Discussion Paper No. 790

EQUILIBRIUM, PRICE FORMATION AND THE VALUE OF INFORMATION IN ECONOMIES WITH PRIVATELY INFORMED AGENTS

by

Matthew O. Jackson
Kellogg Graduate School of Management
Northwestern University, Evanston IL 60208

First Draft: April 1987
Revised: May 1988

Abstract

In this paper, the allocation of goods and the endogenous acquisition of information are studied in an economy with a finite number of agents. More specifically, an economy is analyzed in which agents first choose to acquire information, at a cost, concerning the return to a risky asset, and then choose demand functions which determine the allocation of assets. Exchange is modeled as a game, permitting agents to act strategically. Equilibria are demonstrated in which the price of the risky asset fully reveals the relevant information of all agents, and yet agents still wish to acquire (costly) information. This resolves a well known paradox. The equilibria provide a new notion of the value of information (which is discussed with respect to a conventional notion of the value of information).

---

*This is a chapter of my dissertation, Jackson (1988). I thank the members of my dissertation committee Darrell Duffie, Dilip Mookherjee, and Bob Wilson for their generous help in developing this work. I also thank Anat Admati, Paul Pfleiderer, and Hugo Sonnenschein for helpful conversations. I wish to acknowledge the support of a National Science Foundation fellowship, the Graduate School of Business at Stanford University, and a State Farm Doctoral Dissertation Fellowship.

1. Introduction

In economies in which agents initially possess private and diverse information, rational expectations models have been used to analyze the allocation of goods.[1] In the rational expectations paradigm, traders understand that prices convey information. They condition their expectations upon prices and thus the information in prices is used in the formation of demand functions. At the same time, however, agents are assumed to be price takers. This latter assumption is generally justified as representing competitive behavior. The combination of these assumptions leads in some instances to discontinuities. As Hellwig (1980) terms it, agents are acting "schizophrenically": when forming expectations, agents understand that prices may convey their information and the information of other agents, but as price takers, they may not understand the role of their information in price formation.

There are several types of problems associated with the discontinuities resulting from the "schizophrenic" behavior. We focus on two of these problems which are closely related. The first is an issue discussed by Beja (1977). He examines situations in which the price is fully revealing.[2] Beja points out that if agents are price takers and prices are fully revealing, then demands are functions of prices alone. That is, since prices are fully revealing, an agent's private information is redundant given the price, and therefore need not be included in the formation of demand. However, if individual demands are independent of private information, then the price cannot incorporate any private information. Beja shows that if we require that the price contains only information which can be inferred from the actions of agents, and if agents are price takers, then there will not exist a fully revealing equilibrium price. The second issue is a non–existence result discussed by Grossman (1976) and Grossman and Stiglitz (1980). It concerns the value of information when prices are fully revealing. If agents are price takers and prices are fully revealing, then no agent will wish to purchase information, for all relevant information is revealed by prices. However, if agents are price takers and no one purchases information, then prices reveal nothing and an agent will wish to purchase information. It is important to note that the two issues are related. In the model of Grossman (1976) price formation is not explicit, and the equilibrium price

[1] For surveys see Admati (1985), Allen (1985), and Grossman (1981).

[2] Loosely, a price is fully revealing if the probability measure on states of the world conditional on all private information and the price is the same as the probability measure conditional only on the price.

contains more information than generated by the demands of the agents.[3] From the results of Beja, we know this must be the case since in Grossman's model agents are price takers and the price is fully revealing.

The most common method of circumventing the difficulties described above has been to add noise to the system. This may be done in various ways[4], all leading to prices that are noisy and avoiding the discontinuities associated with fully revealing prices. In this manner, non–existence problems can be overcome and private information is not redundant given prices and is therefore valuable. Given that noisy prices seem to be realistic, this is a reasonable approach.

Although these approaches avoid the issues, they do not help us to understand them. At the heart of the two non–existence results is the price taking assumption. In the rational expectations paradigm, an agent who is considering whether to acquire information or not does not compare the different equilibria which will result from different choices. The agent takes the price as given and only compares the changes in information which result and their direct impact on his or her expected utility. Furthermore, in much of the rational expectations literature, including Grossman (1976) and Grossman and Stiglitz (1980), price formation is not made explicit.[5] This means that even if agents are permitted to take into account the implications of their actions, it is unclear what the implications actually are since price formation is not made explicit. [This is discussed with respect to Grossman's (1976) model in Section 3.]

In this paper the strategic behavior of agents is analyzed. The economy is modeled as a game, and hence, the price formation is explicit. In particular, agents understand how an equilibrium is formed and take this into account in making their decisions. This means that, in choosing whether or not to acquire a signal, the agent compares the expected utility associated with the allocation in which he or she has an additional signal, to the one in which he or she does not. We find that when we have made the price formation explicit and

[3] This is discussed in detail in Section 3.

[4] Examples are: adding noise traders, assuming that the aggregate endowment is imperfectly observed, or considering uncertainty which has dimension greater than that of the price.

[5] Notable exceptions are Diamond and Verrecchia (1981) and Anderson and Sonnenschein (1982). However, given the results of Beja, these models cannot permit fully revealing prices.

dropped the price taking assumption, the non–existence result of Grossman is overturned. As a bonus, we find a new and interesting notion of the value of information.

We examine a private information economy with a finite number of agents who act strategically. Formally, this is modeled as a game in which agents submit demand functions and understand how a price is chosen to clear markets. No noise is added and an equilibrium is demonstrated in which the equilibrium price is fully revealing and yet contains only information present in the demands of the agents. [This point is emphasized, since it is critical. In order to understand the Grossman paradox, we need a framework in which the price formation is explicit and in which the price is fully revealing. According to Beja's results, this requires dropping the price taking assumption.] Furthermore, the price is fully revealing in a single round of trade.[6] Information will be shown to be valuable, even though the price is fully revealing. Specifically, in this setting the distribution of information and the beliefs of agents concerning the total amount of information present in the economy is important in determining the equilibrium allocations. This provides us with a new understanding of the value of information. In an exchange economy, the value of information has, in the past, been associated with an advantage it gives to an agent in forming expectations. Here, the value of information to a particular agent is a function of the beliefs of the other agents concerning the acquisition of information. We find situations in which agents acquire information because other agents expect them to, and because they understand that their allocations depend on how they respond to the actions of the other agents. [This is made more precise in Section 5.] We also find that when information acquisition is a public event, the equilibria we describe are (ex–ante) Pareto efficient, while when information acquisition is a private event, the equilibria may not be (ex–ante) Pareto efficient. In this model, the competitive case is analyzed by letting the number of agents in the economy become arbitrarily large. In some situations, the results will turn out to be markedly non–competitive: a small group of agents acquire information and submit demands markedly different from those of other agents, despite the growing number of agents in the economy.

[6] Dubey, Geanakoplos, and Shubik (1987) analyze a game in which prices are fully revealing in the second round of trade. The value of information is associated with the advantage it gives agents before the price becomes fully revealing. Informed traders benefit in the first period, before the information is known to all through the price. Milgrom (1981) also examines the Grossman paradox. See Section 4 for a discussion.

The paper is organized as follows. First, a simple framework is described in which rational expectations equilibria and two demand submission games are defined. In the first game, information acquisition is publicly observed prior to trade, while in the second game, it is not. Next, the importance of the measurability of price with respect to agents' demands is discussed and Grossman's equilibrium is re-examined. We then examine a share auction in which agents are risk-neutral and the return to the auctioned risky asset is exponentially distributed. Agents purchase signals which are correlated with the return to the risky asset. We begin by presenting a set of optimal trading strategies (as functions of observed signals) which result in a fully revealing price. Given this analysis, we describe the dependence of an agent's ex-ante expected utility on signal purchase. We then obtain an expression for the value of information and describe equilibrium signal acquisition, for both games. Signals are valuable even though the price is fully revealing. The intuition for this result is discussed along with its relation to the Grossman paradox. It is shown that the equilibrium total number of signals acquired is constant, even as the number of agents in the economy becomes arbitrarily large. Finally, we discuss the advantages and disadvantages of a game theoretic approach.

