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BERTRAND COMPETITION WITH SUBCONTRACTING

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## BERTRAND COMPETITION WITH SUBCONTRACTING

### Abstract

We investigate a two stage game in which in its first stage two firms engage in price competition to supply a market and in the second stage may subcontract production to each other. It is supposed that the firms produce the identical product with the same strictly convex cost function. A firm is obliged to supply the entire quantity demanded at its quoted price. In the event of a tie each firm supplies one-half the quantity demanded at that price.

Our analysis discloses that if the winner of the game's first stage determines the terms of the subcontract in its second stage then there exists a unique subgame perfect Nash equilibrium (SPNE) in pure strategies in which the firms bid the same price in the first stage and both realize zero profits. On the other hand, if the loser of the game's first stage sets the terms of the subcontract in the second stage then there exists a unique SPNE in pure strategies in which the firms bid the same price in the first stage and both realize positive profits. The presence of the potential for subcontracting supports the unique SPNE in pure strategies even though no actual subcontracting occurs in these two cases. The SPNE price is below the socially optimal price in the first case and above it in the second case. We also consider other modes of sharing the gains from subcontracting between the two firms such as the Nash bargaining solution. Expansion of the set of strategies to include mixed strategies confirms that the pure strategy SPNE are unique. Finally, we show a case in which subcontracting does occur in the SPNE.



## 1. Introduction

Subcontracting is a commonly employed practice for reducing production costs. The aspect of it of interest to us is the situation in which rivals compete for a market or production contract and then the potential for subcontracting among the very same rivals exists. Recognition of the linkage between the two rounds may cause the rivals to behave in a tacitly collusive manner. This in turn gives rise to the question of whether or not subcontracting is beneficial from the standpoint of society as a whole. For, on the one hand, subcontracting may benefit society through lowering the total cost of production while on the other hand it may be detrimental by supporting tacit collusion.

We address these issues in terms of a two stage game involving two firms producing the identical product according to an identical strictly convex cost function. It is the strict convexity of the cost function that creates the incentive for subcontracting, as total costs are lowered when production is split. In the first stage of the game each firm submits a price at which it is willing to supply the entire quantity demanded, given by the product's demand function. In the event of a tied price, each firm supplies one-half of the quantity demanded at that price. Thus, the firms are engaged in Bertrand competition in the first stage of the game. In its second stage, the firms may subcontract production to each other. There are any number of ways that the amount to be subcontracted and the price to be paid can be arrived at. We focus on two polar cases. In the first case, the winner of the first stage of the game acts as a Stackelberg leader in the second stage. That is, he determines the quantity to be subcontracted

and the price to be paid to the loser, so as to maximize his own profit subject to the loser's opportunity cost, which is zero. In the second case, the loser of the first stage is the Stackelberg leader in the second stage. That is, he chooses the quantity to be subcontracted to the winner and its price so as to maximize his own profit subject to the winner's opportunity cost, which is the profit he can realize if he produces the entire quantity demanded himself. We also indicate what happens if the terms of the subcontract are determined by other means, such as the Nash bargaining solution.

We employ the subgame perfect Nash equilibrium (SPNE) as our solution concept. Thus, we conduct the analysis from the second stage of the game back to its first stage. Our analysis discloses that for a downward sloping demand function each version of the second stage game exhibits a unique pure SPNE in which both firms choose the identical price in the first stage and therefore each produces exactly one-half of the quantity demanded. Thus, in equilibrium, there is no actual subcontracting. The equilibrium price when the winner of the first stage is assumed to be the Stackelberg leader in the second stage equals the average cost of producing one-half of the quantity demanded. Thus, both firms realize zero profits at the equilibrium. On the other hand, if the loser of the first stage of the game is the Stackelberg leader in the second stage, the equilibrium price is higher and both firms realize positive profits. The intuitive reason for this difference in the equilibrium prices, is that if the winner of the bidding stage gets to set the terms of the subcontract in the second stage, then being the loser of the first stage is very costly, it results in zero profit for the loser, and therefore the price in the first stage is bid down to the level at which the

winner's profit equals the loser's. However, if the loser of the first stage gets to set the terms of the subcontract in the second stage, then he is assured a positive profit, and the price in the first stage is only bid down to the level that assures both firms that profit level. The interesting result here is that it is advantageous to both of them to let the loser of the first stage have the power to determine the subcontracting terms in the second stage. Yet this arrangement does not necessarily yield the firms the maximum profit they could realize from complete cooperation, the monopoly profit of a single firm with two identical production facilities. Our analysis of mixed-strategy possibilities indicates that no SPNE other than the pure SPNE exist.

We also consider the welfare implications of subcontracting. For the duopoly case, the price equal to the marginal cost of producing one-half of the quantity demanded maximizes consumer plus producer surplus. Thus, the Nash equilibrium price that obtains if the winner of the first stage sets the subcontract price is too low and it is too high if the loser of the first stage determines the terms of the subcontract in the second stage. However, there does exist a means of sharing the gains from subcontracting such that the SPNE price is socially optimal.

Finally, our analysis of competition for a contract to produce a fixed quantity discloses that if the loser of the first stage is the leader in the second stage then there exists an SPNE in which there is a single winner in the first stage and subcontracting occurs.

