary has at most a finite number of turning points.

These regularity conditions are satisfied in a generic problem. Specifically, equip the set D of pairs of distribution functions (F_1, F_2) that satisfy the assumptions stated in Section 2 with the Whitney C^2 topology. The pairs for which the associated function G satisfies (6.06-07) on $G^{-1}(0)$ form an open and dense subset of D.

Theorem 6.1. If the regularity conditions (6.06-07) hold at all points in $\gamma_{n,m} = G^{-1}(0)$, then there exists a piecewise smooth function B that defines an equilibrium $<\tilde{S},B>$.

Before proving this theorem, we first recall from Section 2 that if B is an increasing function, then $\pi(v,b;B)$ does not depend upon the definition of $B(v)$ for $v > B^{-1}(b)$. This is now important for the following reason. The proof of the existence of an equilibrium strategy B is essentially constructive, and the construction proceeds from left to right through the triangle. To insure that (5.01-02) hold at a discontinuity of B, we consider $\pi(v,b;B)$ for $b \in [B(v),v]$ even though B may at that point only be defined over $[0,v]$. The value of $\pi(v,b:B)$ is well-defined if it is assumed that B will be defined later in the construction over $(v,1]$ to be strictly larger than the bid b under consideration. Similarly, if $b \in (B(v),v)$, then $d\pi/db(v,b;B)$ exists under this same assumption about how the definition of B shall be completed, and its value is given by (2.12). This is how these functions should be interpreted in the proof.

Proof of Theorem 6.1. The proof is by induction on the number k of turning points. We have the following

> Induction Statement: For any $y \in (0,1]$, consider the restriction of the vector field (3.01-02) to the triangle $0 \leq b \leq v \leq y$. If $\gamma_{n,m}$ has no more than k turning points in the interior of this triangle, then there exists a function B on $[0,y]$ that satisfies all of the hypotheses of Theorem 5.1, except perhaps (5.04). The function B also has the following property:

(6.08) $\pi(y,b;B)$ is maximized over $b \in [B(y),y]$ at $b = b(y)$.

Because (5.04) is a condition on $B(1)$, it is not meaningful for a function that is only defined over a proper subinterval of $[0,1]$. Condition (6.08) reduces to (5.04) when $y = 1$. Once the induction statement has been proven to be true for all values of k regardless of the value of y, the theorem follows by applying this result in the $y = 1$ case.

When $k = 0$, $\gamma_{n,m}$ in the triangle $0 \leq b \leq v \leq y$ consists of the graph of a smooth function of v and perhaps several isolated points on the $v = y$ edge. The argument in Example 6.1 can be adapted to prove that the graph of the solution curve through (0,0) in this triangle defines a function B that has the properties listed in the induction statement.

Assuming that the induction statement holds for k, we now prove that it holds for k+1. Suppose that $\gamma_{n,m}$ has k+1 turning points in the interior of the triangle $0 \leq b \leq v \leq y$, and let x be the largest v-coordinate in this set of k+1 points. As illustrated in Figure 6.4, $\gamma_{n,m}$ has at most k turning points in the interior of the triangle $0 \leq b \leq v \leq x$, and no turning points in the interior of the strip $x \leq v \leq y$, $0 \leq b \leq v$. The induction hypothesis defines a function B over $[0,x]$. The proof will now be completed by extending B over the interval $[x,y]$.

The induction hypothesis states that $\pi(x,b;B)$ is maximized over $b \in [B(x),x]$ at $b = B(x)$. By redefining B at x to be the largest value of b in this interval at which this maximum occurs, we can assume that $\pi(x,B(x);B)$ is strictly larger than $\pi(x,b;B)$ for $b \in (B(x),x]$. This implies that $\dot{v}(x,b) > 0$ for $b \geq B(x)$ near $B(x)$. The graph of B can therefore be continued by following the solution curve through $(x,B(x))$.

Over the interval (x,y), $\gamma_{n,m}$ is a finite union of graphs of smooth functions. Some of these functions have values at x larger than B(x); as in Figure 6.4, label these function $\lambda_1(v) \le \lambda_2(v) \le \ldots \le \lambda_s(v)$. The solution curve through (x,B(x)) may intersect $b = \lambda_1(v)$ at some point $(y',\lambda_1(y'))$, at which point it would enter $\Gamma_{n,m}$. Example 6.2 shows how a discontinuity should be introduced into the definition of B between x and y' to avoid this difficulty. For $v \in (x,y')$, define B_v as the function on [0,v] that equals B on [0,x] and whose values on [x,v] are given by the solution curve through (x,B(x)). Because $\pi(x,B(x);B) > \pi(x,\lambda_i(x);B)$ for $1 \le i \le s$ and $\pi(y',\lambda_1(y');B) < \pi(y',\lambda_2(y');B)$, there exists a $v \in (x,y')$ such that for some i:

(6.09) $\pi(v,\lambda_i(v);B_v) = \pi(v,B_v(v);B_v)$;

(6.10) $\pi(v,b;B_v)$ is maximized over $b \in [B_v(v),\lambda_i(v)]$ at $b = B_v(v)$, and $\pi(v,B_v(v);B_v) > \pi(v,b;B_v)$ for $b \in (\lambda_i(v),v]$.

For this value of v, extend the definition of B over [0,v] by setting $B \equiv B_v$. Now relabel v as x, and repeat the above procedure. In a finite number of steps, the definition of B can be extended over [0,y] so that it has the properties in the interior of this interval by the induction statement. One may need to redefine B near the v = y edge so that (6.08) holds. This can also be accomplished in a finite number of steps using the procedure outlined in Example 6.3. Q.E.D.

7. The Monopsonist and Monopolist Cases.

The monopsonist and monopolist cases are part of a complete analysis of the BBDA. They are also of interest because the provide a simple setting in which the ideas and methods of this paper can be illustrated. Finally, these cases are where auction theory and the theory of double auctions overlap, and each of these theories can provide insight into each other. This last point is illustrated here by showing that a familiar "monotonicity condition" from auction theory is really a hypothesis about the shape of $\gamma_{n,m}$ in these special cases.

Consider first the monopsonist case ($m = 1$). In this market, the BBDA selects the second highest offer/bid as the price, and trade occurs whenever the buyer's bid is as large as at least one offer. The first order condition (2.11) on the buyer's strategy $b = B(v)$ is

$$(7.01) \qquad nf_1(b)(1-F_1(b))^{n-1}(v-b) - nF_1(b)(1-F_1(b))^{n-1} = 0,$$

or equivalently,

$$(7.02) \qquad v = b + F_1(b)/f_1(b).$$

As noted in Section 2, the buyer's strategy B in the monopsonist must be nondecreasing. It is clear from (7.02) that it must in fact be increasing, for any nondecreasing function $b = B(v)$ that satisfies (7.02) clearly has this property.