2. A general framework

In this section we describe the basic framework of the paper.

The economy consists of N traders. There are two traded assets: a risk-free asset with known gross return $R > 0$, and a risky asset with unknown[7] return P. A signal is the value of a random variable, y_k, which is jointly distributed with P. At a cost, agents may observe signals at time $t = 0$. The set of signals observed by agent i is indexed by the set S_i (so $\{y_k\}_{k \in S_i}$ are the signals observed by agent i). Let $n_i = \#S_i$. The cost of purchasing signals incurred by agent i is given by $C_i : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I}$ denotes the non-negative integers. Define n to be the total number of signals observed across all agents and y to be the vector $(y_1, \ldots, y_n)$. The collection $(y, S_1, \ldots, S_N)$ is called an *information structure.* The values of y_k and y_j, $k \neq j$, are independently and identically distributed conditional upon P. We will consider two situations, the first in which signal acquisition is publicly observed (that

[7] All random variables in the model are defined on a common probability space. Expectation and conditional expectation statements are relative to this probability space, and "almost surely" is suppressed for simplicity.

is, $(n_1, \ldots, n_N)$ is common knowledge), prior to trade, and the second in which agents know only their own n_i, prior to trade. At time $t = 1$, trade occurs and then, at $t = 2$, the return P is revealed and agents consume. This is described in the figure below.

Figure I

$t = 0$	$t = 1$	$t = 2$
Signals are purchased.	Trade is conducted.	P is revealed and agents consume.

Agent i's wealth available for trade at $t = 1$ is the market value of w_i and e_i, agent i's endowment of the risk–free and risky assets, respectively. The trading prices of the risk–free and risky assets are 1 and p, respectively. After trading is completed, agent i's risk–free and risky asset holdings are denoted by z_i and x_i, respectively. At time $t = 2$, P is revealed and the agent's wealth is $Rz_i + Px_i$. The agent's budget constraint is

$$w_i + pe_i = z_i + C_i(n_i) + px_i,$$

which leads to the following expression for z_i as a function of x_i:

$$z_i(x_i) = w_i + pe_i - C_i(n_i) - px_i.$$

The agent obtains utility dependent upon the post–trade wealth. Agents are assumed to act so as to maximize a Von Neumann–Morgenstern utility of the wealth, U_i, such that the expectation of $U_i(Rz_i(x_i) + Px_i)$ exists for all $x_i \in I\!R$.

RATIONAL EXPECTATIONS EQUILIBRIA

Let us define a demand function to be a mapping from the price of the risky asset and an agent's signals to units of the risky asset, $d_i : I\!R \times I\!R^{n_i} \to I\!R$.[8] A price function is a mapping from the set of all signals to the price of the risky asset, $\widehat{p} : I\!R^n \to I\!R$. In this framework, a rational expectations equilibrium for a given information structure, $(y, S_1, \ldots, S_N)$, is defined as a set of demand functions and a price function, $(d_1(\cdot), \ldots, d_N(\cdot);\ \widehat{p}(\cdot)\ )$ such that

(i) $d_i(\widehat{p}(y), \{y_k\}_{k \in S_i}) \in \text{argmax}_{x_i \in I\!R} E\left[U_i(Rz_i(x_i) + Px_i) | \{y_k\}_{k \in S_i}, \widehat{p}(y)\right]$, and

(ii) $\sum_{i=1}^{N} d_i(\widehat{p}(y), \{y_k\}_{k \in S_i}) = \overline{X}$,

[8] To simplify the exposition, the budget constraint has been assumed to hold with equality. Hence, the demand for the riskless asset is implicitly determined by $z_i[d_i(p)]$.

where $\overline{X}$ is the supply of the risky asset available for trade, which is assumed to be fixed.

Since in the rational expectations paradigm agents are price takers, the above definitions apply whether signal acquisition is publicly observed or not.

DEMAND SUBMISSION GAMES

We first describe trade. The set of possible trading actions for each agent is the set $C(\mathbb{R})$ of all continuous (demand) functions mapping $\mathbb{R}$ into $\mathbb{R}$. Agents simultaneously submit $D_i \in C(\mathbb{R})$, $i \in \{1, \dots, N\}$. Define the set $A(D) = \{ \ p \mid \sum_i D_i(p) = \overline{X}, \ p \geq 0 \ \}$, where $D = (D_1, \dots, D_N)$. The payoff to agent i is $U_i(Px_i, Rz_i)$ (realized at $t = 2$), where the net trade (x_i, z_i) is determined according to

(i) $(x_i, z_i) = (e_i, w_i - C_i(n_i))$ if $A(D) = \emptyset$, and

(ii) $(x_i, z_i) = (D_i(p^*), w_i - p^*[D_i(p^*) - e_i] - C_i(n_i))$ if $A(D) \neq \emptyset$,

where $p^* = min_{p \in A(D)} p$ when $A(D) \neq \emptyset$. [9]

We now consider the game where agents choose how well to become informed (at $t = 0$) and choose a trading action (at $t = 1$).

PUBLICLY OBSERVED SIGNAL PURCHASE.

With publicly observed signal purchase, $(n_1, \dots, n_N)$ is common knowledge at $t = 1$, and thus is known by agent i when i chooses a trading action D_i. Therefore, in this situation, a (pure) strategy is defined to be a choice of n_i and a choice of a map from observed signals and signal purchase choices to a trading action [that is, from $\mathbb{R}^{n_i} \times \mathbb{I}^N$ to $C(\mathbb{R})$]. An *equilibrium* (with publicly observed signal purchase) is a subgame perfect Bayesian Nash equilibrium of the game. In other words, an equilibrium is a set of strategies such that, for each i: (a) for any $(n_1, \dots, n_N)$, agent i's trading action maximizes i's expected utility, given $\{y_k\}_{k \in S_i}$ and the strategies of the other agents,[10] and (b) n_i maximizes i's ex–ante expected utility, given the strategies of the other agents and i's trading action.

---

[9] Since each $D_i(\cdot)$ is continuous, $A(D)$ is closed and p^* is well defined. The particular description of how p^* is chosen is arbitrary. Continuity of each D_i is required solely for the purpose of assuring that p^* is well defined.

[10] Information can be inferred from the market clearing restriction implicit in the structure of the game. For example, consider an economy with two agents each having one signal. Suppose $\sigma_2(y_2) = y_2 - p$. Then agent 1 can deduce that $y_2 = \overline{X} + p^* - x_1$. This will become clear in the following sections.

PRIVATE SIGNAL PURCHASE.

With private signal purchase, agents only know their own n_i and $\{y_k\}_{k \in S_i}$ when they choose a trading action. Therefore, in this situation, a (pure) strategy is a choice of n_i and a map from signals to a trading action $[\mathbb{R}^{n_i} \to \mathrm{C}(\mathbb{R})]$. An *equilibrium* is defined to be a Bayesian Nash equilibrium of the game (with private signal purchase). In this case, an equilibrium is a set of strategies such that agent i's strategy maximizes i's ex–ante expected utility, given the strategies of the other agents.

3. The Resolution of the Paradoxes and Price Taking Behavior

Let us begin by re–examining the model described by Grossman (1976). There are a finite number of types of agents, n, and a continuum (of mass 1) of agents of each type. Agents of type i have negative exponential utility with risk aversion parameter a_i and observe the same signal, y_i. The return to the risky asset is normally distributed and signals (of different types) are independently, identically, and normally distributed conditional on P.