We are unaware of any previous work on Bertrand competition with subcontracting. The most closely related work appears to be that dealing with price competition in the presence of convex costs and/or capacity

constraints, summaries of which are provided by Allen and Hellwig (1986), Dixon (1984), and Maskin (1986). In that work only the existence of a mixed strategy Nash equilibrium is established whereas here the potential for subcontracting allows the existence of unique pure strategy SPNE. This result obtains even though there is no actual subcontracting at the SPNE. Stahl (1988), in a paper more closely related to ours, also showed the existence of a pure strategy Nash equilibrium.

In the next section we present the model for the case of competition for a market. In the following section the case of competition for a fixed quantity is presented. A brief summary follows in the subsequent section.

## 2. Competition to Supply a Market

We posit two firms that produce an identical divisible product, the demand function for which is  $Q(P)$ , with the same cost function  $C(Q)$ . The assumed properties of  $Q(P)$  and  $C(Q)$  are:

- A.1 The demand function  $Q(P)$  is defined for  $P \geq 0$ , nonnegative, differentiable, downward sloping, ( $Q'(P) < 0$ ), and
 
$$\lim_{P \rightarrow \infty} Q(P) = 0.$$
- A.2 The cost function,  $C(Q)$ , is defined for  $Q \geq 0$ , differentiable, strictly increasing ( $C'(Q) > 0$ ), strictly convex, ( $C''(Q) > 0$ ), and  $C(0) = 0$ .
- A.3 Production is profitable, i.e.,  $Q(C'(0)) > 0$ .

The game involves two stages. In its first stage the two firms choose prices and compete for the production of the total quantity demanded at that price. The firm with the lowest price wins and is obliged to provide the

entire quantity demanded at that price. In the event of a tie, each firm produces one-half of the total quantity demanded at that price and the game ends. In the second stage of the game we allow subcontracting to take place. One of the firms is a Stackelberg leader who offers the other firm a quantity to produce and the unit price to be charged. We study two versions of the game. In the first,  $\Gamma_1$ , the leader is the firm which won in the first stage. In the second version,  $\Gamma_2$ , the loser of the first stage is the leader in the second stage.

As we seek subgame perfect equilibria we begin with the second stage. We start with  $\Gamma_1$ , in which the winner is the leader.

Suppose there was a single winner in the first stage who offered the lower price  $P$ . He chooses the quantity,  $q$ , to subcontract to the loser and the price,  $p$ , to offer so as to

$$(1) \quad \max_{p,q} PQ(P) - pq - C(Q(P) - q)$$

$$(2) \quad \text{s.t. } pq - C(q) \geq 0, \quad p, q \geq 0$$

Note that  $P$ , and  $Q(P)$ , are fixed from the first stage and that  $Q(P) - q$  is the amount to be produced by the winner. Also, the constraint represents the loser's profit from subcontracting and cannot be driven below zero as this is his opportunity cost. For  $q > 0$ , the constraint can be rewritten as

$$(3) \quad p \geq C(q)/q$$

Since the objective function (1) is strictly decreasing in  $p$ , it is set



equal to its lowest value, namely  $p = C(q)/q$ . Thus, the maximum at (1) is attained by solving

$$(4) \quad \max_q PQ(P) - C(q) - C(Q(P) - q) \\ \text{s.t. } q \geq 0$$

which, upon differentiation with respect to  $q$ , yields

$$(5) \quad C'(q) = C'(Q(P) - q)$$

Since the objective function in (4) is strictly concave in  $q$ , the first order condition is necessary and sufficient for a maximum. As the cost function is strictly increasing, (5) implies that  $q = Q(P)/2$ . Thus the winner subcontracts one-half of the total output at the unit price,  $p = C(Q(P)/2)/(Q(P)/2)$ , the loser's average cost of production. The payoffs to the winner and loser, respectively, as a function of the winning price  $P$  are

$$(6) \quad W_1(P) = PQ(P) - 2C(Q(P)/2)$$

$$(7) \quad L_1(P) = 0.$$

The winner realizes the entire benefit,  $C(Q(P)) - 2C(Q(P)/2)$ , the cost saving from splitting production, of subcontracting in this case.

If the first stage ends with a tie, the payoffs to both players are the same and are given by

$$(8) \quad T(P) = PQ(P)/2 - C(Q(P)/2).$$

Before proceeding to the analysis of the first stage of the game we record the following result:

Lemma 1: There exists a unique price  $P_1$  such that

$P_1 = C(Q(P_1)/2)/(Q(P_1)/2)$ ,  $W_1(P) < 0$  for each  $P < P_1$ ,  $W_1(P) \geq 0$  for each  $P > P_1$  and  $dW_1/dP > 0$ , for  $P \leq P_1$ .

Proof: Let  $\bar{P}$  be the lowest price for which  $Q(P) = 0$ . If  $Q$  is positive for all  $P$  then  $\bar{P} = \infty$ . For  $P < \bar{P}$ ,  $Q(P) > 0$  and (6) can be rewritten as

$$(9) \quad W_1(P) = Q[P - C(Q/2)/(Q/2)]$$

where we write  $Q$  for  $Q(P)$ .