Equation (7.02) is familiar from the auction literature, and it is also the equation that determines $\gamma_{n,1}$. Suppose the following <u>monotonicity condition</u> is added:

$$(7.03) \qquad b + F_1(b)/f_1(b) \text{ is a nondecreasing function on } [0,1].$$

With this additional hypothesis, equation (7.02) defines b as an increasing smooth function B of v, and $<\tilde{S},B>$ is an equilibrium. The monotonicity condition (7.03) is often used in the auction literature to insure the existence of a smooth equilibrium (e.g., see Myerson (1981)). When (7.03) does not hold, one must properly choose between the critical values given by

(7.02) in order to construct the buyer's strategy, which may introduce discontinuities. As in multiple buyer cases, the shape of $\gamma_{n,m}$ (here $\gamma_{n,1}$) may prevent the existence of a smooth equilibrium strategy. It is also interesting to note that the monopsonist's equilibrium strategy does not depend upon the number of sellers.

Now consider the case of monopoly (n = 1 seller, and m ≥ 2 buyers). In this market, the BBDA specifies the highest offer/bid as the price. It is therefore a first price auction in which the seller submits an offer that must be outbid by at least one buyer for trade to occur. This is a commonly used procedure for auctioning an item.[2] The outcome of the BBDA in this market is also a subgame perfect outcome of a two-stage auction in which after first receiving bids from the buyers the seller may choose to either sell the item at the highest bid or keep the item for himself.

The first order condition (2.11) on a buyer's strategy B is

$$(7.04) \qquad \left[f_1(b)F_2(v)^{m-1} + \frac{f_2(v)}{B'(v)} (m-1)F_1(b)F_2(v)^{m-2} \right](v-b) - F_1(b)F_2(v)^{m-1} = 0.$$

The term $f_1(b)F_2(v)^{m-1}$ in brackets is the marginal probability that the selected buyer will pass the seller's offer at b, when all other buyers bid less than b. In the standard model of the first price auction, the seller's reservation value is commonly known to be less than every buyer's reservation value, and the seller does not submit an offer. The first order condition for that model is therefore obtained by omitting the $f_1(b)F_2(v)^{m-1}$ term from (7.04). This produces a first order linear differential equation in B that is

---

[2] For instance, in May of 1988 a repeated version of this procedure was used by the International Olympic Committee to auction off the American television rights to the 1992 Winter Olympics (New York Times, May 24, 1988).

easily solvable by standard methods (e.g., see Ross (1974), p. 46).[3]

Returning to the model of this paper, one cannot obtain an exact solution B to (7.04) for generic distributions F_1 and F_2. One can, however, make several statements about an equilibrium strategy of buyers by investigating the vector field representation (3.10-02) of (7.04). This representation reduces to

$$\dot{v} = \frac{F_2(v)[F_1(b) - f_1(b)(v-b)]}{f_2(v)(m-1)F_1(b)(v-b)} \tag{7.05}$$

and

$$\dot{b} = 1. \tag{7.06}$$

From (7.05) it is clear that $\gamma_{1,m}$ is the solution set of equation (7.02). If the monotonicity condition (7.03) holds, then $\gamma_{1,m}$ is the graph of a smooth function of v and therefore does not have any turning points; the argument in Example 6.1 then shows that there exists a unique smooth equilibrium strategy. An equilibrium strategy in the general case can be constructed by the procedure described in Examples 6.2-3.

Intuitively, competition should force the equilibrium common strategy of buyers towards truthful revelation as the number of buyers increases. The method of Theorem 4.3 cannot be used to prove this, for the boundary $\gamma_{1,m}$ is independent of the number m of buyers. Convergence can be established with a slightly more general procedure that involves selecting a different family of curves to bound equilibrium strategies. Consider the line $b = tv$ where $t \in (0,1)$ is sufficiently large that this line lies above the solution set of

---

[3] An even simpler derivation of the optimal common strategy of buyers in a first price auction can be found in Maskin and Riley (1986), Milgrom and Weber (1985), and McAfee and McMillan (1987).

(7.02). For fixed t, we shall show that for m sufficiently large the optimal common strategy of buyers is unique, smooth and lies above the line $b = tv$. Because t can be chosen arbitrartily close to one, this is sufficient to establish convergence to truthful revelation. Theorem 3.1 states that there exists a solution to (7.05-06) that enters the triangle through the vertex (0,0) with slope $m/(m+1)$. We restrict our attention to m such that $m/(m+1) > t$. It is easy to show that for each m, $\dot{v}$ can be defined at (0,0) so that $\dot{v}$ is continuous on the line $b = tv$. It is then clear from formula (7.05) that for m sufficiently large the vector $(\dot{v},1)$ lies above this line at each of its points. For sufficiently large m, the solution curve to (7.05-06) through (0,0) therefore enters above the line $b = tv$ and stays above this line as it proceeds through the triangle. Theorem 5.1 then implies that this solution curve defines an equilibrium strategy of buyers.

When the distributions are appropriately restricted, this procedure can be used to obtain tighter bounds on the amount of misrepresentation. Assume, for instance, that $[F_2/f_2]'$ is nonnegative on [0,1], and consider $(\dot{v},1)$ along the curve

$$(7.07) \qquad v - b = F_2(v)/(m-1)f_2(v).$$

It is easy to show that if $(F_2/(m-1)f_2)' < 1$ on [0,1], then $(\dot{v},1)$ lies above this curve at each of its points. For m sufficiently large, an equilibrium strategy therefore lies above the curve given by (7.07), which means that the amount of misrepresentation in this market is $O(1/m)$.

References

Chatterjee, K. and W. Samuelson. 1983. Bargaining under incomplete information. Operations Research 31: 835-51.

Gresik, T. and M. Satterthwaite. 1986. The rate at which a simple market becomes efficient as the number of traders increases: An asymptotic result for optimal trading mechanisms. CMSEMS DP No. 708, Northwestern University, October.

Harsanyi, J. 1967-68. Games with incomplete information played by Bayesian players. Parts I, II, and III. Management Science 14: 159-82, 320-34, 486-502.

Leininger, W. P. Linhart, and R. Radner. 1986. The sealed bid mechanism for bargaining with incomplete information. AT&T Bell Laboratories.

Maskin, E. and J. Riley. 1986. Existence and uniqueness of equilibrium in sealed high bid auctions. mimeo., UCLA, March.

McAfee, R. P. and J. McMillan. 1987. Auctions and bidding. J. of Econ. Lit. 25: 699-738.

Milgrom, P. and R. Weber. 1985. Distributional strategies for games with incomplete information. Math. Operations Res. 10: 619-32.

Myerson, R. 1981. Optimal auction design. Math. Operations Res. 6: 58-73.

Myerson, R. and M. Satterthwaite. 1983. Efficient mechanisms for bilateral trade. J. of Econ. Theory 29: 265-281.

New York Times. 1988. "2 Networks to Bid for '92 Games." May 24.

Ross, S. 1974. Differential Equations. New York: John Wiley & Sons.

Satterthwaite M. and S. Williams. 1987. Bilateral trade with the sealed bid k-double auction: existence and efficiency. CMSEMS DP No. 723R, Northwestern University, December. Forthcoming in J. of Econ. Theory.

-----. 1988. The rate of convergence to efficiency in the buyer's bid double auction as the market becomes large. CMSEMS DP No. 741, Northwestern University, February.