Grossman describes the following fully revealing rational expectations equilibrium. The demand of an agent of type i is given by

$$d_i(p, y_i) = \frac{E[P|y_i, \widehat{p}(y)] - Rp}{a_i V},$$

where V is the variance of P conditional on y, and the fully revealing equilibrium price is

$$\widehat{p}(y) = \frac{E[P|y]}{R} - \frac{V\overline{X}}{R\sum_i \frac{1}{a_i}}.$$

Given the fully revealing price we can write agents' demands as

$$d_i(p, y_i) = \frac{\overline{X}}{a_i \sum_k \frac{1}{a_k}}.$$

Notice that the equilibrium price is not measurable with respect to the agents' demand functions. That is, it incorporates more information than is present in the demands of the agents. It is unclear how the price comes to incorporate the signals of the agents.

In this situation, the revealing quality of the price does not depend upon the number of agents who are informed, since in the model there is no description of how the price is formed. Let us consider the situation in which no agents are informed and an agent considers becoming informed. If the agent does become informed there exists an equilibrium in which the price is fully revealing! It is simply the equilibrium with demands $d_i(p, y_i) = \overline{X}/(a_i \sum_k \frac{1}{a_k})$ and price $\widehat{p}(y_i) = E[P|y_i]/R - V\overline{X}/R \sum_i \frac{1}{a_i}$. There also exists an equilibrium in which the price reveals nothing (and the market clears almost surely). Again, demands are $d_i(p, y_i) = \overline{X}/(a_i \sum_k \frac{1}{a_k})$, except for the informed agent whose demand is $d_i(p, y_i) = (E[P|y_i] - Rp)/a_i V$, and the market clearing price is $\widehat{p}(y_i) = \overline{P}/R - V_P \overline{X}/R \sum_i \frac{1}{a_i}$, where V_P is the unconditional variance of P. The Grossman argument (and hence paradox) relies on the fact that the equilibrium is of the second type. For equilibria of the first type, there is no existence problem: there is an equilibrium where no agent chooses to become informed.

Part of the difficulty in analyzing this situation stems from the fact that the price is not measurable with respect to agents' demands, and thus we do not know how the price is formed, and which equilibrium will arise if an agent becomes informed. If the price were measurable with respect to agents' demands, we would not have such a difficulty. Beja's definition of a genuine trading process requires that the price be measurable with respect to agents' demands. The Beja 'paradox' is then simply that there do not exist genuine trading processes in which the price is fully revealing. As we shall show, however, this is not true if agents act strategically. In the strategic case, there exist equilibria with fully revealing prices which are measurable with respect to agents' demands.

In order to make an analysis of information acquisition, it is critical that agents understand how allocations are affected by their actions (even if they turn out to be unaffected), and hence, the price formation needs to be explicit. In the setup of a demand submission game the price is determined by the agents' actions. Hence, since there is a price associated with each set of actions, the equilibrium price is measurable with respect to the agents' strategies. We could, alternatively, examine the genuine trading processes which are essentially rational expectations equilibria with the additional requirement that prices be measurable with respect to demand. However, in order to address the issues associated with fully revealing prices, raised by Grossman, we have to drop the price taking assumption (by the results of Beja) and thus consider the game theoretic approach.

We now examine a demand submission game under specific parametric assumptions.

4. An Equilibrium with Fully Revealing Prices.

In this section we examine a share auction with risk neutral agents.[11] It is assumed that there is a seller who is auctioning a quantity $\overline{X}$ of the risky asset. [In Section 6 it is shown that the model can be completed by explicitly describing sellers and examining a fully specified exchange economy, rather than a share auction. The sellers, however, do not affect the actions of the buyers, the acquisition of information, or the price. Hence, we defer the description of sellers until we begin a welfare analysis.] Agents each submit a demand schedule indicating a quantity of the asset desired for each price. A price is then chosen which clears the market and shares are allocated accordingly.

Milgrom (1981) analyzes strategic behavior in a Vickrey auction.[12] He shows that there is value to information, even if the price is 'fully revealing' in a suitable sense. The price considered is the k–th highest bid of the other agents and is fully revealing in the sense that it reveals the information of the other agents only. Although the agent cannot condition upon this price, Milgrom shows that the agent would not change bids were this price known when the bid was made. In the Vickrey auction, however, the value of a signal is in part due to the fact that the signal is not redundant given this price. In this sense, the Grossman paradox is not resolved. The price in the Vickrey auction is not fully revealing in the same manner described by Grossman and Stiglitz, since it reveals only the information of the other agents and hence, the agent's signal is not redundant given the price.

In the share auction described here, agents submit demand functions, which allows them to condition directly on the price. The determination of the price and allocations are as specified in the description of a demand submission game in Section 2. Furthermore, the price is fully revealing in the same sense as defined by Grossman and Stiglitz. We will be able to resolve the Grossman paradox by showing that information is valuable to agents even though the price is fully revealing (and their signals are redundant given the price). In the share auction, an agent's allocation depends upon the agent's signals and the beliefs of other agents concerning the total number of signals purchased. The value

[11] The analysis in this section will build on Wilson (1979). For more discussion concerning the auction of shares the reader is referred to Wilson's paper. The share auction is well suited to analyzing the issues associated with the rational expectations paradigm because of its similarity to an exchange economy.

[12] A Vickrey auction is one in which k identical objects are auctioned. The k agents with the highest bids each receive one object and each pay the $k+1$–th highest price.

of information will derive from this relationship. The value of information we find differs from a conventional notion of the value of information (including that of Milgrom (1981)) in fundamental and important ways. This will become clear as we proceed.

In this model we make the assumptions: agents are risk–neutral, P is exponentially distributed with mean $\overline{P} > 0$, and signals are independently and identically distributed with an exponential distribution having mean $\overline{P}^{-1}$.[13]

We begin by analyzing trading strategies. The following proposition will apply to the game in which signal purchase is publicly observed, as well as to the game in which signal purchase is private. We will use this analysis to study the complete games with signal acquisition. Proposition 1 is partially derived from Wilson (1979). In Wilson (1979), the signal structure is common knowledge (each agent is endowed with one signal). Proposition 1 is designed to allow us to analyze endogenous signal acquisition, in a game where the signal structure is not common knowledge, at $t = 1$. Hence, it does not assume that the total number of signals to be acquired is known to any agent. It describes an optimal trading action for a given agent, who believes that a total of n signals have been acquired, while other agents believe that a total of $\hat{n}$ signals have been acquired. [In equilibrium, we will find that $\hat{n} = n$. The structure of Proposition 1 will allow us to examine deviations by a single agent.]

PROPOSITION 1. *Suppose that the strategies of agents $j \neq i$ have trading actions described by*

$$\frac{\overline{X}}{N-1}\left[1 - 2Rp\left(\frac{\Sigma_{k\in S_j} y_k \overline{P} + \frac{1}{N}}{\overline{P}(\hat{n}+1)}\right)\right], \tag{1}$$

for some $\hat{n} \in \mathit{I\!I}$. If overall strategies involve a total of n signals being purchased, then the trading action

$$\left(\frac{n+1}{\hat{n}+1}\right)\left(\frac{\overline{X}}{N-1}\right)\left[1 - 2Rp\left(\frac{\Sigma_{k\in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)}\right)\right],$$

is a best response for agent i. For these trading actions, the (fully revealing) market clearing price is

$$p^* = \frac{E[P|y]}{2R}.$$

[13] This can be extended to the more general Weibull distribution. The calculations for the value of information are a bit more complicated.

The proof of Proposition 1 is presented in Appendix 1.

It follows from Proposition 1 that if $(n_1, \ldots, n_N)$ is common knowledge, then the trading strategies described by (1) with $\hat{n} = n$ form a Bayesian Nash equilibrium of the trading subgame. Having dropped the price taking assumption, the non–existence result of Beja no longer holds: the equilibrium price is fully revealing, and yet price is explicitly determined by the actions of the agents (agents' demands are functions of their own signals). The share auction analyzed by Wilson (1979) was chosen as a basis for the analysis here, for this reason: To resolve the Grossman paradox we need to analyze an equilibrium in which the price is fully revealing and where price formation is explicit and understood by the agents.