Consider the function

$$(10) \quad G(P) = P - C(Q/2)/(Q/2)$$

Upon differentiation we find

$$(11) \quad G'(P) = 1 - Q'[C'(Q/2) - C(Q/2)/(Q/2)]/Q$$

By the strict convexity of  $C$  and  $Q'(P) < 0$ ,  $G'(P) > 0$  for all  $P$ . Also  $G(0) < 0$  and  $\lim_{P \rightarrow \bar{P}} G(P) = \bar{P} - C'(0) > 0$  by A.3 and A.2. Thus, there exists a  $P_1$ ,  $0 < P_1 < \bar{P}$  such that  $G(P_1) = 0$ ,  $G(P) < 0$  for each  $P < P_1$  and  $G(P) > 0$ .

for  $P > P_1$ . From (9) it follows that  $W_1(P_1) = 0$  and  $W_1(P) < 0$  for each  $P < P_1$ , while  $W_1(P) \geq 0$  for  $P > P_1$ . And from (10) we have  $P_1 = C(Q(P_1)/2)/(Q(P_1)/2)$ . Differentiating  $W_1$  gives  $dW_1/dP = Q + Q'[P - C'(Q/2)]$ . But  $P - C'(Q/2) < G(P)$  and therefore for  $P \leq P_1$ ,  $P - C'(Q/2) < 0$ , which shows that for such  $P$ ,  $dW_1/dP > 0$ . Clearly the properties of  $P_1$  imply that there can be at most one such point.

Proposition 1: In the game  $\Gamma_1$ , i.e., if the winner of the first stage of the game is the Stackelberg leader in the second stage, there exists a unique SPNE in pure strategies, in which both firms offer the price  $P_1$  that is uniquely determined by  $P_1 = C(Q(P_1)/2)/(Q(P_1)/2)$  in the first stage. Each firm produces one-half of the total quantity demanded at that price and they both realize zero profit.

Proof: Note that  $T(P) = W_1(P)/2$ . Let  $\underline{P}$  be the lowest price offered in the first stage. Suppose  $\underline{P} < P_1$ , then both  $T(\underline{P})$  and  $W(\underline{P})$  are negative and therefore a firm that offered that price will be better off by bidding a higher price and being the loser. If  $\underline{P} > P_1$ , then either there is a losing firm which makes zero profit or there is a tie, in which case both firms realize  $T(\underline{P})$ . In the first case the loser could be better off choosing a price  $P$ ,  $P_1 < P < \underline{P}$  becoming the winner and obtaining  $W_1(P) > 0$ , since  $dW/dP(P_1) > 0$ . In the second case one of the firms could bid a price  $\underline{P} - \epsilon$  for small enough  $\epsilon > 0$  that satisfies  $W_1(\underline{P} - \epsilon) > W_1(\underline{P})/2 = T(\underline{P})$ . Thus,  $\underline{P} = P_1$ . If there is a winner in the game then he can gain by choosing a winning price  $P > P_1$  close enough to  $P_1$  and getting  $W_1(P) > W_1(\underline{P})$ . So the only possible SPNE is one in which both firms offer the price  $P_1$ . It is easy to

see that this is indeed a SPNE.

We turn next to the game  $\Gamma_2$  where the loser of the first stage of the game is the Stackelberg leader in the second stage. The loser, if there is one, determines the subcontract terms by solving the problem

$$(12) \quad \max_{p,q} pq - C(q)$$

$$(13) \quad \text{s.t. } PQ(P) - pq - C(Q(P) - q) \geq PQ(P) - C(Q(P))$$

$$p, q \geq 0$$

The constraint (13) indicates that the winner's profit cannot be reduced below the level he could realize by supplying the total quantity demanded alone. For  $q > 0$ , it can be rewritten as

$$(14) \quad p \leq [C(Q(P)) - C(Q(P) - q)]/q$$

As the objective function (12) increases with  $p$ , the subcontract price is set at its upper limit. Thus, (12) and (13) reduce to

$$(15) \quad \max_q C(Q(P)) - C(Q(P) - q) - C(q)$$

$$\text{s.t. } q \geq 0$$

upon substitution for  $p$  from (14). Taking the derivative of (15) with respect to  $q$  and setting it equal to zero yields

$$(16) \quad C'(Q(P) - q) = C'(q)$$

from which it follows that  $q = Q(P)/2$ . Thus, the payoffs to the winner and loser, respectively, as a function of the winning price  $P$ , are

$$(17) \quad W_2(P) = PQ(P) - C(Q(P))$$

$$(18) \quad L_2(P) = C(Q(P)) - 2C(Q(P)/2).$$

The loser realizes the entire benefit of subcontracting in this case but both firms realize positive profits. If a tie occurs the profit function  $T(P)$  is given as before by (8). The counterpart of Lemma 1, in this case is:

Lemma 2: There exists a unique price  $P_2$  such that

$P_2 = [C(Q) - C(Q/2)]/(Q/2)$  where  $Q = Q(P_2)$ ,  $W_2(P) < L_2(P)$  for each  $P < P_2$ ,  $W_2(P) \geq L_2(P)$  for each  $P > P_2$ ,  $dW_2/dP(P) > 0$  for each  $P \leq P_2$  and  $dL_2/dP(P) \leq 0$ , for each  $P$ .