Wilson, R. 1982. Double auctions. IMSSS technical report no. 391, Stanford University, December.

-----. 1983. Efficient trading. IMSSS technical report no. 432, Stanford University, October.

-----. 1985. Incentive efficiency of double auctions. Econometrica 53: 1101-15.

-----. 1987. Game-theoretic analyses of trading processes. In Advances in Economic Theory, Fifth World Congress, ed. by T. Bewley, 33-70. Cambridge: Cambridge University Press.

8. Appendix.

This section contains a proof of Lemma 4.2, the completion of the proof of Theorem 3.1, and an example in which a smooth equilibrium strategy is computed. These results appear together at the end of the paper because the analysis is purely computational and because they all require the same special notation. We begin by developing this notation.

Define the function $z(v,b)$ and the polynomials $K^*_{n,m}(z)$, $L^*_{n,m}(z)$ and $M^*_{n,m}(z)$ with the formulas

$$(8.01)\quad z(v,b) \equiv \frac{F_2(v)(1 - F_1(b))}{F_1(b)(1 - F_2(v))},$$

$$(8.02)\quad K^*_{n,m}(z) \equiv \sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}(m-i)z^i,$$

$$(8.03)\quad L^*_{n,m}(z) \equiv \sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}iz^i,$$

$$(8.04)\quad M^*_{n,m}(z) \equiv \sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}z^i.$$

Formula (3.01) reduces to

$$(8.05)\quad \dot{v}(v,b) = \frac{F_2(v)}{f_2(v)}\;\frac{M^*_{n,m}(z) - K^*_{n,m}(z)(v-b)f_1(b)/F_1(b)}{(v-b)L^*_{n,m}(z)},$$

and a pair (v,b) is outside $\Gamma_{n,m}$ if and only if

$$(8.06)\quad v - b \le \frac{F_1(b)}{f_1(b)}\;\frac{M^*_{n,m}(z)}{K^*_{n,m}(z)}.$$

The inequality (8.06) is equivalent to (4.02). Formulas (8.05-06) are obtained by the following steps: (i) the index i in (2.07) is replaced with i-1; (ii) (2.06-08) are rewritten using only the index i; (iii) the identities

$$(8.07)\qquad \binom{n-1}{m-1-i} = \frac{m-i}{n}\binom{n}{m-i}, \text{ and}$$

$$(8.08)\qquad \binom{m-2}{i-1} = \frac{i}{m-1}\binom{m-1}{i}$$

are substituted into (2.06) and (2.07), respectively; (iv) the reduced forms of (2.07-08) are substituted into (3.01) and the right-hand side of the inequality in (4.02), which are then reduced using arithmetic. While tedious, this reduction is elementary, and it is therefore omitted. All of the results in this section are obtained using formula (8.05) for $\dot{v}$ and the inequality (8.06) that defines the complement of $\Gamma_{n,m}$.

<u>Example 8.1</u>. To verify that $b = mv/(m+1)$ is a solution curve to (3.01-02) when F_1 and F_2 are uniform on [0,1], it is sufficient to show that (8.05) holds along this line. Leaving z alone, substitution of

$$(8.10)\qquad F_2(v)/f_2(v) = v = (m+1)b/m,$$

$$(8.11)\qquad f_1(b)/F_1(b) = 1/b, \text{ and}$$

$$(8.12)\qquad v - b = b/m$$

into the right-hand side of (8.05) produces

$$(8.13)\qquad \frac{(m+1)\sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}[1-(m-i)/m]z^i}{\sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}iz^i}.$$

After multiplying both its top and bottom by m and then simplifying, (8.13) reduces to $(m+1)/m$, which is $\dot{v}$. This completes the argument. Theorems 3.1 and 5.1 then imply that $b = B(v) = mv/(m+1)$ is the unique smooth equilibrium strategy of buyers when there are n sellers, m buyers, and both F_1 and F_2 are uniform.

<u>Proof of Theorem 3.1</u>. Recall that for $\delta > 0$, $\tau_\delta(b) \equiv F_2^{-1}(\delta F_1(b))$, and $\tau_\delta'(b) = \delta f_1(b)/f_2(\tau_\delta(b))$. To complete the proof of Theorem 3.1, we show that

$$(8.14)\qquad \lim_{b\downarrow 0} \dot{v}(\tau_\delta(b),b) - \tau_\delta'(b) = 0$$

has $\delta = (m+1)f_2(0)/mf_1(0)$ as its unique solution, and that

$$(8.15)\qquad \lim_{b\downarrow 0} \frac{\partial\dot{v}}{\partial b}(\tau_\delta(b),b) = \infty$$

for this value of δ.

We begin by solving (8.14). The following limits along $v_2 = \tau_\delta(b)$ are elementary:

$$(8.16)\qquad \lim_{b\downarrow 0} z(v,b) = \delta;$$

$$(8.17)\qquad \lim_{b\downarrow 0} \frac{d}{db}(F_1(b)/f_1(b)) = 1;$$

$$(8.18)\qquad \lim_{b\downarrow 0}\ (v-b)f_1(b)/F_1(b) = [\delta f_1(0) - f_2(0)]/f_2(0);$$

$$(8.19)\qquad \lim_{b\downarrow 0}\ F_2(v)/f_2(v)(v-b) = \delta f_1(0)/[\delta f_1(0) - f_2(0)].$$

From formula (8.05) for $\dot{v}$ and (8.16-19) we obtain

$$(8.20)\qquad \lim_{b\downarrow 0}\ \dot{v} = \frac{\delta f_1(0)}{\delta f_1(0) - f_2(0)}\ \frac{M^*_{n,m}(\delta) - K^*_{n,m}(\delta)[\delta f_1(0) - f_2(0)]/f_2(0)}{L^*_{n,m}(\delta)}\ .$$

Substitute $\tau'_\delta(0) = \delta f_1(0)/f_2(0)$ and (8.20) into (8.14); after multiplying through by $f_2(0)/\delta f_1(0)$, we obtain

$$(8.21)\qquad \frac{f_2(0)M^*_{n,m}(\delta) - K^*_{n,m}(\delta)[\delta f_1(0) - f_2(0)]}{[\delta f_1(0) - f_2(0)]L^*_{n,m}(\delta)} - 1 = 0,$$

or equivalently,

$$(8.22)\qquad f_2(0)M^*_{n,m}(\delta) - [\delta f_1(0) - f_2(0)][K^*_{n,m}(\delta) + L^*_{n,m}(\delta)] = 0.$$

After substituting (8.02-04) for $K^*_{n,m}$, $L^*_{n,m}$, and $M^*_{n,m}$, (8.22) reduces to

$$(8.23)\qquad \sum_{i=0}^{m-1} \binom{n}{m-i}\binom{m-1}{i}\Big[(m+1)f_2(0) - m\delta f_1(0)\Big]\delta^i = 0.$$

It is clear that the unique positive solution to (8.23) is $\delta = (m+1)f_2(0)/mf_1(0)$.

