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Northwestern University, Evanston, Illinois 60208

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A THEORY OF VOTING EQUILIBRIA

by

Roger B. Myerson and Robert J. Weber*

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Abstract

A voting equilibrium arises when the voters in an electorate, acting in accordance with both their preferences for the candidates and their perceptions of the relative chances of each candidate being in contention for victory, generate an election result which justifies their perceptions. Voting equilibria always exist, and the set of equilibria can vary substantially with the choice of voting system. Equilibria under plurality rule, approval voting, and the Borda system are compared. A candidate-positioning game is considered, and it is found that plurality rule imposes little restriction on the position of the winning candidate in three-candidate races, while approval voting leads to a winner positioned at the median of the voter distribution. Campaign activities intended to influence voter preferences are contrasted with activities meant to influence only perceptions of candidate viability.

*Both of the Department of Managerial Economics and Decision Sciences, J. L. Kellogg Graduate School of Management, Northwestern University, Evanston, Illinois 60208.

Introduction

Voters in an election typically have not only personal preferences for the various candidates, but also perceptions (influenced by news reports, public opinion polls, and commonly-heard talk "on the street") of the candidates' relative chances of winning the election. The behavior of a voter may depend both on his preferences, and on his perceptions of candidate viability. For example, when many candidates are running for the same office under plurality rule, a voter might not vote for the candidate that he most prefers if this candidate is considered unlikely to be in contention for victory.

A voter's honest response to a poll asking his voting intentions might well depend upon perceptions he has formed from viewing the results of previous polls. Voting intentions may change from day to day as the voters' perceptions evolve during a campaign (even if their preferences remain unchanged), and so a poll, when published, may invalidate itself. A *voting equilibrium* arises when the perceptions arising from the publication of a poll lead the voters to behave in a manner that in turn justifies the predictions of the poll.

Voting equilibria based on polls are somewhat analogous to competitive equilibria based on prices in economic markets. In a pure exchange economy, buyers have preferences for various bundles of commodities. Trading behavior, which generates prices, is driven by the buyers' perceptions of the prices which they will confront. At equilibrium, prices summarize demand, and simultaneously generate that same demand. The study of competitive equilibria does not require specification of the precise mechanism by which individual market participants adjust their behavior as the market moves toward equilibrium. Similarly, a detailed model of the process by which a voting equilibrium might be reached is not essential to the definition and computation of voting equilibria. Yet still, just as one expects markets to eventually clear at equilibrium prices, one might expect that voters will ultimately behave in accordance with a voting equilibrium after observing the series of public reports which accompany an extended campaign (i.e., the campaign is, in part, a political "tâtonnement" process).

Following the terminology of Simon 1954, we say that an election displays a *bandwagon effect* if voters become more inclined to vote for a given candidate as his standing in pre-election polls improves. Conversely, it displays an *underdog effect* if the voters become less inclined to vote for a candidate as his standing in the polls improves. The possibility of bandwagon or underdog effects has been recognized and investigated in the received literature on voting. In particular, Simon showed that there can be many different predictions that are self-fulfilling prophecies, especially when there is a

bandwagon effect. (See also Bowden 1987; Grunberg and Modigliani 1954; McKelvey and Ordeshook 1985.)

In most of the literature on elections involving three or more candidates, however, bandwagon and underdog effects are exogenously specified, rather than endogenously derived from the voters' preferences and the nature of the electoral system. Here, we develop a simple decision-theoretic model to describe how voters' actions might be based on their preferences and on perceptions derived from pre-election polls, and we characterize equilibrium outcomes under different electoral systems for multi-candidate elections. We use this model to show that different electoral systems may have very different sets of voting equilibria, even when the voters' preferences are held fixed. We also present a simple model of spatial competition among candidates, and show that the choice between plurality rule and approval voting can lead to substantial differences in the equilibrium positioning of candidates. Finally, we discuss the difference between campaign activities intended to present the candidates' positions to the electorate (and hence help voters determine their preferences), and activities which attempt to influence the selection of an equilibrium outcome (by manipulating the voters' perceptions of relative candidate viability).

A Model of Multi-Candidate Elections

We begin by defining an abstract model of an election. There are k candidates, who we number consecutively to form the set $K = \{1, 2, \dots, k\}$. Each voter submits a ballot, which is a vector of the form $v = (v_1, \dots, v_k)$, where v_1 votes are given to candidate 1, v_2 votes to candidate 2, and so on.¹ The winner of the election is the candidate who receives the largest total number of votes on all submitted ballots.