Proof: Let  $\bar{P}$  be the lowest price for which  $Q(P) = 0$  ( $\bar{P} = \infty$  if there is none). For each such  $P < \bar{P}$  we can write, for  $Q = Q(P)$

$$(19) \quad \begin{aligned} W_2(P) - L_2(P) &= PQ - 2[C(Q) - C(Q/2)] \\ &= Q\{P - [C(Q) - C(Q/2)]/(Q/2)\} = QG(P) \end{aligned}$$

where

$$(20) \quad G(P) = P - [C(Q) - C(Q/2)]/(Q/2)$$

Upon differentiating  $G(P)$  we have:

$$(21) \quad G'(P) = 1 - 2Q'\{C'(Q) - C'(Q/2)/2 - [C(Q) - C(Q/2)]/Q\}/Q.$$

We recall that by strict convexity of  $C(Q)$ ,  $[C(Q) - C(Q/2)]/Q < C'(Q)/2$ , and thus by substitution

$$\{C'(Q) - C'(Q/2)/2 - [C(Q) - C(Q/2)]/Q\} > [C'(Q) - C'(Q/2)]/2 > 0.$$

Since  $Q'(P) < 0$ , it follows that  $G'(P) > 0$ . But  $G(0) < 0$  and  $\lim_{P \rightarrow \bar{P}} G(P) = \bar{P} - C'(0) > 0$  and therefore there exists a  $P_2$ ,  $0 < P_2 < \bar{P}$  such that  $G(P_2) = 0$ ,  $G(P) < 0$  for  $P < P_2$  and  $G(P) > 0$  for  $P > P_2$ . Clearly  $P_2 = [C(Q) - C(Q/2)]/(Q/2)$  where  $Q = Q(P_2)$  and from (19)  $W_2(P) < L_2(P)$  for  $P < P_2$  and  $W_2(P) \geq L_2(P)$  for  $P > P_2$ . Finally, to show that  $dW_2/dP > 0$  for  $P \leq P_2$  observe that

$$(22) \quad dW_2/dP = Q + Q'[P - C'(Q)] > 0$$

since for each  $P \leq P_2$ ,  $P - C'(Q) < G(P) < 0$  and  $Q' < 0$ . To show  $dL_2/dP \leq 0$ , simply observe that

$$(23) \quad dL_2/dP = Q'(P)[C'(Q(P)) - C'(Q(P)/2)] \leq 0$$

by convexity of  $C(Q)$  and  $Q'(P) < 0$ .

We can now state the counterpart to Proposition 1:

Proposition 2: In the game  $\Gamma_2$ , i.e., if the loser in the first stage of the game is the Stackelberg leader in the second stage, there exists a unique SPNE in pure strategies in which both firms bid the price  $P_2$  which is uniquely determined by

$$P_2 = [C(Q(P_2)) - C(Q(P_2)/2)] / (Q(P_2)/2),$$

in the first stage. In equilibrium, each firm supplies one-half of the total quantity demanded at that price and they both realize a positive profit,  $C(Q(P_2)) - 2C(Q(P_2)/2)$ .

Proof: The proof is similar to the proof of Proposition 2. We observe first that  $T(P) = (W_2(P) + L_2(P))/2$ . The lowest price,  $\underline{P}$ , in the first stage in equilibrium cannot be lower than  $P_2$  since if only one firm bids  $\underline{P}$  it can benefit by increasing its bid a little ( $W_2$  is increasing below  $P_2$ ) and if both firms bid  $\underline{P}$  then a firm can gain by increasing its bid and becoming the loser as  $L_2(\underline{P}) > T(\underline{P})$ . This lowest price  $\underline{P}$  cannot exceed  $P_2$ . Since if there is a losing player he can choose  $P$  close enough to  $P_2$  with  $P_2 < P < \underline{P}$ , become a winner and gain, as  $W_2(P) > W_2(P_2) = L_2(P_2) \geq L_2(\underline{P})$  by Lemma 2.

Consider next a tie situation. Two cases are possible. In the first,  $W_2(\underline{P}) = L_2(\underline{P})$ . By deviating to  $P > P_2$ , close enough to  $P_2$ , a player can become a winner and obtain  $W_2(P) > W_2(P_2) = L_2(P_2) \geq L_2(\underline{P}) = T(\underline{P})$ . In the second,  $W_2(\underline{P}) > L_2(\underline{P})$ . By deviating to  $P < \underline{P}$ , close enough to  $\underline{P}$ , a player can become a winner and obtain  $W_2(P) > (W_2(\underline{P}) + L_2(\underline{P}))/2 = T(\underline{P})$ . Therefore the lowest price is  $P_2$ . There cannot be a winning player since he could

still win by bidding  $P > P_2$  close enough to  $P_2$  and obtain  $W_2(P) > W_2(P_2)$ . So the game can only end in a tie where they both bid  $P_2$ . It is easy to show that this is indeed a SPNE.

In the Appendix we analyze mixed strategy equilibria in  $\Gamma_1$  and  $\Gamma_2$ . We show that even in this wider set of strategies there exists a unique SPNE in each game which is the pure strategy SPNE of Propositions 1 and 2.

We now turn to social welfare implications of subcontracting.