Fixing δ at this value, we now prove (8.15). For this value of δ, we have the following limits along $v = \tau_\delta(b)$:

$$(8.24)\qquad \lim_{b\downarrow 0}\ (v - b)f_1(b)/F_1(b) = 1/m;$$

$$(8.25) \qquad \lim_{b\downarrow 0} F_2(v)/f_2(v)(v - b) = m;$$

$$(8.26) \qquad \lim_{b\downarrow 0} \frac{\partial z}{\partial b}(v - b) = -\delta/m.$$

It can also be shown that $d/db[(v-b)f_1(b)/F_1(b)]$ approaches a finite limit. Treating $F_2(v)/f_2(v)$ as a constant, we compute $\partial\dot{v}/\partial b$ by applying the quotient rule to the remaining term in (8.05). The denominator of the resulting fraction is $[(v-b)L^*_{n,m}(z)]^2$. Applying (8.16) and (8.25), we obtain as a limit along $v = r_\delta(b)$

$$(8.27) \qquad \lim_{b\downarrow 0} \frac{F_2(v)}{f_2(v)[(v - b)L^*_{n,m}(z)]^2} = \frac{m}{[L^*_{n,m}(\delta)]^2} \quad \lim_{b\downarrow 0} \frac{1}{v - b} = \infty.$$

To prove (8.15), it is therefore sufficient to show that the numerator of the fraction obtained by the quotient rule has a positive finite limit. The limit of this numerator along $v = r_\delta(b)$ is

$$(8.28) \qquad [M^{*\prime}_{n,m}(\delta) - K^{*\prime}_{n,m}(\delta)/m](-\delta/m)L^*_{n,m}(\delta)$$
$$- [M^*_{n,m}(\delta) - K^*_{n,m}(\delta)/m][L^{*\prime}_{n,m}(\delta)(-\delta/m) - L^*_{n,m}(\delta)].$$

Reindex formula (8.03) for $L^*_{n,m}$ by replacing i with j, and then write (8.28) as a single polynomial in δ. For $0 \le t \le 2(m-1)$, the coefficient of δ^t in this polynomial is

$$(8.29) \qquad \sum_{\substack{i+j=t \\ 0\le i,j\le m-1}} \binom{n}{m-i}\binom{m-1}{i}\binom{n}{m-j}\binom{m-1}{j}(-i^2j + ij^2 + mij)/m^2.$$

The proof is now completed by showing that all of these coefficients are nonnegative, and some are strictly positive. An $i = j$ term in (8.29) is nonnegative, if it is present; the coefficient of $\delta^{2(m-1)}$ is clearly positive. We pair the remaining terms in (8.29); for $u \neq v$, the $i = u$, $j = v$ and the $j = u$, $i = v$ term sum to

$$(8.30)\quad \binom{n}{m-u}\binom{m-1}{u}\binom{n}{m-v}\binom{m-1}{u}\frac{2uv}{m},$$

which implies that the sum (8.29) is nonnegative. Q.E.D.

<u>Proof of Lemma 4.2</u>. Define $N_{n,m}(v,b)$ as

$$(8.31)\quad N_{n,m}(v,b) \equiv M^{*}_{n,m}(v,b)/K^{*}_{n,m}(v,b).$$

It is sufficient to prove that (4.03-05) are satisfied by $N_{n,m}$. Substitution of $n = 1$ into (8.02) and (8.04) shows that $N_{1,m}(v,b) = 1$, which establishes (4.05). Statements (4.03-04) are proven by showing that $N_{n,m} - N_{n+1,m}$ and $N_{n,m} - N_{n,m+1}$ are positive. The sign of each of these expressions is determined by its numerator, which is a polynomial in z. The proofs of (4.03-04) are completed by showing that all of the coefficients of each of these polynomials are nonnegative, and some are strictly positive.

We begin with (4.03). Reindex the denominator $K^{*}_{n,m}$ of $N_{n,m}$ by replacing i with j. The numerator of $N_{n,m} - N_{n+1,m}$ is

$$(8.32)\quad \left[\sum_{i=0}^{m-1}\binom{n}{m-i}\binom{m-1}{i}z^{i}\right]\left[\sum_{j=0}^{m-1}\binom{m-1}{j}\binom{n+1}{m-j}(m-j)z^{i}\right]$$

$$-\left[\sum_{j=0}^{m-1}\binom{m-1}{j}\binom{n}{m-j}(m-j)z^{j}\right]\left[\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{n+1}{m-i}z^{i}\right].$$

For $0 \le t \le 2(m-1)$, the coefficient of z^t is

$$(8.33)\qquad \sum_{\substack{i+j=t \\ 0\le i,j\le m-1}} \binom{m-1}{i}\binom{m-1}{j}\left[\binom{n}{m-i}\binom{n+1}{m-j} - \binom{n}{m-j}\binom{n+1}{m-i}\right](m-j).$$

The terms in (8.33) are now paired, and each pair is then shown to be nonnegative. The $i = u$, $j = v$ term is added to the $i = v$, $j = u$ term to obtain

$$(8.34)\qquad \binom{m-1}{u}\binom{m-1}{v}\left[\binom{n}{m-u}\binom{n+1}{m-v} - \binom{n}{m-v}\binom{n+1}{m-u}\right](u-v).$$

Four cases are now considered that depend upon which of the two terms in brackets in (8.34) are nonzero. If both are zero, then (8.34) is zero. If both are nonzero, then the following formula shows that the expression in brackets has the same sign as $u-v$:

$$(8.35)\qquad \binom{n}{m-u}\binom{n+1}{m-v} = \binom{n}{m-v}\binom{n+1}{m-u}\frac{n-m+1+u}{n-m+1+v}.$$

If the first term is zero but the second term is not, then $m-u = n+1$ and $m-v \le n$, which implies that $v \ge u+1$; the expression in brackets therefore has the same sign as $u-v$. A similar argument applies when the first term is nonzero and the second term is zero.

Each coefficient of the numerator of $N_{n,m} - N_{n+1,m}$ is therefore nonnegative. When $m \ge 2$, a positive coefficient is produced by substituting $u = m-1$, $v = m-2$ into (8.34). This completes the proof of (4.03).

Now consider (4.04). The numerator of $N_{n,m} - N_{n,m+1}$ is

$$(8.36)\qquad \left[\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{n}{m-i}z^i\right]\left[\sum_{j=0}^{m}\binom{m}{j}\binom{n}{m+1-j}(m+1-j)z^j\right]$$

$$-\left[\sum_{i=0}^{m-1}\binom{m-1}{i}\binom{n}{m-i}(m-i)z^i\right]\left[\sum_{j=0}^{m}\binom{m}{j}\binom{n}{m+1-j}z^j\right],$$

where the terms from $N_{n,m+1}$ have been reindexed using j instead of i. For $0 \le t \le 2m-1$, the coefficient of z^t is

$$(8.37)\qquad \sum_{\substack{i+j=t\\ 0\le i\le m-1\\ 0\le j\le m}}\binom{m-1}{i}\binom{m}{j}\binom{n}{m-i}\binom{n}{m+1-j}(i+1-j).$$

An $i = j-1$ term and a $j = 0$ term are clearly nonnegative, if either is present. A well-defined pairing of the remaining terms in (8.37) is obtained by adding the $i = u$, $j = v$ term to the $i = v-1$, $j = u+1$ term. This sum is

$$(8.38)\qquad (u+1-v)\binom{n}{m-u}\binom{n}{m+1-v}\left[\binom{m-1}{u}\binom{m}{v}-\binom{m-1}{v-1}\binom{m}{u+1}\right].$$

The equality

$$(8.39)\qquad \binom{m-1}{u}\binom{m}{v}=\binom{m-1}{v-1}\binom{m}{u+1}\frac{u+1}{v}$$

shows that the expression in brackets in (8.38) has the same sign as $(u+1-v)$. Each coefficient (8.37) is therefore nonnegative. Finally, in the $n \ge 2$ case substitution of $u = v = m-1$ into (8.38) shows that at least one of these coefficients is positive. Q.E.D.