We can provide an intuitive explanation for why agents' demands are functions of their own signals even though the price is fully revealing. Since agents are not price takers, they understand that it is the *equilibrium* price which is fully revealing. If they do not use the value of their own signals in forming a demand the price will only reveal the value of the signals of the other agents. In forming expectations, given the strategies of the other agents, agents view the price as a function of their own demand, and hence understand that they must make use of their own signals as well as the price.

5. The Value and Acquisition of Information.

We now turn to an examination of the information gathering decisions of the agents and the equilibria of the two games (described in Section 2). We first calculate an agent's ex–ante expected utility for the two information structures.

PUBLICLY OBSERVED SIGNAL PURCHASE.

We first examine the situation in which $(n_1, \ldots, n_N)$ is common knowledge prior to trade. We examine strategies in which agents choose the trading actions given in Proposition 1 in the trading portion of the game. Let $v_i(n_i, n)$ denote the (pre–trade) expected utility of agent i if he or she purchases n_i signals and a total of n signals are purchased by all agents. It should be emphasized that the expression $v_i(n_i, n)$ represents the expected utility of the agent *before* the signals are observed and trade occurs.[14] The expected utility v_i is also a function of $\overline{P}$, $\overline{X}$, N and R. We shall treat these as fixed attributes of the economy.

Proposition 2 develops an expression for the value of a signal to an agent. Using this we are able to characterize the set of equilibria in the overall game including signal gathering. In Proposition 2 it is assumed that agents are making calculations with the understanding that demands in the trading subgame will be the demands described in (1) with $\hat{n} = n$.

PROPOSITION 2.

$$v_i(n_i, n) = \frac{\overline{PX}}{2(N-1)} \left(\frac{n - n_i + 2\frac{N-1}{N}}{(n+2)} \right) + Rw_i - RC_i(n_i). \tag{2}$$

The proof of Proposition 2 is presented in Appendix 1.

PROPOSITION 3. *If C_i is non–decreasing and non–negative valued, Then the strategies with $n_i = 0$ and trading actions*

$$\frac{\overline{X}}{N-1} \left[1 - 2Rp \left(\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}} \right) \right]$$

form an equilibrium of the overall game with publicly observed signal purchase. Furthermore, this is the only equilibrium with the given trading actions.

PROOF: From Proposition 1 we know that the given trading actions form a Nash equilibrium of the trading subgame for any given $(n_1, \ldots, n_N)$. Therefore, to prove Proposition 3 we need to show that $n_i = 0$ is the correct choice for each agent i given that the given trading actions are to be played in the trading round. From Proposition 2 we know that

$$v_i(n_i, n) - v_i(0, n - n_i) = \frac{\overline{PX}}{2(N-1)} \left(\frac{-n_i(n - n_i + 2\frac{N-1}{N})}{(n+2)(n - n_i + 2)} \right) - R[C_i(n_i) - C_i(0)].$$

Since this expression is negative for all $n_i > 0$ $(n_i \leq n)$ it follows that no agent will wish to purchase information. ∎

Proposition 3 shows that there is an equilibrium in information acquisition even though the equilibrium price is fully revealing, and so the Grossman paradox does not arise. In

[14] At time $t = 0$, (n_i, n) is a sufficient statistic for $(y, S_1, \ldots, S_N)$ with respect to agent i's utility.

this situation it turns out that the equilibrium involves no information purchase. This is not surprising, given that signal purchase is publicly observed and prices are fully revealing. We now turn to the game in which signal purchase is a private event. There we will find that there are equilibria with valuable information, even though the price is fully revealing.

UNOBSERVED SIGNAL PURCHASE.

We now examine the game in which no agent knows how many signals any other agent has purchased.

Consider the situation in which agent i chooses to purchase $n_i + k$ signals and the other agents anticipate that a total of $\widehat{n}$ signals are to be purchased in the economy. Furthermore, in this case assume that the true number of signals to be purchased is $n = \widehat{n} + k$. [This may be interpreted as a deviation by agent i to purchase k extra (or fewer) signals from the situation in which agent i was to purchase n_i signals and a total of $\widehat{n}$ signals were to be purchased.] Define $V_i(n_i, \widehat{n}, k)$ to be the ex–ante expected utility of agent i in this situation given that the trading strategies are those described in Proposition 1. [Again, the expression $V_i(n_i, \widehat{n}, k)$ represents the expected utility of the agent *before* the signals are observed and trade occurs.[15]] The expected utility V_i is also a function of $\overline{P}$, $\overline{X}$, N and R. We fix these for now and discuss comparative statics later.

PROPOSITION 4.

$$V_i(n_i, \widehat{n}, k) = \left(\frac{\widehat{n} + k + 1}{\widehat{n} + k + 2}\right) \frac{\overline{PX}}{2(N-1)} \left(\frac{\widehat{n} - n_i + 2\frac{N-1}{N}}{\widehat{n} + 1}\right) + Rw_i - RC_i(n_i + k). \quad (3)$$

The proof of Proposition 4 is presented in Section 8.

We can see the tension in V_i, between the benefits from having a greater number of signals ($[\widehat{n} + k + 1]/[\widehat{n} + k + 2]$ increases), and the losses due to increased costs (C_i increases).

The following proposition describes an equilibrium of the game.

PROPOSITION 5. *If C_i is convex and non–negative valued for each i, then the strategies described by $(n_1, \ldots, n_N)$ and the trading actions described by (1) (with $\widehat{n} = n = \sum_{k=1}^{N} n_k$)*

[15] At time $t = 0$, $(n_i, \widehat{n}, k)$ is a sufficient statistic for $(y, S_1, \ldots, S_N)$ with respect to agent i's utility.

form an equilibrium of the private signal purchase game, where $(n_1, \ldots, n_N)$ is characterized by

$$\frac{\overline{PX}}{2(N-1)}\left(\frac{n - n_i + 2\frac{N-1}{N}}{(n+1)(n+2)(n+3)}\right) \leq R[C_i(n_i+1) - C_i(n_i)]$$

and, when $n_i \geq 1$,

$$\frac{\overline{PX}}{2(N-1)}\left(\frac{n - n_i + 2\frac{N-1}{N}}{(n+2)(n+1)^2}\right) \geq R[C_i(n_i) - C_i(n_i-1)],$$

for each i.

PROOF: Given Proposition 1, we need only to show that the choice of $(n_1, \ldots, n_N)$ is as described above.

We consider a deviation by agent i of k signals from the situation in which $\widehat{n}$ signals were to be purchased in total and i was to purchase n_i signals. First, we verify that if C_i is convex and non–negative valued and if $k > 0$, then $V_i(n_i, \widehat{n}, k) - V_i(n_i, \widehat{n}, 0) \geq 0$ only if $V_i(n_i, \widehat{n}, 1) - V_i(n_i, \widehat{n}, 0) \geq 0$. It follows from (3) that
$V_i(n_i, \widehat{n}, k) - V_i(n_i, \widehat{n}, 0) =$

$$\frac{k}{\widehat{n}+k+2}\left(\frac{\overline{PX}}{2(N-1)}\right)\left[\frac{\widehat{n} - n_i + 2 - \frac{2}{N}}{(\widehat{n}+2)(\widehat{n}+1)}\right] - R\left[C_i(n_i+k) - C_i(n_i)\right].$$

Since C_i is convex and $k \geq 1$, it follows that
$V_i(n_i, \widehat{n}, k) - V_i(n_i, \widehat{n}, 0) \leq$

$$\frac{k}{\widehat{n}+3}\left(\frac{\overline{PX}}{2(N-1)}\right)\left[\frac{\widehat{n} - n_i + 2 - \frac{2}{N}}{(\widehat{n}+2)(\widehat{n}+1)}\right] - kR\left[C_i(n_i+1) - C_i(n_i)\right].$$

Therefore, $V_i(n_i, \widehat{n}, k) - V_i(n_i, \widehat{n}, 0) \leq k[V_i(n_i, \widehat{n}, 1) - V_i(n_i, \widehat{n}, 0)]$. Our claim follows from this inequality.