Let V denote the set of all possible ballots that a voter could submit, under the rules of the election. In order to simplify notation, we will only consider voting systems for which V is finite. If the election is run under *plurality rule*, then each voter can cast a single vote for at most one candidate. Thus, in a three-candidate election with plurality rule,²

$$V = \{(1,0,0), (0,1,0), (0,0,1), (0,0,0)\}.$$

Under the rules of *approval voting*, a voter can give one or zero votes independently to each of the candidates. (For example, see Brams and Fishburn 1978, 1983; Merrill 1988; Weber 1977.) In a three-candidate election with approval voting,

$$V = \{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1), (0,0,0)\}.$$

Under *Borda-rule* voting, a voter who does not abstain gives zero votes to one candidate, one vote to another, two votes to a third, and so on, so that with three candidates,

$$V = \{(2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,1,2), (0,2,1), (0,0,0)\}.$$

To represent formally the diversity of interests present within the electorate, we assume that each voter has preferences that can be summarized by a utility vector of the form $u = (u_1, \dots, u_k)$, where u_i denotes the payoff (on some von Neumann-Morgenstern utility scale) that the voter would receive if candidate i were to win the election.³ We refer to a voter's vector of utility payoffs as his *type*. We further assume that there is some finite set T such that $T \subseteq \mathfrak{R}^k$ and every voter's type is in the set T .⁴ Let $f(u)$ denote the fraction of all voters in the population whose type is the vector u . Then $f(\cdot)$ is a probability distribution over T which summarizes the preferences of the electorate.⁵

We assume that each voter follows the principle of expected utility maximization to determine which ballot in V he will submit in the election. Under our assumptions, his vote only influences his payoff if it changes the winner of the election. Therefore, to apply the criterion of expected utility maximization, a voter needs to estimate the probability that any given pair of candidates may be in a sufficiently close race for first place that his ballot alone could swing the election from one to the other. In order to simplify the following discussion, we will say that two candidates are "tied" if their vote totals differ by $1/2$ vote or less. (This convention saves us from needing to give special consideration to voting systems which permit the casting of fractional votes.)

Let H denote the set of all (unordered) pairs of distinct candidates. We denote a pair $\{i,j\}$ in H as ij for short (so $ij = ji$); there are $k(k-1)/2$ such pairs. For each pair of candidates i and j , the *ij-pivot probability* p_{ij} is the probability (perceived by a voter) of the event that candidates i and j will be tied for first place in the election. We assume that a voter perceives the probability that he will change the winner of the election from any candidate i to another candidate j , by casting a ballot v with $v_i \geq v_j$, to be linearly proportional to $v_i - v_j$, and that the constant of proportionality (the *ij-pivot probability*) is the same for the perceived chance of changing the winner from j to i if $v_j \geq v_i$. (This is equivalent to assuming that, conditional on two candidates being in a close race for

first place, the difference in their vote totals is perceived to be approximately uniform near 0 on the lattice of possible vote differences, so that the probability of i and j being tied for first place is essentially the same as the probability of i being in first place one vote ahead of j , which is also essentially the same as the probability of j being in first place two votes ahead of i , etc. For this assumption to be reasonable, it is only necessary that the electorate be not too small.) We also assume that the perceived probability of three candidates being tied with one another for first place is infinitesimal in comparison to the probability of a two-candidate tie.

A vector that lists the perceived pivot probabilities for all pairs of candidates is denoted $p = (p_{ij})_{ij \in H}$. A voter with ij -pivot probability p_{ij} will perceive the probability that he might change the winner of the election from j to i by submitting the vote vector v to be $p_{ij} \cdot \max(v_i - v_j, 0)$.

Let $G(p, v, u)$ denote the utility gain expected by a voter of type u from submitting the ballot v , when p is his vector of perceived pivot probabilities. It follows from our assumptions that

$$G(p, v, u) = \sum_{ij \in H} p_{ij} (v_i - v_j) (u_i - u_j).$$

That is, for each ij -pair, if $v_i > v_j$ then by casting the vote vector v (rather than abstaining) the voter has probability $p_{ij} (v_i - v_j)$ of changing the winner of the election from j to i ; this gives him a net utility gain of $u_i - u_j$. Multiplying the utility gains by the corresponding probabilities and summing over all pairs of candidates gives the expected gain.