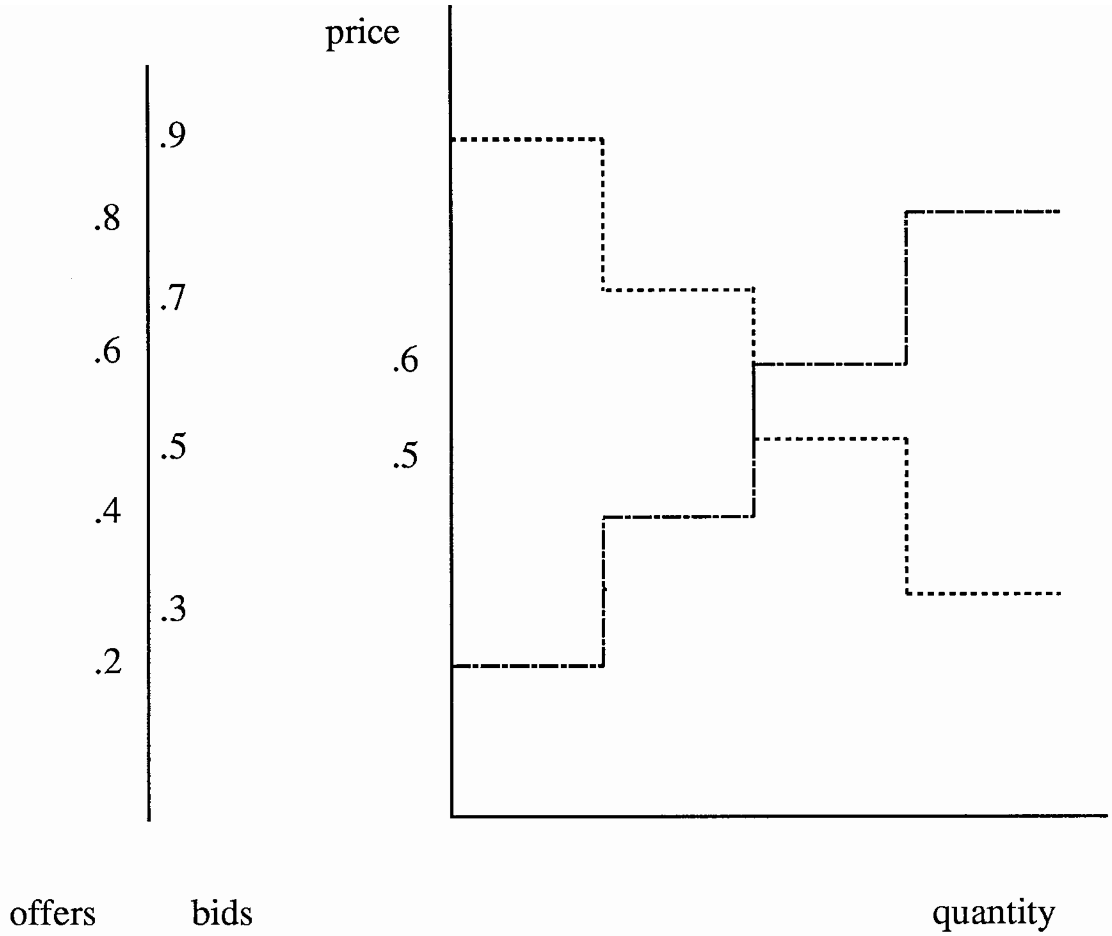

Figure 1.1. On the left is a list of offers/bids in a market with n = 4 sellers and m = 4 buyers with the corresponding supply and demand curves on the right. Any number between the mth largest offer/bid of .5 and the (m+1)st largest offer/bid of .6 is a market-clearing price. The BBDA selects the (m+1)st largest offer/bid as the price.