Similarly, for $k < 0$ $(n_i + k \geq 0)$, $V_i(n_i, \widehat{n}, k) - V_i(n_i, \widehat{n}, 0) \geq 0$ only if $V_i(n_i, \widehat{n}, -1) - V_i(n_i, \widehat{n}, 0) \geq 0$. Thus, an equilibrium is characterized by $V_i(n_i, n, 1) - V_i(n_i, n, 0) \leq 0$ and when $n_i \geq 1$, $V_i(n_i, n, -1) - V_i(n_i, n, 0) \leq 0$. Direct calculation of these expressions, using (3), gives Proposition 5. ∎

Proposition 5 shows that this framework also provides a resolution to Grossman's paradox: agents purchase signals even though the signals are costly and the price fully reveals the relevant content of the signals. It is important that agents understand that the equilibrium allocations are determined by the distribution of signals and the beliefs of agents concerning the total number of signals purchased. In particular, as we saw in (3), agents gathering signals which other agents do not anticipate, (expect to) receive a larger allocation than the other agents. The benefits from this, however, are partially offset by the costs of acquisition. The equilibrium exists where there is a balance between these factors.

It follows from Proposition 5 that if all agents face the same convex cost structure C, then in equilibrium any agent purchases at most one more signal than any other agent.

COROLLARY 1. *If $C_i = C$, for all i, and C is convex, then in the equilibrium described in Proposition 5, $|n_i - n_j| \leq 1$ for any i and j.*

The intuition behind Corollary 1, is simple. Given that agents face the same concave gains in expected utility net of costs (see (3)), and that they face the same convex cost structures, they face the same expected utility changes when choosing to acquire information. The proof of Corollary 1 is presented in section 8. Next, we remark that if agents face identical linear cost structures, then the equilibrium number of signals is unique.

COROLLARY 2. *If agents have identical linear signal costs, then all equilibria of the type described in Proposition 5 have the same total number of signals.*

Corollary 2 shows that, with identical linear cost structures, the equilibria of the type in Proposition 5 are unique, up to a permutation of the agents. For instance, we may have an equilibria with $n = 3$ and $N = 4$. Corollary 2 then implies that all equilibria, of the type in Proposition 5, must have $n = 3$. There are four such equilibria, however: one with each of the agents not acquiring a signal. The proof of Corollary 2 is presented in Appendix 1.

A third interesting implication of Proposition 5 concerns sequences of equilibria as the size of the economy becomes large.

COROLLARY 3. *Consider a sequence of economies for which agents face identical convex information cost structures and the per–capita supply, $\frac{\overline{X}}{N}$, is constant. There exists a number $\bar{N}$ of agents such that if there is an equilibrium of the type described in Proposition 5 with equilibrium number of signals n (for the economy $\bar{N}$), where $\bar{N} > n > 0$, then n is an equilibrium number of signals for all economies with $N > \bar{N}$.*

PROOF: This follows from an inspection of the inequalities in Proposition 5. ∎

Indeed, even as the economy becomes large, the number of signals observed remains small. This is illustrated in Example 1.

EXAMPLE 1.

Let $\overline{X} = N$, $\bar{P} = 2$, $R = 1$, and $C_i(n_i) = \frac{n_i}{12}$, for all i. We verify below that $n_i = 1$ for $i \leq 2$ and $n_i = 0$ for $i > 2$, with $n = 2$, is an equilibrium for all $N > 3$. We need to verify that

$$\frac{(\frac{N}{N-1})2 - 2}{60} \leq \frac{1}{12} \leq \frac{(\frac{N}{N-1})2 - 2}{36}$$

and that

$$\frac{(\frac{N}{N-1})1 - 2}{60} \leq \frac{1}{12} \leq \frac{(\frac{N}{N-1})1 - 2}{36}.$$

These inequalities can be combined and simplified to

$$\left(\frac{N}{N-1} + 1\right)\frac{2}{5} \leq 1 \leq \left(\frac{N}{N-1} + 2\right)\frac{1}{3},$$

which is satisfied for all $N > 3$. This ends Example 1.

Example 1 has a distinctly non–competitive flavor. As the number of agents becomes very large, the ex–ante expected utility of agents who do not acquire signals converges to $1 + w_i$, while the ex–ante expected utility of the two signal gatherers converges to $2/3 + w_i$. [The ex–ante expected utility of a signal gatherer converges to $2/3 + w_i$, even if the agent deviates and fails to collect a signal.]

6. Information Acquisition and Welfare Issues.

In this section we investigate the welfare characteristics of the equilibria described in Propositions 3 and 5. The notion of ex–ante Pareto efficiency is used as the measure of welfare. This measure is used, because signal acquisition decisions are made at $t = 0$ (ex–ante). In order to use the notion of Pareto efficiency in conjunction with the behavior we have examined, we need to complete the economies so that we have a pure exchange economy. That is, we need to model the agents who are assumed to auction off assets in the share auction. We add N_S "sellers" to the existing N "buyers" in the economy. The first N agents are referred to as buyers ($i \leq N$), and the remaining N_S agents are referred to as sellers ($N + 1 \leq i \leq N + N_S$), where $N = N_S + 1$. The sellers ($i > N$) have utility

$U_i(z, x_i) = z$ and endowment $e_i = \frac{\overline{X}}{N_S}$. The sellers value only immediate consumption of the risk–free asset. Thus, we have the gains from trade between buyers and sellers necessary to assure the existence of exchange.

PROPOSITION 6. *The strategies described in Proposition 5 (Proposition 3) for $i \leq N$ and by $n_i = 0$ and trading action $D_i(p) = 0$ for all p for $i > N$, form an equilibrium of the demand submission game with private (public) information purchase. Furthermore, the equilibrium price $p^* = \frac{E[P|y]}{2}$ is fully revealing.*

The proof of Proposition 6 is given in Appendix 1.

Proposition 6 shows that there is an equilibrium for the complete exchange economy, in which the sellers sell all of their endowment of the risky asset and the buyers behave as we described in the share auction.[16]

For $i > N$, it is straightforward to show that[17]

$$v_i(n_i + k, \widehat{n} + k) = V_i(n_i, \widehat{n}, k) = w_i + \frac{\overline{P}}{2}\frac{\overline{X}}{N-1} - C_i(n_i + k).$$

This implies that if $C_i(n_i) > 0$ for $n_i > 0$, then no seller will wish to purchase information since it does not enter his ex–ante expected utility except through its cost. This is true whether signal purchase is observed or not.

We are now ready to examine the efficiency of information acquisition. From Corollary 1, for $i \leq N$,

$$V_i(n_i, n, 0) - V_i(0, n - n_i, 0) = \frac{\overline{PX}}{2(N-1)}\left(\frac{-n_i(n - n_i + 2\frac{N-1}{N})}{(n+2)(n - n_i + 2)}\right) - R[C_i(n_i) - C_i(0)].$$

Likewise, for $j \neq i$, $j \leq N$,

$$V_j(n_j, n, 0) - V_j(n_j, n - n_i, 0) = \frac{\overline{PX}}{2(N-1)}\left(\frac{n_i(n_j + \frac{2}{N})}{(n+2)(n - n_i + 2)}\right).$$

We begin by analyzing the situation where the number of signals an agent acquires is not known to any other agent. From the calculations above, we see that agents wish to have

[16] Although in our analysis of the share auction we assumed that the buyers were not endowed with the risky asset, this assumption is not important. If buyers are endowed with the risky asset, then there is an equilibrium in which buyers' strategies are those given in (1) plus their endowment. This does not change any of our analysis in any substantial way.