Proposition 3:  $P_1 < P^* = C'(Q(P^*)/2) < P_2$ , where  $P^*$  is the price that maximizes consumer surplus plus producer surplus.

Proof: The price  $P^*$  and the quantities to be produced in the two identical production facilities is determined from the problem

$$(24) \quad \max_{Q^*, q} \int_0^{Q^*} P(Q) dQ - C(q) - C(Q^* - q),$$

where  $P(Q)$  is the inverse demand function. Partial differentiation with respect to  $Q^*$ , and  $q$ , yields

$$(25) \quad P(Q^*) - C'(Q^* - q) = 0, \quad -C'(q) + C'(Q^* - q) = 0$$

and thus

$$(26) \quad P(Q^*) = C'(Q^*/2)$$

But by the strict convexity of  $C(Q)$  it follows that



$$(27) \quad P_1 = C(Q(P_1)/2)/(Q(P_1)/2) < C'(Q^*/2) \\ < P_2 = [C(Q(P_2)) - C(Q(P_2)/2)]/(Q(P_2)/2).$$

Thus, from society's standpoint the price,  $P_1$ , is too low and production too high if the winner in the first stage gets to set the terms of the subcontract in the second stage and  $P_2$  is too high and production too low if the loser in the first stage is the Stackelberg leader in the second stage.

Remark: Suppose the terms of the subcontract are determined by a bargaining process in which the winning and losing firms share the potential gains from subcontracting in proportions  $s$  and  $1 - s$ , respectively, with  $0 \leq s \leq 1$ . The gain from subcontracting  $C(Q(P)) - 2C(Q(P)/2)$  is realized entirely by the winner in the game  $\Gamma_1$  and by the loser in the game  $\Gamma_2$ . Then it can be shown that:

- a. the quantity subcontracted is always  $Q(P)/2$ ;
- b. the SPNE price in pure strategies is

$$P_s = \frac{sC(Q(P_s)/2) + (1 - s)[C(Q(P_s)) - C(Q(P_s)/2)]}{Q(P_s)/2}$$

The Nash bargaining solution occurs at  $s = 1/2$  and  $P_s = C(Q(P_s))/Q(P_s)$ , the average cost of producing the entire quantity demanded. It can be shown that there exists an  $s$ , say  $s^*$ , such that  $P_{s^*} = P^*$ , the socially optimal price and an  $s$ , say  $s^{**}$ , such that  $P_{s^{**}} = P_m$ , the monopoly price, providing

$$P_m \leq P_2.$$

Thus, comparing Propositions 1 and 2 it is evident that were the firms able to choose the means of subcontracting, they would both prefer that the loser in the first stage be the one that sets the terms of the subcontract in the second stage. The intuitive reason for this is that if the winner of the first stage gets to set the subcontract in the second stage, the loser realizes zero profit and therefore being the loser is very disadvantageous. Thus, bidding in the first stage is aggressive to the point that both firms realize zero profit in equilibrium. On the other hand, if the loser of the first stage gets to set the terms of the subcontract in the second stage, then being the loser is not disadvantageous and both firms bid less aggressively in the first stage. While both firms are better off if the loser of the first stage is the leader in the second stage, these profits do not necessarily coincide with those that obtain under full cooperation, i.e., the profit of a monopolist with two identical production facilities. Examples can be constructed with explicit cost and demand functions in which the monopoly price is above, equal to, or below the SPNE price if the loser of the first stage is the leader in the second stage. It should be noted that we have not indicated which sharing rule would be chosen by the firms if they could. This would require adding a previous stage to the game in which the sharing rule was determined strategically.

### 3. Competition to Supply a Fixed Quantity

The description of the games  $\Gamma_1$  and  $\Gamma_2$  remain the same in this section, but we assume now that the demand function is of the form

$$(28) \quad Q(P) = \begin{cases} 0 & \text{if } P > P_0 \\ Q_0 & \text{if } P \leq P_0 \end{cases}$$

for the positive price  $P_0$  and quantity  $Q_0$ . This is descriptive of the situation in which the first stage involves bidding for production of a fixed quantity as in a government contract. The analysis of the second stage of the game remains the same since  $P$  is assumed to be fixed in this stage. Thus the loser's and winner's payoffs as a function of the winning price  $P$  are given by

$$\text{For } P \leq P_0: \quad W_1(P) = PQ_0 - 2C(Q_0/2)$$

$$L_1(P) = 0$$

$$W_2(P) = PQ_0 - C(Q_0)$$

$$L_2(P) = C(Q_0) - 2C(Q_0/2)$$

$$\text{For } P > P_0 \quad W_1(P) = L_1(P) = W_2(P) = L_2(P) = 0.$$

The payoffs for both players in the case of a tie are

$$T(P) = \begin{cases} PQ_0/2 - C(Q_0/2), & \text{for } P \leq P_0, \\ 0, & \text{for } P > P_0. \end{cases}$$

The prices

$$P_a = C(Q_0/2)/(Q_0/2),$$

$$P_b = C(Q_0)/Q_0,$$

$$P_c = [C(Q_0) - C(Q_0/2)]/(Q_0/2),$$

play an important role in describing the equilibria in the games  $\Gamma_1$  and  $\Gamma_2$ . Note that  $W_1$  is an increasing linear function vanishing at  $P_a$ .  $W_2$  is an increasing linear function vanishing at  $P_b$  and coincides with the constant function  $L_2$  at  $P_c$ .