The expected gain formula can be rewritten as

$$G(p, v, u) = \sum_{i \in K} v_i \sum_{j \neq i} p_{ij} (u_i - u_j).$$

The expression

$$\sum_{j \neq i} p_{ij} (u_i - u_j)$$

is called the *prospective rating* of candidate i , with respect to the voter type u and the pivot-probability vector p . Under plurality rule, where a voter can set one component of v equal to 1, and must set all other components to 0, a voter of type u maximizes his expected gain $G(p, v, u)$ over all permissible ballots v in V by voting for the candidate who has the highest prospective rating. Under approval voting, where a voter can give one vote each to as many candidates as he wants, $G(p, v, u)$ is maximized by giving a vote to every candidate who has a positive prospective rating and by not voting for any candidate who has a negative prospective rating. Under Borda rule, $G(p, v, u)$ is maximized by ranking the candidates on the Borda ballot in order of their prospective ratings.

An *election result* is a probability distribution μ over the set $V \times T$ which summarizes the aggregate voting behavior of voters of each type. For any v in V and any u in T , $\mu(v, u)$ is the

fraction of the electorate consisting of voters of type u who cast the ballot v . Therefore, $\mu(v,T) = \sum_{u \in T} \mu(v,u)$ is the fraction of all voters in the electorate who cast the ballot v . Furthermore, the marginal distribution of μ on the type set T is the population distribution f , i.e., for every voter type u , $\sum_{v \in V} \mu(v,u) = f(u)$. In what follows, we shall view an election as the result of a large number of random draws from μ . This will allow us to differentiate between two-candidate races which are "taken seriously" (i.e., the perceived chance of a tie between the candidates is positive), and races which are "close" (i.e., the actual outcome of the election is a near-tie for victory between the candidates).

Given an election result μ , the *predicted score* (i.e., the expected per-capita vote total) of candidate i is

$$S_i(\mu) = \sum_{v \in V} v_i \cdot \mu(v,T) .$$

A *likely winner* is a candidate whose predicted score is maximal. We denote by $W(\mu)$ the set of likely winners when the election result is μ , i.e., candidate i is in $W(\mu)$ if $S_i(\mu) = \max_{j \in K} S_j(\mu)$.

For any pivot-probability vector p , we define the *voter-response set* $R(p)$ to be the set of all election results in which all of the voters cast ballots which maximize their expected gains, given that their common perception of the pivot probabilities is p . (To this point, we have been considering a pivot-probability vector p to represent the perceptions of an individual voter concerning the likelihood of various pairs of candidates being the primary contenders to victory. In what follows, we shall be assuming that all voters hold the same perceptions.) An election result μ is in $R(p)$ if and only if, for every ballot v in V and voter type u in T ,

$$\text{if } \mu(v,u) > 0 \text{ then } G(p,v,u) = \max_{w \in V} G(p,w,u) .$$

If p denotes the vector of pivot probabilities as perceived by the voters at the time of the election, and all voters choose their ballots according to the principle of expected utility maximization, then the result of the election should be a distribution of votes by voter type which lies in the voter-response set $R(p)$. If an election result μ is in $R(p)$, then we say that the pivot-probability vector p *justifies* μ . For future reference, we note here that if p justifies μ , and if q is another pivot-probability vector satisfying $q_{ij} = \alpha \cdot p_{ij}$ for some $\alpha > 0$ and all candidate pairs ij , then q also justifies μ , because the expected-gain function G is homogeneous in p .

Voting Equilibria

The voter-response mapping $R(\cdot)$ indicates how potential election results depend on the voters' perception of the pivot probabilities, and the likely-winner mapping $W(\cdot)$ indicates how the set of candidates with serious chances of winning depends on the election result. It remains for us to characterize the pivot-probability vectors that may be perceived by the voters in equilibrium, given an electoral situation described by (K, V, T, f) .

When the electorate is large, the probability of an election being nearly tied is typically very small. However, if the pivot probabilities were all perceived to be zero, then an individual voter's action would not matter, and so any ballot would be optimal for any voter. We can avoid this trivial situation by requiring that, for every pair of candidates $\{i, j\}$, the ij -pivot probability is a positive number, although it may be very small. (Weakening this assumption does not eliminate any voting equilibria, but we will see in the next section that it can allow new equilibria to arise.)