[17] In the proof of Proposition 6 it is evident that the sellers' trading actions described in Proposition 6, are a best response regardless of the beliefs of agents concerning n.

more information present in the economy, however, only when other agents are performing the acquisition. The loss in expected utility, net of costs to agent i, is completely transferred into gains in expected utility for the other agents ($j \neq i$). This follows from above, since

$$V_i(n_i, n, 0) - V_i(0, n - n_i, 0) + \sum_{j \neq i} V_j(n_j, n, 0) - V_j(n_j, n - n_i, 0) = -R[C_i(n_i) - C_i(0)].$$

One wonders, then, why agent i is acquiring information. Recall that the relevant calculation for the agent considering a deviation from acquiring n_i signals to $n_i - k$ signals is $V_i(n_i - k, n, -k) - V_i(n_i, n, 0)$. This is negative in equilibrium according to Corollary 2. What is important in determining how many signals an agent acquires, is the beliefs of the other agents. If the other agents expect that n signals are acquired and the agent does not acquire any signals reducing the total to $n - n_i$, the agent may be worse off, even though the agent would prefer that other agents expect $n - n_i$ signals in total.

It follows from above that an agent would prefer that other agents be able to verify how many signals the agent is observing. [Simple announcements of signal purchase away from an equilibrium number would not be credible, however, by the calculations of Proposition 4.]

The equilibria in which information is acquired are ex–ante Pareto inefficient, since resources are lost in the acquisition of information, and there is no aggregate expected utility gain, net of costs.[18] [To see this, consider a combination of a sufficiently large tax on signal acquisition to assure that the equilibrium strategies involve $n = 0$, and a lump sum tax used to redistribute wealth so that each agent's ex–ante expected utility is the same as the ex–ante expected utility when signals are acquired, gross of costs.] If, however, we assume gathering information is costless, then all equilibria are Pareto efficient.[19]

[18] This may be, in part, due to the fact that agents are risk neutral. In a situation with risk averse agents, all buyers might gain from the reduction in uncertainty associated with signal acquisition, even given the cost. However, if we also allow sellers to be risk averse, the issue is unclear. Finding an equilibrium set of strategies for agents who are risk averse, even with very restrictive assumptions, seems intractable.

[19] We could also imagine a situation in which information acquisition leads to an increase in $\overline{P}$. Depending on the extent to which n affects $\overline{P}$, and on the cost structure, the equilibria may or may not be efficient. It is also worth noting, that in this case, the sellers would also benefit from signal acquisition since the ex–ante expected equilibrium price would then be dependent on n.

In the situation in which n_i is public information for all i, we know from Corollary 1 that the equilibrium is unique and $n = 0$. This equilibrium is Pareto efficient.

7. Remarks on a Game Theoretic Approach

There are various advantages to employing the game theoretic approach used here. First of all, it allows us to analyze situations in which there are only a finite number of agents, including situations in which markets are small and information may be asymmetric. Although fully revealing prices may not seem realistic in situations with large numbers of agents, they seem more reasonable in situations with small numbers of agents. In the analysis provided for the share auction, we were able to isolate a value of information which is associated with the change in price influence it provides to the agent. This is quite different from the 'standard' value of information which we think of, which arises when prices are not fully revealing. For instance, if we examine an exchange between two agents, the price is likely to be 'very' revealing. In such a situation, much of the value of information is associated with the perceptions of the other agent concerning the amount of information the agent has, as opposed to the informational asymmetry it provides. Secondly, a game theoretic approach allows us to analyze the 'competitive' case without making an artificial assumption. That is, we can analyze the behavior of agents as the number of agents becomes large. As we have seen, this may be important since, with fixed costs per unit of information, there may be some markedly non–competitive behavior, even in economies with arbitrarily large numbers of agents.

A disadvantage of the approach we have adopted is that there may exist a multiplicity of equilibria. In so much as our objective is to understand the value of information at the limit (with fully revealing prices), the game theoretic approach is useful. The multiplicity of equilibria for demand submission game can be a problem in general, since the behavior associated with the different equilibria can vary dramatically. This appears to be due (at least in part) to the extreme size of the action space in the demand submission game. This does not, however, mean that the game theoretic approach itself is to blame, but rather the description of the game. In this paper, we chose to analyze a demand submission game because it is similar to the rational expectations setup, thus allowing a direct comparison and an analysis of the non–existence issues associated with rational expectations equilibria.

For this purpose, the multiplicity of equilibria is not a difficulty. In general, difficulties with multiple equilibria may be overcome by using a model which pays close attention to the institutional detail of how exchange is conducted, or by using an equilibrium selection criterion.[20] In sum, the value of information in various market situations remains largely unexplored. Our results in this paper indicate that modeling the strategic actions of agents might play an important role in the exploration.

[20] Of course, it may be the case that the multiplicity is a reality.

References

ADMATI, A. [1985], "Information in Financial Markets: The Rational Expectations Approach," Working Paper 836, Stanford Graduate School of Business.

——————[1985b], "A Noisy Rational Expectations Equilibrium for Multi–Asset Securities Markets," *Econometrica*, Vol. 53 , pp. 629–657.

ADMATI, A. AND PFLEIDERER, P. [1984], "The Value of Information in Speculative Trading," Working Paper 782, Stanford Graduate School of Business.

——————[1985], "On the Allocation of Information in Speculative Markets," Working Paper 782R, Stanford Graduate School of Business.

——————[1986], "A Monopolistic Market for Information," Working Paper 797, Stanford Graduate School of Business.

ALLEN, B. [1981], "Generic Existence of Completely Revealing Equilibria for Economies with Uncertainty when Prices Convey Information ," *Econometrica*, Vol., pp. 1173–1199.

——————[1985], "General Equilibrium with Rational Expectations," University of Pennsylvania.

ALLEN, F. [1983], "The Social Value of Assymetric Information," Unpublished, University of Pennsylvania.

ANDERSON, R. AND H. SONNENSCHEIN [1982], "On the Existence of Rational Expectations Equilibria," *Journal of Economic Theory* , Vol. 26 , pp. 261–278.

BEJA, A. [1977], "The Limits of Price Information in Market Processes," Working Paper 61, Research Program in Finance, University of California at Berkeley.

DIAMOND, D. AND R. VERRECCHIA [1981], "Information Aggregation in a Noisy Rational Expectations Economy," *Journal of Financial Economics,* Vol. 9, pp. 221–235.

DUBEY, P., J. GEANAKOPLOS, AND M. SHUBIK [1987], " The Revelation of Information in Strategic Market Games: A Critique of Rational Expectations Equilibria," Working Paper , Yale University .

GALE, D. AND M. HELLWIG [1987], "Informed Speculation in Large Markets ," Working Paper 220, University of Pittsburg.

GROSSMAN, S. [1976], "On the Efficiency of Competitive Stock Markets Where Agents have Diverse Information," *Journal of Finance*, Vol. 31 , pp. 573–585.

——————[1978], "Further Results on the Informational Efficiency of Competitive Stock Markets," *Journal of Economic Theory*, Vol. 18 , pp. 81–101.

——————[1981], "An Introduction to the Theory of Rational Expectations under Assymetric Information," *Review of Economic Studies*, Vol. 48 , pp. 541–559.

GROSSMAN, S. AND STIGLITZ, J. [1980], "On the Impossibility of Informationally Efficient Markets," *The American Economic Review*, Vol. 70 , pp. 393–408.

HELLWIG, M. [1980], "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory*, Vol. 22 , pp. 477–498.

Jackson, M.O. [1988], "Private Information and Exchange: The Impications of Strategic Behavior," Ph.D. Dissertation: Graduate School of Business, Stanford University.

Jordan, J. and R. Radner [1982], "Rational Expectations in Microeconomic Models: An Overview," *Journal of Economic Theory*, Vol. 26 , pp. 201–223.

Kreps, D. [1977], "A Note on 'Fulfilled Expectations' Equilibria," *Journal of Economic Theory*, Vol. 44 , pp. 32–43.

Kyle, A. [1985], "Continuous Auctions and Insider Trading," *Econometrica*, Vol. 53, pp. 1315–1335.

——————[1986], "Informed Speculation with Imperfect Competition," Working Paper 66, Financial Research Center, Princeton University.

Milgrom, P. [1981], "Rational Expectations, Information Acquisition, and Competitive Bidding," *Econometrica*, Vol. 49, pp. 921–944.