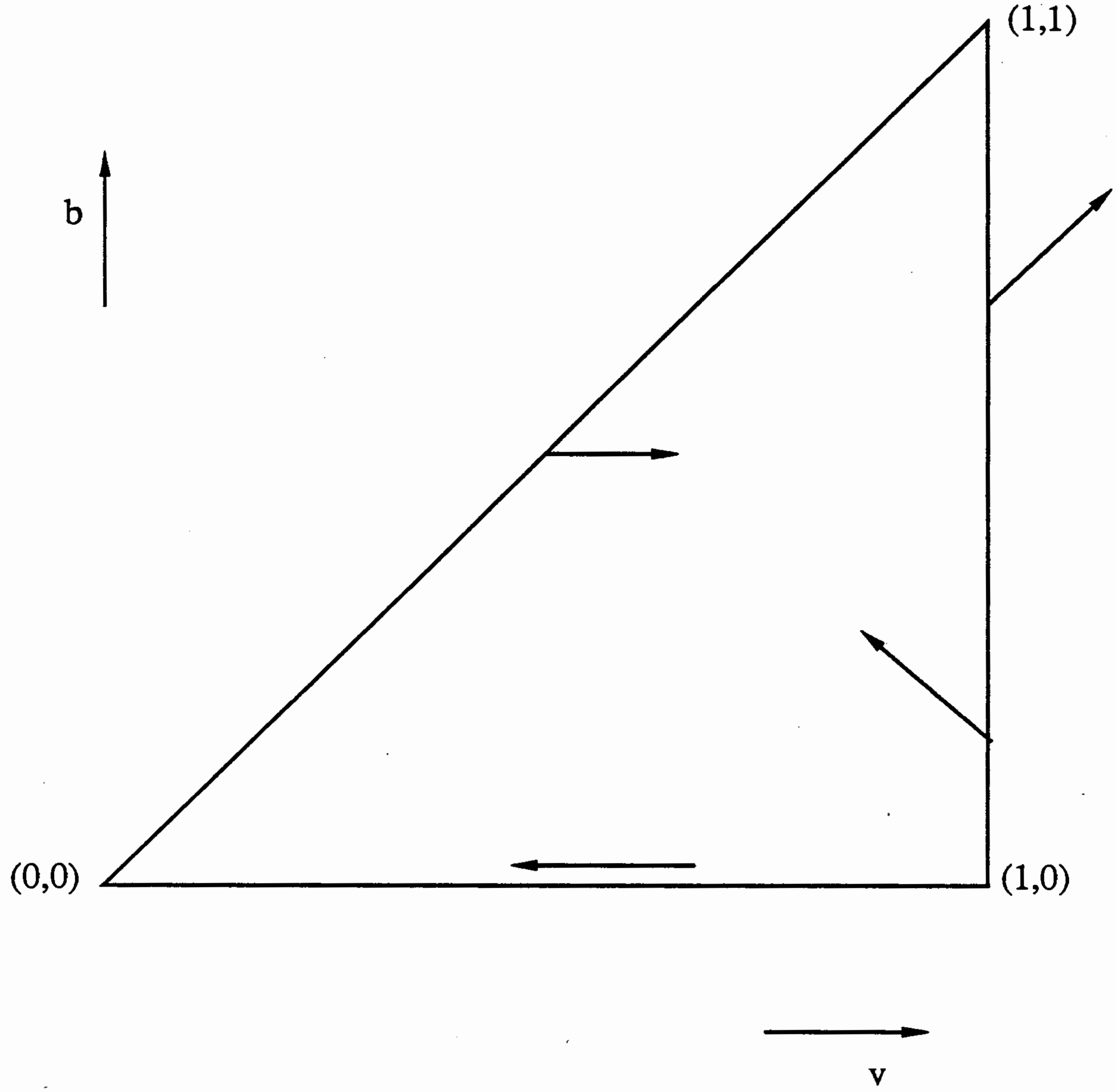

Figure 3.1. The vectors depict the direction of the vector field defined by (3.01-02) on the sides of the triangle $0 \le b \le v \le 1$.

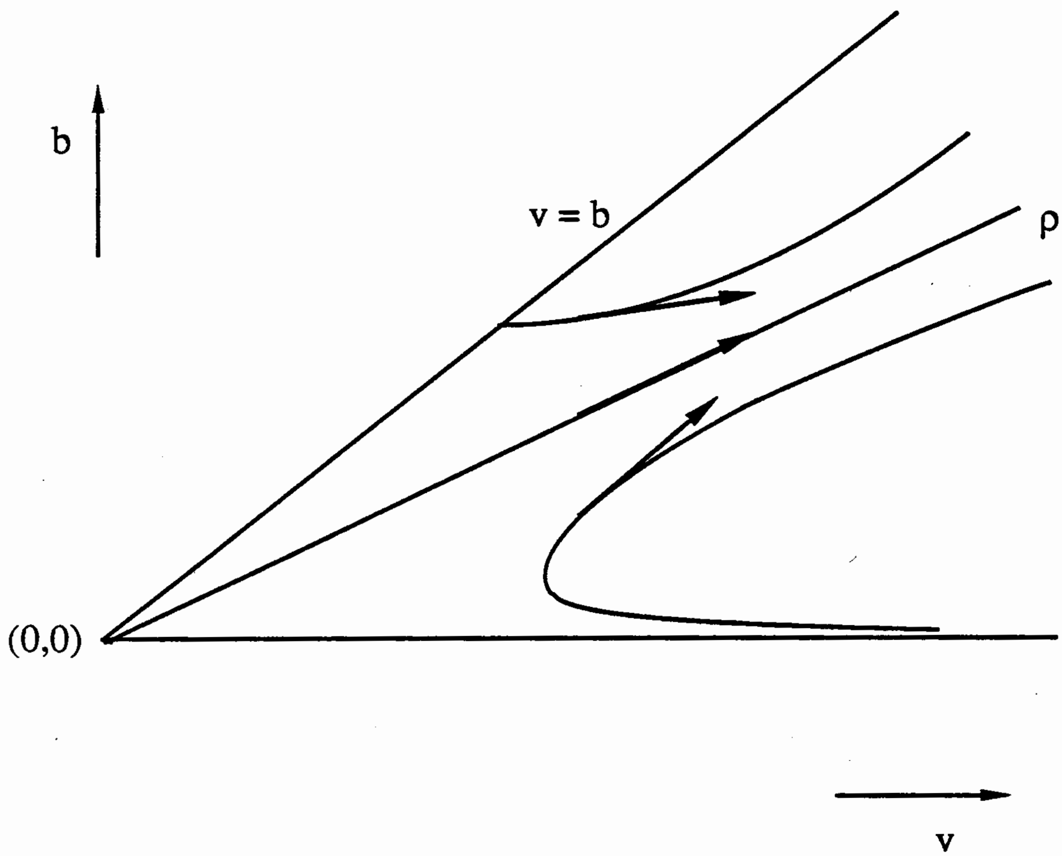

Figure 3.2. The solution curve ρ to (3.01-02) passes through (0,0), and its tangent at (0,0) is $b = mv/(m+1)$. Solution curves above ρ enter the triangle through the $v = b$ edge, while those below ρ enter through the $v = 1$ edge.