Note also that as above  $T = (W_1 + L_1)/2 = (W_2 + L_2)/2$ .

Proposition 4: In the game  $\Gamma_1$ , i.e., if the winner of the first stage of the game is the Stackelberg leader in its second stage the following hold.

- (a) If  $P_0 > P_a$  there exists a unique SPNE in which both firms bid the price  $P_1 = P_a$  and end up with payoff zero.
- (b) If  $P_0 < P_a$  then in any SPNE both firms bid prices higher than  $P_0$  in the first stage. Moreover any two bids higher than  $P_0$  support an SPNE. In all these SPNE's, the firms produce zero and receive zero payoffs.
- (c) If  $P_0 = P_a$  then all SPNE's in (b) are still SPNE and there is an additional SPNE in which both firms bid the price  $P_0$ , each produces  $Q_0/2$  and receives a zero payoff.

Proof:

(a) Conditions similar to those of Lemma 1 hold here for  $P_1 = C(Q_0/2)/(Q_0/2)$  and the proof now follows that of Proposition 1.

(b) For any  $P \leq P_0$ ,  $W_1(P) < T(P) < 0$ . Thus the lower price in the first stage in an SPNE cannot be lower than  $P_0$  since in this case at least one of the firms receives a negative payoff and it can always guarantee a zero payoff. So in any SPNE the two bids are above  $P_0$ . Indeed, any such

two prices constitute an SPNE since the payoffs in this case are zero and no one can improve upon it.

(c) Clearly, bidding  $P_0$  is also an equilibrium.

Proposition 5: In the game  $\Gamma_2$ , i.e., if the loser of the first stage is the leader in the second stage, the following hold.

- (a) If  $P_0 \geq P_c$ , then there exists a unique SPNE where the firms bid the same price  $P_c$  in the first stage and each receives the positive payoff  $C(Q_0) - 2C(Q_0/2)$ .
- (b) If  $P_b < P_0 < P_c$ , then all the SPNE's are of the following type. One firm wins by bidding  $P_0$  while the other bids a higher price. This results in subcontracting in the second stage. Both firms enjoy a positive profit, the loser's being higher.
- (c) If  $P_0 = P_b$ , then all SPNE's of (b) are also SPNE here. In addition, any pair of prices higher than  $P_0$  are also SPNE prices resulting in zero payoffs for both firms.
- (d) If  $P_0 < P_b$ , any pair of prices higher than  $P_0$  is an SPNE. These are all the SPNE's.

Proof:

(a) Conditions similar to those of Lemma 2 hold here and the proof is similar to that of Proposition 2.

(b) Similar to Proposition 2, the lowest bid,  $\underline{P}$ , cannot be below  $P_0$ .  $\underline{P}$  cannot exceed  $P_0$  since in this case both realize a zero profit and a firm can gain by bidding  $P_0$ , become the winner and receive  $W_2(P_0) > 0$ . Thus  $\underline{P} = P_0$ . It is impossible in equilibrium that both bid  $P_0$  since by

increasing the bid and becoming a loser, a firm receives  $L_2(P_0) > T(P_0)$ . Therefore one firm bids  $P_0$  and receives  $W_2(P_0)$  while the other bids a price higher than  $P_0$  and receives  $L_2(P_0)$ . To see that this is indeed an equilibrium we note that the loser receives the highest payoff in the game and would not deviate and the winning firm can only lower its payoff by bidding a lower price or receive zero by increasing its bid.

The proofs of (c) and (d) are simpler and left to the reader.

#### 4. Summary

We have investigated a Bertrand duopoly with the potential for subcontracting production to the rival as a two stage game. The presence of convex production costs together with the requirement that each rival stands ready to supply the entire quantity demanded at his quoted price creates the incentive for subcontracting. Our analysis of the SPNE of the game discloses that if the terms of the subcontract favor the loser of the game's first stage then both rivals are better off than if they favor the winner. (A result that should be of some comfort to all the losers in the world.) Indeed, if the winner of the first stage sets the terms of the subcontract in the second stage then competition to be the winner dissipates away all the profits. On the other hand, if the first stage loser sets the terms of the subcontract then both firms realize the full benefits of subcontracting.

The presence of subcontracting possibilities does not in general assure that the SPNE price will be socially optimal. However, the potential for subcontracting does give rise to pure strategy SPNE, even though there is no actual subcontracting in equilibrium, instead of only mixed strategy equilibria that occur in the absence of subcontracting possibilities.

There are a large number of possible extentions of our analysis. These include considerations of more than two firms, firms with different cost functions, repeated play of the game, the absence of complete information, and the possibility of information sharing. Expansion of the game to include the choice of the mode of subcontracting as a strategic variable would also be interesting.

Appendix

We consider the games  $\Gamma_1$  and  $\Gamma_2$  (in the case of competition to supply a market) when the firms use mixed strategies in the first stage. (Allowing mixed strategies in the second stage does not change the analysis since in this stage the leader has a dominant strategy.) We show that the results of Section 2 do not change. The pure strategy equilibria in Propositions 1 and 2 remain the only equilibria also in the bigger set of mixed strategies. We study a single game  $\Gamma$  which generalizes both  $\Gamma_1$  and  $\Gamma_2$ .