On the other hand, if the voters can predict the scores $(S_1(\mu), S_2(\mu), \dots, S_k(\mu))$ of the various candidates relatively accurately (e.g., as a result of pre-election polls that have very small standard errors), then some pivot probabilities should be perceived to be much smaller than others. It seems reasonable to assume that voters would expect candidates with lower predicted scores to be much less likely to be serious contenders for first place than candidates with higher predicted scores. That is, if the predicted score for some candidate i is strictly less than the score for some other candidate j , then the voters would perceive that the event of candidate i being tied for first place with any third candidate h is much less likely than the event of candidate j being tied for first place with candidate h . Given an election result μ , and any $0 \leq \epsilon < 1$, we say that a pivot-probability vector p satisfies the *ordering condition* for ϵ (with respect to μ) if, for every three distinct candidates i, j , and h ,

$$\text{if } S_i(\mu) < S_j(\mu) \text{ then } p_{ih} \leq \epsilon \cdot p_{jh} .$$

We are at last prepared to define the central concept of this paper: An election result μ is a *voting equilibrium* if and only if, for every positive number ϵ , there exists some vector p of positive pivot probabilities which justifies μ , and which satisfies the ordering condition for ϵ . That is, an election result μ is a voting equilibrium if the voters would be willing to cast their votes in accordance with μ , while holding perceptions that candidates with low predicted scores under μ have very small chances of being serious contenders for victory relative to candidates with higher predicted scores under μ .

Our first basic result is that voting equilibria always exist. The proof is at the end of this section.

Theorem 1. In every electoral situation, the set of voting equilibria is nonempty.

Demonstrating that an election result μ is a voting equilibrium requires the construction of an infinite sequence of voter perceptions which justify μ and satisfy the ordering condition for ever-smaller values of ϵ . However, we can provide a simple limit condition which greatly facilitates the search for voting equilibria.

Given any voting equilibrium μ , select for each positive number ϵ a pivot-probability vector $p(\epsilon)$ that justifies μ and satisfies the ordering condition for ϵ . Typically, the voters will perceive the pivot probabilities to be very small. However, the ordering condition is homogeneous in p , i.e., multiplying all of the components of $p(\epsilon)$ by the same positive constant yields another vector which still satisfies the ordering condition for ϵ . Furthermore, as noted at the end of the previous section, the resulting vector will still justify μ . In particular, we can rescale each vector $p(\epsilon)$ to make its components sum to 1, i.e., to make each $p(\epsilon)$ a probability distribution over H . The set of probability distributions over H is a closed and bounded set, and hence there will exist a subsequence of the rescaled $p(\epsilon)$ -vectors that converges, as ϵ goes to zero, to some vector q which is also a probability distribution over H . Since the expected gain function G is continuous, q will justify μ . This limiting q will also satisfy the ordering conditions for all positive ϵ , and therefore a pair of candidates ij must have pivot probability q_{ij} equal to zero if there exists some other candidate h with a predicted score higher than at least one of them. (For example, if $S_i(\mu) < S_h(\mu)$, then $q_{ij} \leq \epsilon \cdot q_{hj}$ for all positive ϵ .) This gives us the following necessary condition for a voting equilibrium.

Theorem 2. If μ is a voting equilibrium then there exists some probability distribution $q = (q_{ij})_{ij \in H}$ which justifies μ , and for which, whenever $q_{ij} > 0$, $S_i(\mu) = \max_{h \neq j} S_h(\mu)$ and $S_j(\mu) = \max_{h \neq i} S_h(\mu)$.

We say that a probability vector q *supports* the voting equilibrium μ if it satisfies the conditions in Theorem 2 for μ . If q supports a voting equilibrium μ , then each component q_{ij} can be interpreted as the perceived conditional probability of a tie for first place between candidates i and j , given that some pair of candidates is tied for first place, and given that very accurate pre-election polls predicted the election result μ . The conditions in the theorem imply that q_{ij} can be positive only if one of the following conditions is satisfied: Either candidates i and j are both in the set of likely winners, or one of these two candidates is the unique likely winner and the other candidate has the second-highest predicted score (i.e., only if each is a relatively "serious threat" to the other).

Proof of Theorem 1. The proof employs a straightforward fixed-point argument, and is similar to the existence proof for proper equilibria of non-cooperative games (see Myerson 1978). Take any number ϵ such that $0 < \epsilon < 1$. For any election result μ , let $Q^\epsilon(\mu)$ be the set of all pivot-probability vectors p which satisfy the ordering condition for ϵ and for which

$$p_{ij} \geq \epsilon^{k^2}/k^2, \text{ for all } ij \in H.$$

It is easily shown that $Q^\epsilon(\cdot)$ maps every μ into a nonempty⁶ convex set, and is upper-hemicontinuous in μ . Similarly, the voter-response mapping $R(\cdot)$ maps every pivot-probability vector p into a nonempty convex set of election results, and is upper-hemicontinuous in p . Thus, by the Kakutani fixed-point theorem, there exists a pair $(p^\epsilon, \mu^\epsilon) \in Q^\epsilon(\mu^\epsilon) \times R(p^\epsilon)$.