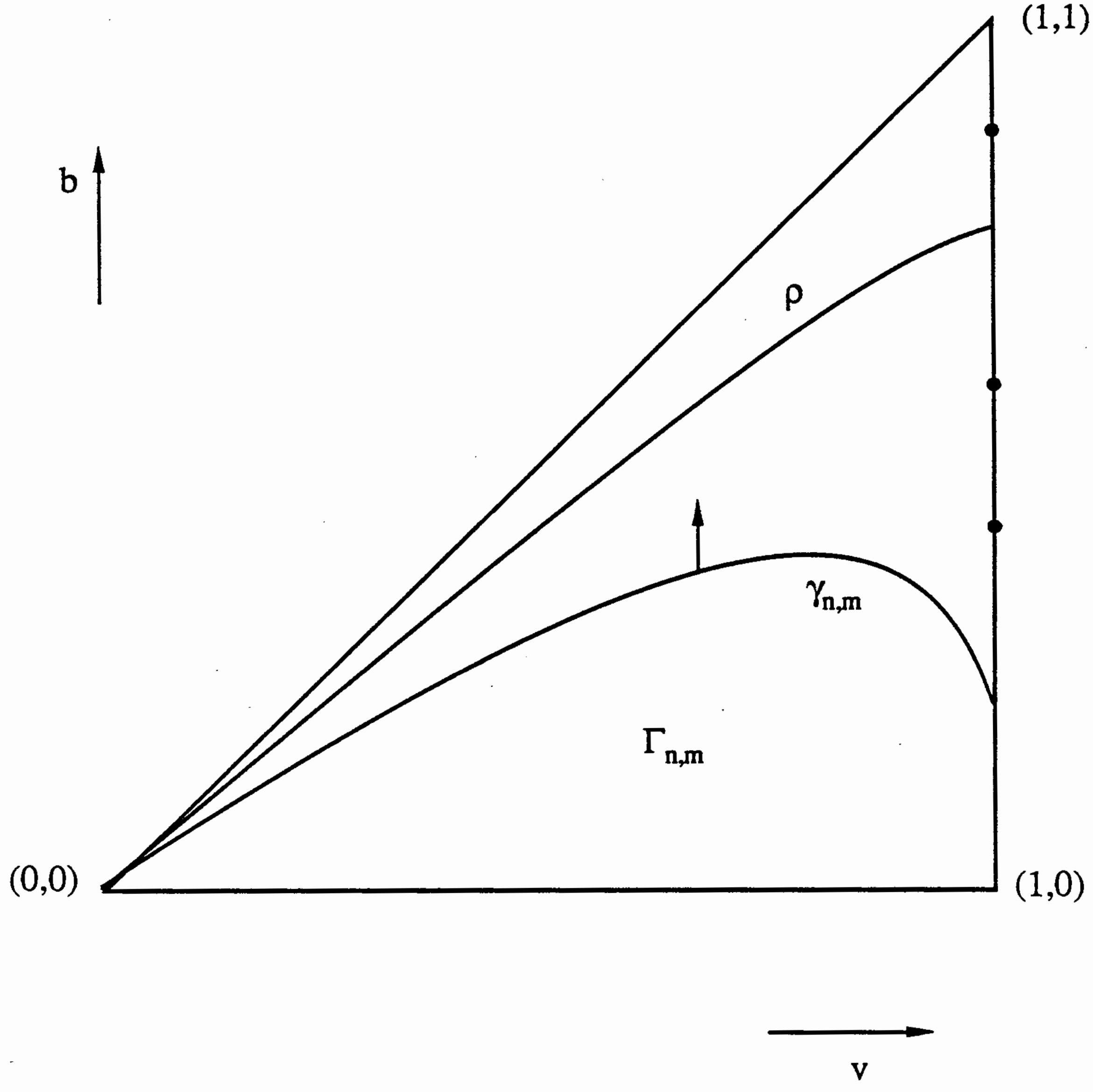

Figure 6.1. If $\gamma_{n,m}$ consists of the graph of a C^0 function plus several isolated points on the v = 1 edge, then the solution curve to (3.01-02) through (0,0) defines a smooth equilibrium strategy of buyers.

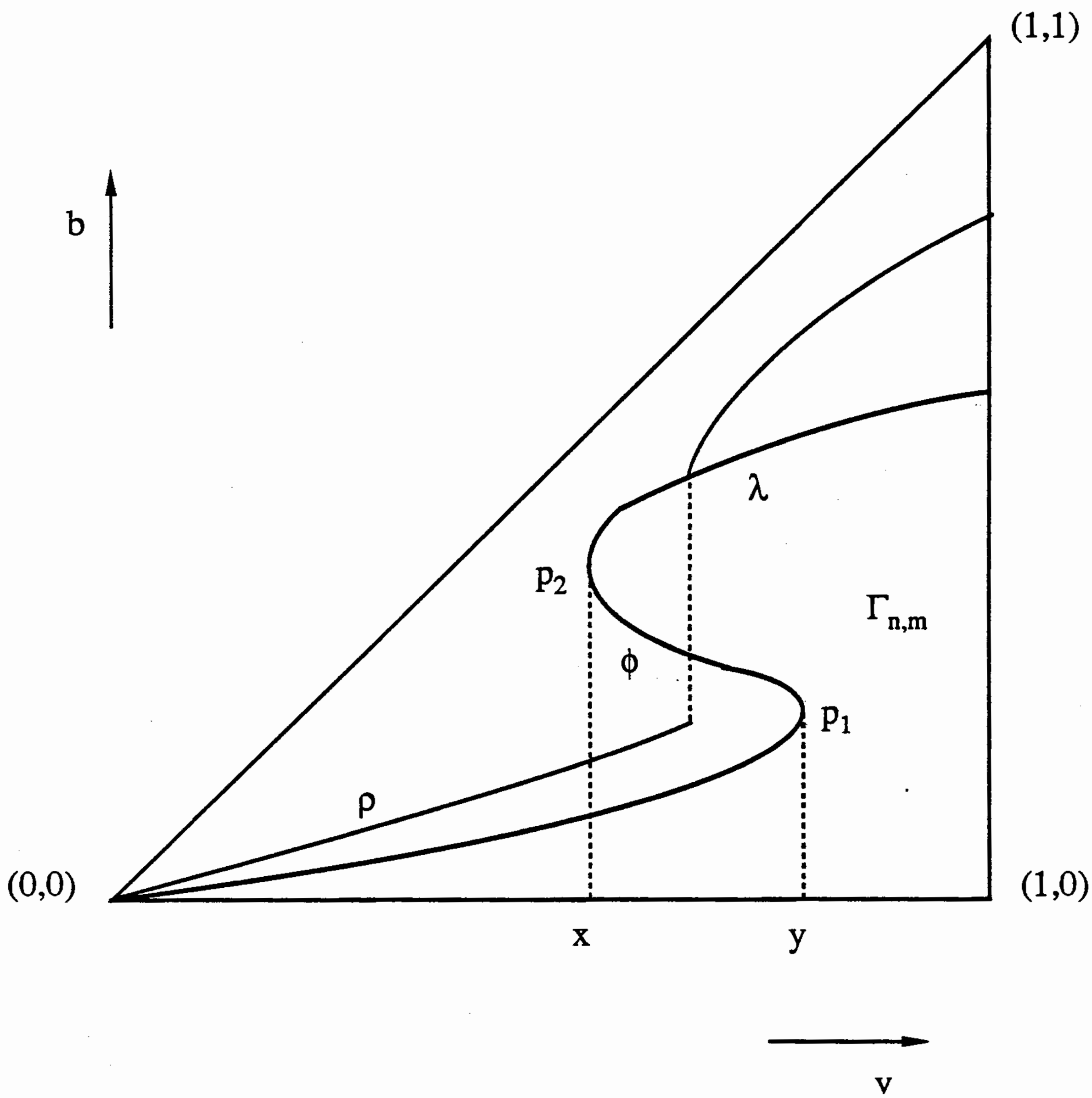

Figure 6.2. Suppose the solution curve ρ through (0,0) passes below p_2. An equilibrium strategy of buyers can be defined by jumping from ρ to an appropriate point on $\gamma_{n,m,}$ and continuing along the solution curve through that point.