Consider a two person game  $\Gamma$  in which pure strategies for both players are prices in  $[0, \infty)$ . We denote the players by 1 and 2 and when we use  $i$  and  $j$  to refer to the players we always assume  $i \neq j$ . The payoff function to player  $i$  is  $v_i(P_i, P_j)$  where the price chosen by  $i$  appears always as the first argument. The payoff functions are given by three continuous functions on  $[0, \infty)$ ,  $L$ ,  $T$ , and  $W$ , as follows.

$$v_i(P_i, P_j) = \begin{cases} L(P_j) & \text{if } P_i > P_j \\ T(P_i) & \text{if } P_i = P_j \\ W(P_i) & \text{if } P_i < P_j \end{cases}$$

Clearly,  $v_i$  is continuous at each point  $(P_i, P_j)$  when  $P_i \neq P_j$ . We assume that the functions  $L$ ,  $T$  and  $W$  satisfy the following requirements. There exists  $P^0 > 0$  such that

- (1) for  $P < P^0$ ,  $L(P) > T(P) > W(P)$ ;



- (2) For  $P > P^0$ ,  $W(P) \geq T(P) \geq L(P)$ , where strong inequality holds in a neighborhood of  $P^0$  and both inequalities are strict whenever one of them is;
- (3)  $W(P)$  is not decreasing for  $P < P^0$ ;
- (4)  $L(P)$  is not increasing for  $P > P^0$ ;
- (5)  $W$  and  $L$  are differentiable at  $P^0$ .

Observe that by continuity of  $L$ ,  $T$  and  $W$ ,  $L(P^0) = T(P^0) = W(P^0)$  by (1) and (2). A mixed strategy is a measure  $\mu$  on  $[0, \infty)$  which we represent also by its cumulative distribution function  $F$ . We identify each  $P \in [0, \infty)$  with the mixed strategy with all mass at  $P$ . The payoff functions  $v_i$  are extended naturally to mixed strategies. For a pair of mixed strategies  $F_i$  and  $F_j$

$$v_i(F_i, F_j) = \int_0^\infty \int_0^\infty v_i(P_i, P_j) dF_i(P_i) dF_j(P_j).$$

The support of  $F$ , denoted  $\text{supp}(F)$ , contains all points which do not have a neighborhood of measure 0 (i.e., where  $F$  is constant). The set of atoms of  $F$  (i.e., where  $F$  has a jump) is denoted by  $J(F)$ . For  $P \in J(F)$ ,  $\mu(P)$  is the measure of  $P$ . A pair,  $(F_1, F_2)$ , is an equilibrium if for each player  $i$ ,  $v_i(F_i, F_j) = \max_F v_i(F, F_j)$ . Note that since  $v_i$  is continuous whenever  $P_i \neq P_j$ , it follows that  $v_i(P, F_j) (= \int_0^\infty v_i(P, P_j) dF_j(P_j))$  is continuous in  $P$  whenever  $P \notin J(F_j)$ . This remark can be easily used to prove the following proposition.

Proposition 6: Let  $(F_1, F_2)$  be an equilibrium. Then for each price  $P$  and player  $i$ ,  $v_i(P, F_j) \leq v_i(F_i, F_j)$ . Moreover, equality holds for each

$P \in \text{supp}(F_i)$  whenever either  $P \in J(F_i)$  or  $P \notin J(F_j)$ .

Corollary 1: Let  $(F_1, F_2)$  be an equilibrium. If  $P_i \in \text{supp}(F_i)$  and either  $P_i \in J(F_i)$  or  $P_i \notin J(F_j)$  then for each  $P$ ,  $v_i(P_i, F_j) \geq v_i(P, F_j)$ .

Theorem 1:  $\Gamma$  has a unique equilibrium in mixed strategies given by  $(P^0, P^0)$ .

We note that the functions  $W_1, L_1$  and  $T$  as well as the functions  $W_2, L_2$  and  $T$  satisfy all the requirements (1)-(5), and therefore by Theorem 1,  $\Gamma_1$  and  $\Gamma_2$  each have a unique SPNE in mixed strategies.

We prove the theorem through Lemmas 3 and 4. We assume throughout the proof that  $(F_1, F_2)$  is an equilibrium of  $\Gamma$ .

Lemma 3:  $\min(\text{supp}(F_1) \cup \text{supp}(F_2)) \geq P^0$ .

Proof: Let  $\underline{P} = \min(\text{supp}(F_1) \cup \text{supp}(F_2))$  and assume  $\underline{P} < P^0$ . Now either  $\underline{P} \in J(F_1) \cap J(F_2)$  or for at least one  $j$ ,  $\underline{P} \notin J(F_j)$ . In either case, by Corollary 1,  $v_i(\underline{P}, F_j) \geq v_i(P^0, F_j)$ . We finish the proof by contradicting the last inequality. Indeed, for  $P_j \geq P^0$ ,  $v_i(\underline{P}, P_j) = W(\underline{P}) \leq W(P^0) = v_i(P^0, P_j)$  by (3). For  $\underline{P} \leq P_j < P^0$ ,  $v_i(\underline{P}, P_j) \leq T(P_j) < L(P_j) = v_i(P^0, P_j)$  by (1). Thus  $v_i(\underline{P}, F_j) = \int_{\underline{P}}^{\infty} v_i(\underline{P}, P_j) dF_j(P_j) < v_i(P^0, P_j)$ .