Now consider a sequence of values for ϵ that converges to zero. Since the sets $V \times T$ and H are both finite, there must be an infinite subsequence for which the sets

$$\{(v,u) \in V \times T \mid \mu^\epsilon(v,u) > 0\} \text{ and } \{ij \in H \mid S_i(\mu^\epsilon) < S_j(\mu^\epsilon)\}$$

are constant for all ϵ in the subsequence. Let $\mu = \mu^{\epsilon'}$ for any particular ϵ' in this subsequence. Then for every ϵ in the subsequence, p^ϵ justifies μ (since sets of the first type are constant along the subsequence) and satisfies the ordering condition for ϵ (since sets of the second type are constant). Hence, μ is a voting equilibrium. Q.E.D.

Example 1: Comparison of Voting Systems

We are now prepared to examine how, for a fixed electorate, changes in the voting system can affect equilibrium predictions of election results.

In our first example, there are three candidates, and three types of voters whose utility vectors are given below together with the distribution of types within the electorate:

voter type	utility vector	proportion of electorate
A	$u^A = (10, 9, 0)$	$f(u^A) = 0.3$
B	$u^B = (9, 10, 0)$	$f(u^B) = 0.3$
C	$u^C = (0, 0, 10)$	$f(u^C) = 0.4$

To interpret these parameters, suppose that candidates 1 and 2 are leftist candidates, and candidate 3 is a rightist. Then 60% of the voters are leftists who greatly prefer both candidates 1 and 2 over candidate 3. Of these leftist voters, however, half (30% of the electorate) are A-type voters who have a slight preference for candidate 1, and the other half are B-type voters who have a slight preference for candidate 2. The C-type voters are rightists who strictly prefer candidate 3 over both of the leftist candidates, and are indifferent between the two leftists. These C-type voters form 40% of the electorate. This example is similar to that discussed by Borda 1781. It also models a situation which has arisen on numerous occasions in two-party systems, when the nominee of the party holding majority support is confronted with a third, independent candidate contesting that support. The 1912 U.S. presidential election is an oft-cited instance of this situation.

Under plurality rule, there are three voting equilibria. At one equilibrium, all of the leftists coordinate their support behind candidate 1, voting (1,0,0), and candidate 1 beats candidate 3. In our notation, this equilibrium is written as

$$\mu((1,0,0), u^A) = 0.3, \quad \mu((1,0,0), u^B) = 0.3, \quad \mu((0,0,1), u^C) = 0.4.$$

That is, 30% of the voters are A-type leftists voting for candidate 1, 30% are B-type leftists also voting for candidate 1, and 40% are C-type rightists voting for candidate 3. The election result yields predicted scores (expected per-capita vote totals) of

$$S_1(\mu) = 0.6, \quad S_2(\mu) = 0, \quad S_3(\mu) = 0.4.$$

Thus, candidate 1 is the only likely winner, i.e., $W(\mu) = \{1\}$. This equilibrium is supported by the pivot-probability vector q satisfying $(q_{12}, q_{13}, q_{23}) = (0, 1, 0)$. That is, if there were a close race for first place, the voters would expect (with conditional probability very near 1) this close race to be between candidates 1 and 3, who have the top two predicted scores. Fearing a close race between candidates 1 and 3, the B-type leftists vote for candidate 1, even though they slightly prefer candidate 2. (To a B-type leftist with perceptions p , the prospective rating of candidate 1 is $9p_{13} - p_{12}$, and of candidate 2 is $p_{12} + 10p_{23}$. For every p proportionally near q , the first of these quantities is larger than the second.)

There is a second voting equilibrium under plurality rule, in which the leftist voters instead coordinate behind candidate 2 to beat candidate 3. This equilibrium is

$$\mu((0,1,0), u^A) = 0.3, \quad \mu((0,1,0), u^B) = 0.3, \quad \mu((0,0,1), u^C) = 0.4.$$

The predicted scores and likely winners are then

$$S_1(\mu) = 0, S_2(\mu) = 0.6, S_3(\mu) = 0.4, \text{ and } W(\mu) = \{2\}.$$

This equilibrium is supported by the pivot-probability vector $(q_{12}, q_{13}, q_{23}) = (0, 0, 1)$.