Roberts, D. J., and M. Postlewaite [1976], " The Incentives for Price–Taking Behavior in Large Economies," *Econometrica*, Vol. 44, pp. 151–128.

Wilson, R. [1979], "Auctions of Shares," *Quarterly Journal of Economics*, Vol., pp. 675–689.

APPENDIX 1

Proofs of Propositions 1,2,4,6, and Corollaries 1 and 2.

Under the distributional assumptions given at the beginning of Section 3, the density of a signal y_k conditional on P and the unconditional density of P are

$$f(y_k|P) = Pe^{-Py_k} \quad \text{and} \quad g(P) = \overline{P}^{-1} e^{-P\overline{P}^{-1}}, \tag{4}$$

respectively.

The following lemmas will aid in the proofs of the propositions.

LEMMA 1. *The (unconditional) joint density function of the signals is*

$$f(y) = \frac{n!\,\overline{P}^n}{(\sum_{k=1}^n y_k \overline{P} + 1)^{n+1}}. \tag{5}$$

PROOF:

$$f(y) = \int_0^\infty f(y|P) g(P)\, dP.$$

Since the $(y_1, \ldots, y_n)$ are independent conditional on P, it follows from (4) that

$$f(y) = \int_0^\infty P^n \overline{P}^{-1} \exp[-P(\sum_{k=1}^n y_k + \overline{P}^{-1})]\, dP.$$

With a change of variables,

$$f(y) = \overline{P}^{-1} \left(\frac{1}{\sum_{k=1}^n y_k + \overline{P}^{-1}} \right)^{n+1} \int_0^\infty Y^n e^{-Y}\, dY,$$

which simplifies to (5). ∎

LEMMA 2. *The density function and expectation of P conditional on y are*

$$g(P|y) = \frac{P^n (\sum_{k=1}^n y_k \overline{P} + 1)^{n+1}}{n!\,\overline{P}^{n+1}} \exp[-P(\sum_{k=1}^n y_k + \overline{P}^{-1})]$$

and

$$E[P|y] = \frac{(n+1)\overline{P}}{\sum_{k=1}^n y_k \overline{P} + 1},$$

respectively.

PROOF:

$$g(P|y) = \frac{f(y|P) g(P)}{f(y)}.$$

The stated expression for $g(P|y)$ then follows directly from Lemma 1. $E[P|y]$ is found by direct calculation of $\int_0^\infty Pg(P|y)\, dP$. ∎

PROOF OF PROPOSITION 1.

It will be shown that if agents $j \neq i$ submit demands according to (1), then the response described in Proposition 1 maximizes agent i's expected utility, conditional upon his or her signal(s), the price, market clearing, and the other agents' demands.

The following claim will be useful.

CLAIM 1. *If the trading actions are described by (1) for each $j \neq i$, then the market clearing condition implies*

$$E[P|y] = \left(\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)} + \frac{(N-1)x_i}{2Rp\overline{X}} \frac{(\widehat{n}+1)}{(n+1)} \right)^{-1}. \tag{7}$$

PROOF: The market clearing condition is

$$\overline{X} = x_i + \sum_{j \neq i} x_j.$$

Substituting for x_j, $j \neq i$, from (1),

$$0 = x_i - 2Rp \frac{\overline{X}}{N-1} \left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + \frac{N-1}{N}}{\overline{P}(\widehat{n}+1)} \right). \tag{8}$$

We then have

$$p(x_i) = \frac{x_i(N-1)}{2R\overline{X}} \left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + \frac{N-1}{N}}{\overline{P}(\widehat{n}+1)} \right)^{-1}. \tag{9}$$

From (8) it also follows that

$$\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + \frac{N-1}{N}}{\overline{P}(\widehat{n}+1)} \right) = \frac{x_i(N-1)}{2Rp\overline{X}}. \tag{10}$$

By Lemma 2 the expectation of P given all of the signals is

$$E[P|y] = \frac{(n+1)\overline{P}}{\sum_{k=1}^{n} y_k \overline{P} + 1}.$$

It follows from (10) that

$$E[P|y] = \left(\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)} + \frac{(N-1)x_i}{2Rp\overline{X}} \frac{(\widehat{n}+1)}{(n+1)} \right)^{-1},$$

which is the desired expression ▮

We are now ready to solve the agent's maximization problem. Under the market clearing restriction, if trading strategies are given by (1) for each $j \neq i$, then we can calculate the expected utility of agent i for demand x_i to be

$$E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i]x_i + Rw_i - Rp(x_i)x_i, \tag{11}$$

where $p(x_i)$ is described by (9). The necessary first order conditions are

$$\frac{\partial E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i]}{\partial x_i}x_i + E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i] - Rp - Rp'x_i = 0.$$

It follows from (9) that

$$p'(x_i) = \frac{(N-1)}{2R\overline{X}}\left(\frac{\Sigma_{k\notin S_i} y_k \overline{P} + \frac{N-1}{N}}{\overline{P}(\widehat{n}+1)}\right)^{-1}. \tag{12}$$

From (7) it follows that $E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i] = E[P|y]$. Thus, from (12) and (7) it is easily verified that

$$\frac{\partial E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i]}{\partial x_i} = 0,$$

and from (12) and (9) it follows that

$$Rp'x_i = Rp.$$

Therefore the first order conditions simplify to

$$E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i] - 2Rp = 0. \tag{13}$$

Substituting from (7), we rewrite (13) as

$$\left(\frac{\Sigma_{k\in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)} + \frac{(N-1)x_i}{2Rp\overline{X}}\frac{(\widehat{n}+1)}{(n+1)}\right) = \frac{1}{2Rp}.$$

Inverting the expression for x_i, we have the expression in Lemma 3.

To verify that the first order conditions are indeed sufficient, we verify the second order conditions

$$2\frac{\partial E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i]}{\partial x_i} + \frac{\partial^2 E[P|\{y_k\}_{k\in S_i}, p(x_i), x_i]}{\partial x_i^2}x_i - 2Rp' - x_i Rp'' < 0.$$

These simplify to

$$-2Rp' < 0,$$

which follows from (12) since $\sum_{k \notin S_i} y_k \geq 0$.

We now verify the expression for p^*. Substituting the expression for x_i given in Lemma 3 into the market clearing assumption in (8), we find that

$$0 = \left(\frac{n+1}{\widehat{n}+1}\right)\left[1 - 2Rp^*\left(\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)}\right)\right] - 2Rp^*\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + \frac{N-1}{N}}{\overline{P}(\widehat{n}+1)}\right).$$

The desired expression for p^* follows directly. This ends the proof of Proposition 1.

PROOF OF PROPOSITIONS 2 AND 4.

Proposition 2 follows from Proposition 4, noting that $v_i(n_i, n) = V_i(n_i, n, 0)$. We now prove Proposition 4.