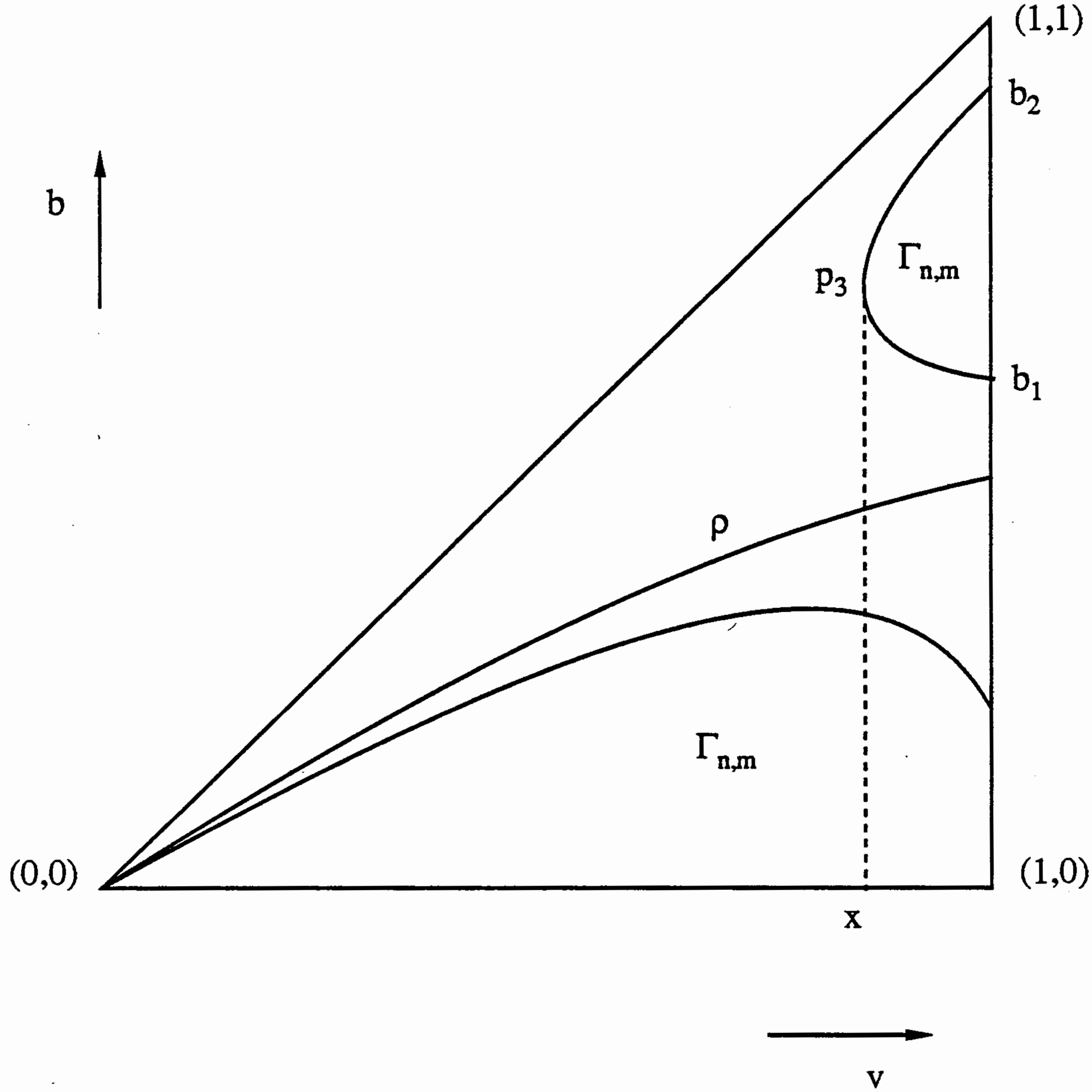

Figure 6.3. The solution curve ρ through (0,0) may or may not define an equilibrium strategy when it passes below a component of $\Gamma_{n,m}$ that borders on the $v = 1$ edge. When it does not, a discontinuity can be introduced to define an equilibrium strategy.

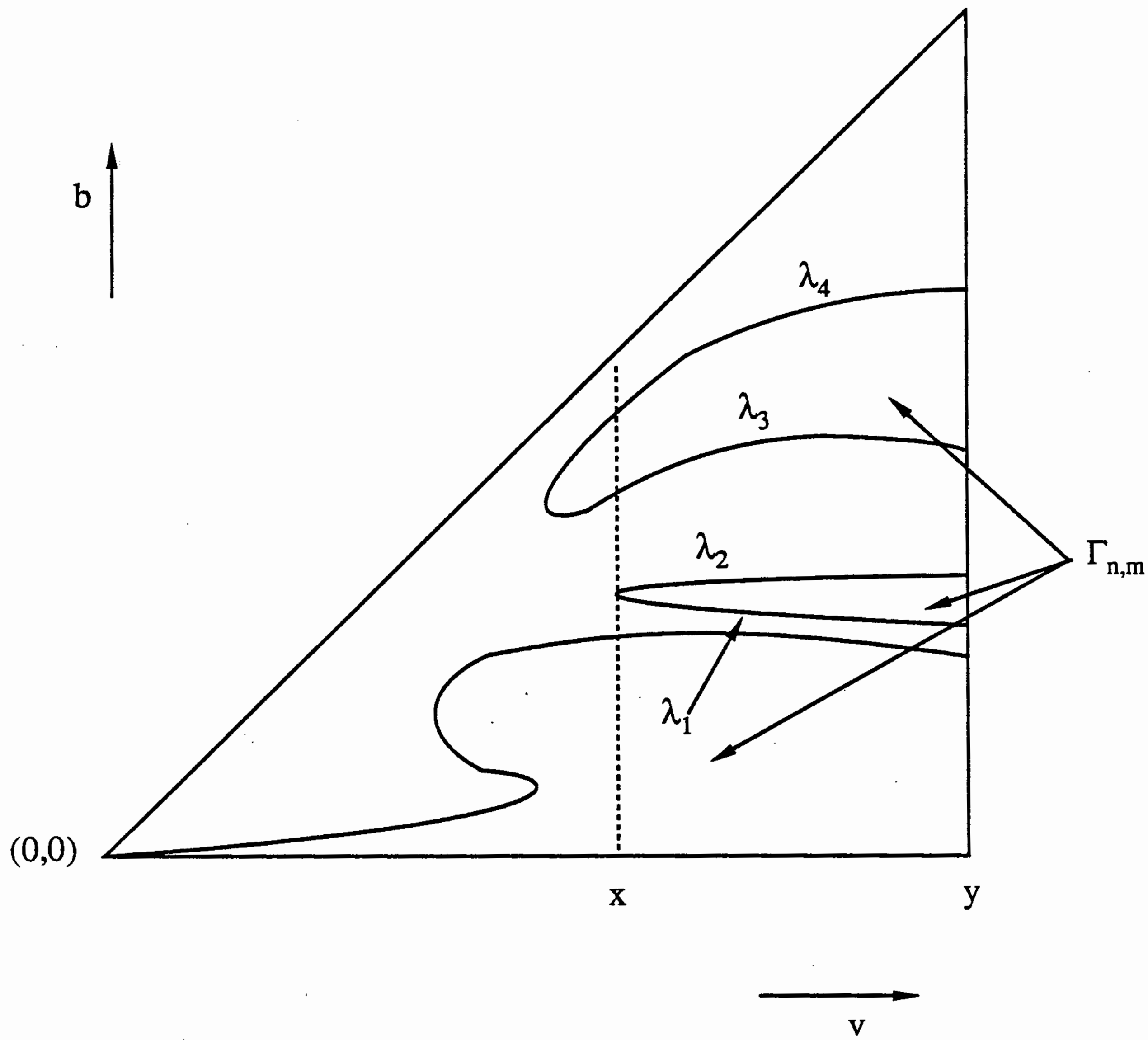

Figure 6.4. For some $x \in (0,y)$, $\gamma_{n,m}$ has no turning points in the strip defined by $x < v < y$ and $0 < b < v < y$.