There is a third plurality-rule equilibrium in which the leftists split their votes between candidates 1 and 2, and candidate 3 is the sole likely winner. This equilibrium is

$$\mu((1,0,0), u^A) = 0.3, \mu((0,1,0), u^B) = 0.3, \mu((0,0,1), u^C) = 0.4,$$

with

$$S_1(\mu) = 0.3, S_2(\mu) = 0.3, S_3(\mu) = 0.4, \text{ and } W(\mu) = \{3\}.$$

This equilibrium is supported by any pivot-probability vector q such that

$$q_{12} = 0, 9/19 \leq q_{13} \leq 10/19, q_{23} = 1 - q_{13}.$$

Here, since candidates 1 and 2 are perceived to have comparable likelihoods of being in contention for victory with candidate 3, the leftist voters fail to coordinate and thus candidate 3 is the likely winner.

This third equilibrium violates Duverger's Law, which asserts that only two candidates should be expected to get positive scores under plurality rule (see Riker 1982). The existence of such a non-Duverger equilibrium is robust to small changes in the parameters. For example, if the proportion of A-type voters were slightly larger than the proportion of B-type voters, say, $f(u^A) = 0.31$ and $f(u^B) = 0.29$, then there would still exist a non-Duverger equilibrium in which candidate 1's chances of seriously challenging candidate 3 were perceived to be somewhat worse than candidate 2's chances, leaving the A-type voters indifferent between voting for 1 and for 2. This equilibrium is

$$\begin{aligned} \mu((1,0,0), u^A) &= 0.30, \mu((0,1,0), u^A) = 0.01, \\ \mu((0,1,0), u^B) &= 0.29, \mu((0,0,1), u^C) = 0.4, \end{aligned}$$

with

$$S_1(\mu) = 0.3, S_2(\mu) = 0.3, S_3(\mu) = 0.4, \text{ and } W(\mu) = \{3\},$$

and is supported by

$$q_{12} = 0, q_{13} = 9/19, q_{23} = 10/19.$$

Thus, Duverger's Law cannot be derived exclusively from analyses of voting equilibria. (This result suggests that some of the interpretive discussion in Palfrey 1989 should be reconsidered.) Any derivation of Duverger's Law would seem to require some additional assumption of dynamic stability or persistence (see, for example, Kalai and Samet 1984), to eliminate equilibria of the type seen above.

Under approval voting, there are three voting equilibria, in none of which is candidate 3 the only likely winner. In one equilibrium, candidate 1 is the only likely winner:

$$\mu((1,0,0), u^A) = 0.3, \quad \mu((1,1,0), u^B) = 0.3, \quad \mu((0,0,1), u^C) = 0.4,$$

with

$$S_1(\mu) = 0.6, \quad S_2(\mu) = 0.3, \quad S_3(\mu) = 0.4, \quad \text{and} \quad W(\mu) = \{1\}.$$

This equilibrium is supported by $(q_{12}, q_{13}, q_{23}) = (0,1,0)$. The voters expect that, if any two candidates are involved in a close race for first place, then they will almost surely be candidates 1 and 3; hence all B-type voters double-vote for candidates 1 and 2. However, voters also consider a close race for first place between candidates 1 and 2 (with predicted scores of 0.6 and 0.3), while much less likely than a close race between candidates 1 and 3, to be much more likely than a close race between candidates 2 and 3 (with predicted scores of 0.3 and 0.4), and therefore A-type voters only cast single votes for candidate 1.

Other election results are "supported" by this q (in the sense that the conditions of Theorem 2 are satisfied), but fail to be equilibria because perceptions satisfying the ordering condition for sufficiently small ϵ do not exist. Consider, for example, the election result

$$\mu((1,0,0), u^A) = 0.25, \quad \mu((1,1,0), u^A) = 0.05, \quad \mu((1,1,0), u^B) = 0.3, \quad \mu((0,0,1), u^C) = 0.4,$$

with

$$S_1(\mu) = 0.6, \quad S_2(\mu) = 0.35, \quad S_3(\mu) = 0.4, \quad \text{and} \quad W(\mu) = \{1\},$$

in which some A-type voters double-vote. This result can be justified only by positive pivot probabilities (p_{12}, p_{13}, p_{23}) satisfying $p_{12}/p_{13} \leq 1/9$ and $p_{23}/p_{12} = 1/9$. (The inequality ensures that B-type voters double-vote, and the equation leaves A-type voters indifferent between casting single or double votes by making their prospective rating of candidate 2 equal to 0.). However, the ordering condition for ϵ requires that we be able to satisfy both $p_{12}/p_{13} \leq \epsilon$ and $p_{23}/p_{12} \leq \epsilon$, and the latter inequality cannot be satisfied (by perceptions justifying μ) for values of ϵ smaller than $1/9$.