We develop the expression for $V_i(n_i, \widehat{n}, k)$. Let $y_{S_i} = \{y_k\}_{k \in S_i}$ and $x_i^* = D_i(p^*)$, where D_i is described in Proposition 1.

$$V_i(n_i, \widehat{n}, k) = E_0\left[\left(E[P|y_{S_i}, p^*, x_i^*] - Rp^*\right) x_i^*\right] + Rw_i - RC_i(n_i + k), \qquad (14)$$

where E_0 indicates expectation with respect to the information available at the beginning of Period 0, before the signals are observed. From Proposition 1 this simplifies to

$$V_i(n_i, \widehat{n}, k) = E_0\left[\frac{E[P|y]}{2} x_i^*\right] + Rw_i - RC_i(n_i + k).$$

The expression $E_0[E[P|y]x_i^*/2]$ expands to

$$\frac{\overline{X}}{2(N-1)}\left(\frac{n+1}{\widehat{n}+1}\right) E_0\left[E[P|y] - \left(\frac{(n+1)\overline{P}}{\sum_{k=1}^{n} y_k \overline{P} + 1}\right)^2 \left(\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{\overline{P}(n+1)}\right)\right].$$

Therefore, we can write $V_i(n_i, \widehat{n}, k)$ as

$$\frac{\overline{PX}}{2(N-1)}\left(\frac{n+1}{\widehat{n}+1}\right)\left(1 - (n+1)E_0\left[\frac{\Sigma_{k \in S_i} y_k \overline{P} + \frac{1}{N}}{(\sum_{k=1}^{n} y_k \overline{P} + 1)^2}\right]\right) + Rw_i - RC_i(n_i + k). \qquad (15)$$

By Lemma 1 we can rewrite (15) as

$$\frac{\overline{PX}}{2(N-1)}\left(\frac{n+1}{\widehat{n}+1}\right)\left(1 - (n+1)\int_0^\infty \frac{(\sum_{k \in S_i} y_k \overline{P} + \frac{1}{N})(n)!\,\overline{P}^n}{(\sum_{k=1}^{n} y_k \overline{P} + 1)^{n+3}}\, dy\right) + Rw_i - RC_i(n_i + k).$$

We now expand the integral. First integrating over y_k for $k \notin S_i$, we rewrite the integral as

$$\int_0^\infty \left(\frac{(\sum_{k \in S_i} y_k \overline{P} + \frac{1}{N})}{(\sum_{k \in S_i} y_k \overline{P} + 1)^{n_i+k+3}} \right) \left(\frac{n!\, \overline{P}^{n_i+k}(n_i + k + 2)!}{(n+2)!} \right) dy_{S_i}.$$

This can be written as

$$\left(\frac{\overline{P}^{n_i+k}(n_i + k + 2)!}{(n+1)(n+2)} \right) \int_0^\infty \frac{1}{(\sum_{k \in S_i} y_k \overline{P} + 1)^{n_i+k+2}} \left[1 - \frac{\frac{N-1}{N}}{(\sum_{k \in S_i} y_k \overline{P} + 1)} \right] dy_{S_i}$$

$$= \frac{n_i + k + 2 - 2\frac{N-1}{N}}{(n+1)(n+2)}.$$

It then follows that

$$V_i(n_i, \widehat{n}, k) = \frac{\overline{PX}}{2(N-1)} \left(\frac{n+1}{\widehat{n}+1} \right) \left(1 - \frac{n_i + k + 2 - 2\frac{N-1}{N}}{(n+2)} \right) + Rw_i - RC_i(n_i + k).$$

This ends the proof of Propositions 2 and 4.

PROOF OF COROLLARY 1.

Suppose there exists an equilibrium with i, j, and $h \geq 2$ such that $n_i = n_j + h$. From Proposition 5 it follows that

$$\frac{\overline{PX}}{2(N-1)} \frac{n - n_j + 2 - \frac{2}{N}}{(n+1)(n+2)(n+3)} \leq R[C_j(n_j + 1) - C_j(n_j)]$$

and

$$R[C_i(n_i) - C_i(n_i - 1)] \leq \frac{\overline{PX}}{2(N-1)} \frac{n - n_i + 2 - \frac{2}{N}}{(n+1)^2(n+2)}.$$

Given that $C_i = C_j = C$ and the convexity of C, this implies that

$$\frac{n - n_j + 2 - \frac{2}{N}}{n+3} \leq \frac{n - n_i + 2 - \frac{2}{N}}{n+1},$$

or

$$0 < 2\left(n - n_j + 2 - \frac{2}{N} \right) - h(n+3).$$

This simplifies to

$$0 < (2-h)n - 3h - 2n_j + 4 - \frac{4}{N},$$

which cannot be satisfied by any $h > 1$. Hence, our supposition was wrong, and Corollary 1 is proven.

PROOF OF COROLLARY 2.

Given any equilibrium n suppose there exists another equilibrium $\tilde{n} = n + h$, where without loss of generality $h \geq 1$. It then follows that there exists some i with $\tilde{n}_i \geq n_i + 1$. According to Proposition 5, it must be that

$$\frac{n - n_i + 2 - \frac{2}{N}}{(n+1)(n+2)(n+3)} \leq \frac{\tilde{n} - \tilde{n}_i + 2 - \frac{2}{N}}{(\tilde{n}+1)^2(\tilde{n}+2)}.$$

It then follows that

$$\frac{(n+h+1)^2(n+h+2)}{(n+1)(n+2)(n+3)} \leq \frac{n + h - n_i - 1 + 2 - \frac{2}{N}}{n - n_i + 2 - \frac{2}{N}}.$$

This can only hold for $h \geq 2$. The above inequality then implies that

$$3(h-1)\frac{(n+2+h-1)^2}{(n+1)(n+2)(n+3)} \leq \frac{h-1}{n - n_i + 2 - \frac{2}{N}}.$$

We therefore have

$$\frac{3(h-1)}{n+1} \leq \frac{h-1}{n - n_i + 2 - \frac{2}{N}},$$

implying that

$$3(n - n_i + 2 - \frac{2}{N}) \leq n + 1. \tag{16}$$

From Remark 1 it follows that if $n \leq N$ then $n_i \leq 1$ for all i. Since $N \geq 3$, it is easily checked that (16) cannot be satisfied by $n \leq N$. When $n > N$ it follows from Proposition 5 that $n_i < \frac{2n+1}{N}$. We then know that $n - n_i + 2 - \frac{2}{N} \geq \frac{(N-2)(n+2)}{N}$. This coupled with (16) implies that

$$3\left[\frac{(N-2)(n+2)}{N}\right] \leq n + 1.$$

Since $N \geq 3$ this is a contradiction. Thus our supposition concerning the existence of the equilibrium $\tilde{n}$ was wrong. This ends the proof of Corollary 2.

PROOF OF PROPOSITION 6.

We note that, given the strategies of the sellers, the strategies given by Proposition 5 (3) are best responses. We verify that the strategies of the sellers are best responses.

We begin by showing that the stated trading actions are a best response, given the strategies of the other agents, regardless of the choices of $(n_1, \ldots, n_{N+N_s})$.

Market clearing implies that, for $i > N$,

$$\overline{X} = x_i + \frac{N}{N-1}\overline{X} - 2Rp\frac{\overline{X}}{N-1}\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + 1}{\overline{P}(\widehat{n}+1)}\right).$$

We then write the price as a function of agent i's demand:

$$p(x_i) = \left(x_i + \frac{\overline{X}}{N-1}\right)\left[2R\frac{\overline{X}}{N-1}\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + 1}{\overline{P}(\widehat{n}+1)}\right)\right]^{-1}. \tag{17}$$

Then

$$p'(x_i) = \left[2R\frac{\overline{X}}{N-1}\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + 1}{\overline{P}(\widehat{n}+1)}\right)\right]^{-1} = p\left(\frac{\overline{X}}{N-1} + x_i\right)^{-1}. \tag{18}$$

The first order necessary conditions for optimization are

$$-p - p'(x_i - e_i) = 0,$$

or, substituting from (17) and (18),

$$\left(x_i + \frac{\overline{X}}{N-1} + x_i - e_i\right)\left[2R\frac{\overline{X}}{N-1}\left(\frac{\Sigma_{k \notin S_i} y_k \overline{P} + 1}{\overline{P}(\widehat{n}+1)}\right)\right]^{-1} = 0.$$

Since $e_i = \frac{\overline{X}}{N-1}$, these are satisfied by $x_i = 0$.

The second derivative, $-2p' - p''(x_i - e_i)$, is always negative, since by (18) $p'' = 0$ and $p' > 0$. The first order necessary conditions are, therefore, also sufficient.

We now show that $n_i = 0$ is a correct choice for $i > N$. In this case it is easily verified that

$$v_i(n_i + k, \widehat{n} + k) = V_i(n_i, \widehat{n}, k) = w_i + \frac{\overline{P}}{2}\frac{\overline{X}}{N-1} - C_i(n_i + k).$$

This ends the proof of Proposition 6.