Lemma 4: For at least one player  $i$ ,  $F_i$  is the pure strategy  $P^0$ .

Proof: Let  $\underline{P} = \max(\text{supp}(F_1) \cup \text{supp}(F_2))$  and suppose  $\underline{P} > P^0$ .

Case 1.  $P \in \text{supp}(F_j)$  and  $P \notin J(F_j)$ .

By Corollary 1,  $v_i(P^0 + \epsilon, F_j) - v_i(P, F_j) \leq 0$ . Evaluating this difference and assuming that  $P^0 + \epsilon \notin J(F_j)$  and that  $P^0 + \epsilon < \bar{P}$  we find, using (4) and (2), that

$$\begin{aligned} & \int_{P^0 + \epsilon}^P [W(P^0 + \epsilon) - L(P_j)] dF_j(P_j) \\ & \geq \int_{P^0 + \epsilon}^P [W(P^0 + \epsilon) - L(P^0 + \epsilon)] dF_j(P_j) \\ & = [W(P^0 + \epsilon) - L(P^0 + \epsilon)] \int_{P^0 + \epsilon}^P dF_j(P_j) \geq 0. \end{aligned}$$

Thus, we conclude that  $[W(P^0 + \epsilon) - L(P^0 + \epsilon)] \int_{P^0 + \epsilon}^P dF_j = 0$ . But for small enough  $\epsilon$ ,  $W(P^0 + \epsilon) - L(P^0 + \epsilon) > 0$  by (2) and therefore  $\int_{P^0 + \epsilon}^P dF_j = 0$ . Letting  $\epsilon \rightarrow 0$  and using Lemma 3, we conclude that  $F_j$  is  $P^0$ .

Case 2.  $P \in J(F_1) \cap J(F_2)$  and  $W(P) = L(P)$ .

By (2) also  $T(P) = L(P)$ , and  $v_i(P, P) = L(P)$ . The computations of Case 1 remain exactly the same.

Case 3.  $P \in J(F_1) \cap J(F_2)$  but  $W(P) > L(P)$ .

We show that this case is impossible. By (2) also  $W(P) > T(P) > L(P)$ . By Corollary 1,  $v_i(P - \epsilon, F_j) - v_i(P, F_j) \leq 0$ . But this difference when  $P - \epsilon \notin J(F_j)$  is

$$\int_{[P - \epsilon, P)} [W(P - \epsilon) - L(P_j)] dF_j(P_j) + \mu_j(P) [W(P + \epsilon) - L(P)]$$

When  $\epsilon \rightarrow 0$  the integral vanishes and the other term converges to  $\mu_j(P)[W(P) - L(P)] > 0$  by the stipulation of the case, which is a contradiction.

Proof of Theorem 1: By Lemma 4 there exists a player  $i$  such that  $F_i$  is  $P^0$ . By Lemma 3,  $\text{supp}(F_j) \subseteq [P^0, \infty)$ . Using Corollary 1 and (4) and (5) for  $P^0 + \epsilon \notin J(F_j)$ ,

$$\begin{aligned}
0 &\geq v_i(P^0 + \epsilon, F_j) - v_i(P^0, F_j) \\
&= \int_{P^0}^{P^0 + \epsilon} L(P_j) dF_j(P_j) + \int_{P^0 + \epsilon}^{\infty} W(P^0 + \epsilon) dF_j(P_j) - L(P^0) \\
&\geq L(P^0 + \epsilon) \mu_j((P^0, P^0 + \epsilon)) + W(P^0 + \epsilon) \mu_j([P^0 + \epsilon, \infty)) \\
&\quad + \mu_j(P^0) L(P^0) - L(P^0) \\
&= [L(P^0) + \epsilon L'(P^0) + o(\epsilon)] \mu_j((P^0, P^0 + \epsilon)) \\
&\quad + [W(P^0) + \epsilon W'(P^0) + o(\epsilon)] \mu_j([P^0 + \epsilon, \infty)) \\
&\quad + \mu_j(P^0) L(P^0) - L(P^0) \\
&= \epsilon [L'(P^0) \mu_j((P^0, P^0 + \epsilon)) + W'(P^0) \mu_j([P^0 + \epsilon, \infty))] + o(\epsilon).
\end{aligned}$$

Since  $\mu_j((P^0, P^0 + \epsilon)) \xrightarrow{\epsilon \rightarrow 0} 0$ , it follows that

$0 \geq \liminf_{\epsilon \rightarrow 0} [v_i(P^0 + \epsilon, F_j) - v_i(P^0, F_j)]/\epsilon \geq W'(P^0)\mu_j((P^0, \infty))$ . But  $W'(P^0) > 0$  and hence  $\mu_j((P^0, \infty)) = 0$ , i.e.,  $F_j$  is  $P^0$ .

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