There is another equilibrium, in which candidate 2 is the only likely winner:

$$\mu((1,1,0), u^A) = 0.3, \quad \mu((1,0,0), u^B) = 0.3, \quad \mu((0,0,1), u^C) = 0.4,$$

with

$$S_1(\mu) = 0.3, \quad S_2(\mu) = 0.6, \quad S_3(\mu) = 0.4, \quad \text{and} \quad W(\mu) = \{2\},$$

supported by $(q_{12}, q_{13}, q_{23}) = (0,0,1)$.

In order to find the third equilibrium under approval voting, let us seek an equilibrium in which (as in the non-Duverger equilibrium in plurality rule) candidates 1 and 2 are considered to have equal chances of being in a close race with candidate 3. That is, let us suppose that the pivot-probability vector supporting the equilibrium is

$$q_{13} = q_{23} = \beta, \quad q_{12} = 1 - 2\beta,$$

with $0 \leq \beta \leq 1/2$.

If β were relatively large, say $\beta = 1/3$ (so that all close two-way races for victory were perceived to be equally likely), then

$$\begin{aligned} G(q, v, u^A) &= (1/3)(v_1 - v_2) \cdot (10 - 9) + (1/3)(v_1 - v_3) \cdot (10 - 0) + (1/3)(v_2 - v_3) \cdot (9 - 0) \\ &= (11/3)v_1 + (8/3)v_2 - (19/3)v_3, \end{aligned}$$

and the A-type voters would vote for both candidates 1 and 2. Similarly, the B-type voters would also vote for both leftist candidates, and the predicted scores would be 0.6 votes per capita for candidates 1 and 2, but only 0.4 votes per capita for candidate 3. Thus the set of likely winners would be $\{1, 2\}$, and the conditions of Theorem 2 would require that $q_{12} = 1$, i.e., $\beta = 0$.

If β were quite small, say $\beta = 0$ (so that the only serious race was perceived to be between candidates 1 and 2), then $G(q, v, u^A) = v_1 - v_2$. The A-type voters would only vote for candidate 1, the B-type voters would similarly vote only for candidate 2, and the predicted scores would be 0.3 for each of the first two candidates and 0.4 for the rightist candidate 3; the conditions of Theorem 2 would require that $q_{12} = 0$, i.e., $\beta = 1/2$, because neither candidate 1 nor 2 would be a likely winner.

To construct an approval-voting equilibrium in which candidates 1 and 2 are treated symmetrically, we cannot have the leftist voters all single-voting for their most-preferred candidate, nor can they all be double-voting. Generally, if $(q_{12}, q_{13}, q_{23}) = (1 - \beta, \beta, \beta)$, we have

$$\begin{aligned} G(q, v, u^A) &= (1 - 2\beta)(v_1 - v_2)1 + \beta(v_1 - v_3)10 + \beta(v_2 - v_3)9 \\ &= (1 + 8\beta)v_1 + (11\beta - 1)v_2 - (19\beta)v_3. \end{aligned}$$

When $\beta = 1/11$, the leftist voters (both A-types and B-types) are indifferent between voting only for their most-preferred candidate and double-voting for both leftist candidates. Then

$$q_{12} = 9/11 \quad \text{and} \quad q_{13} = q_{23} = 1/11,$$

and we obtain our third equilibrium, in which

$$\begin{aligned}\mu((1,0,0), u^A) &= 0.2, & \mu((1,1,0), u^A) &= 0.1, \\ \mu((0,1,0), u^B) &= 0.2, & \mu((1,1,0), u^B) &= 0.1, & \mu((0,0,1), u^C) &= 0.4,\end{aligned}$$

with

$$S_1(\mu) = S_2(\mu) = S_3(\mu) = 0.4, \text{ and } W(\mu) = \{1,2,3\}.$$

Here, a third of the leftist voters double-vote for both leftist candidates, while two-thirds of the leftist voters single-vote only for their most preferred candidate, and so the expected per-capita vote totals of all three candidates are equal to 0.4. In this equilibrium, the race between each pair of candidates is taken seriously, but a close race between candidates 1 and 2 is perceived to be nine times more likely than a close race between either 2 and 3 or 1 and 3. In a later section, we will return to this example, and illustrate by a limit analysis how equal predicted scores for the three candidates can be consistent with unequal pivot probabilities for the three candidate pairs. (Short of the limit, the predicted scores of candidates 1 and 2 will be slightly higher than the predicted score of candidate 3.)

Under Borda rule, there is a set of equilibria, all with the same predicted outcome in which all three candidates are likely winners. The equilibria are of the form

$$\begin{aligned}\mu((2,1,0), u^A) &= \alpha, & \mu((2,0,1), u^A) &= 0.3 - \alpha, \\ \mu((1,2,0), u^B) &= 0.4 - \alpha, & \mu((0,2,1), u^B) &= \alpha - 0.1, \\ \mu((0,1,2), u^C) &= 0.4 - \alpha, & \mu((1,0,2), u^C) &= \alpha,\end{aligned}$$

where $0.1 \leq \alpha \leq 0.3$. For every Borda-rule equilibrium μ , we have

$$S_1(\mu) = S_2(\mu) = S_3(\mu) = 1, \text{ and } W(\mu) = \{1,2,3\}.$$

These equilibria are supported by

$$q_{12} = 28/30, \quad q_{13} = q_{23} = 1/30.$$

All races are taken seriously, but candidate 3 is perceived to be only a slight threat to the other two candidates. (If he were perceived to be a very likely threat, the A-type and B-type voters would all cast 0 votes for him, and the threat would disappear. But if he were perceived to be no threat at all, every leftist would "dump" his 1-vote on candidate 3, in order to maximally differentiate between the more- and less-preferred of candidates 1 and 2, and consequently candidate 3 would win.)

In summary, for the electoral situation studied here we find that under plurality rule, there is an equilibrium in which the minority (rightist) candidate is the only likely winner. Under Borda rule, the

minority candidate is always a likely winner at equilibrium. Only under approval voting do we see simultaneously the existence of equilibria in which the minority candidate is not a likely winner, and the nonexistence of equilibria in which the minority candidate is the only likely winner.

Example 2: The Effect of a Small Minority

We now consider a second example, which differs from the first only in that the number of supporters of candidate 3 is reduced to 2% of the population, while the remaining 98% of the voters are evenly divided between A-type voters who prefer candidate 1 and B-type voters who prefer candidate 2:

voter type	utility vector	proportion of electorate
A	$u^A = (10, 9, 0)$	$f(u^A) = 0.49$
B	$u^B = (9, 10, 0)$	$f(u^B) = 0.49$
C	$u^C = (0, 0, 10)$	$f(u^C) = 0.02$

Under Borda rule, there are still no equilibria in which candidate 3 is not a likely winner. All equilibria are of the form

$$\begin{aligned} \mu((2,1,0), u^A) &= \alpha, & \mu((2,0,1), u^A) &= 0.49 - \alpha, \\ \mu((1,2,0), u^B) &= 0.02 - \alpha, & \mu((0,2,1), u^B) &= 0.47 + \alpha, \\ \mu((0,1,2), u^C) &= 0.02 - \alpha, & \mu((1,0,2), u^C) &= \alpha, \end{aligned}$$

where $0 \leq \alpha \leq 0.02$. As in Example 1, each equilibrium μ has

$$S_1(\mu) = S_2(\mu) = S_3(\mu) = 1 \text{ and } W(\mu) = \{1,2,3\},$$

and is supported by

$$q_{12} = 28/30, \quad q_{13} = q_{23} = 1/30.$$

Even though there are relatively few voters who prefer candidate 3, there are no Borda-rule equilibria in which he is not a likely winner. If q_{12} were greater than $28/30$, while $q_{13} = q_{23}$, then the A-type and B-type voters would want to maximally separate candidates 1 and 2 (voting $(2,0,1)$ and $(0,2,1)$, respectively), and candidate 3 would win with a per-capita vote total of $0.98 \cdot 1 + 0.02 \cdot 2 = 1.02$.

Under approval voting, there is a unique voting equilibrium μ in which everyone votes only for his most-preferred candidate:

$$\mu((1,0,0), u^A) = 0.49, \quad \mu((0,1,0), u^B) = 0.49, \quad \mu((0,0,1), u^C) = 0.02,$$

and so $S_1(\mu) = S_2(\mu) = 0.49$, $S_3(\mu) = 0.02$, and $W(\mu) = \{1,2\}$. This equilibrium is supported by $q_{12} = 1$, and $q_{13} = q_{23} = 0$.

Even though everyone single-votes in the unique approval-voting equilibrium for this example, the ability of the voters to cast votes for more than one candidate does have an impact: The set of equilibria under plurality rule is larger than the set of equilibria under approval voting. For example, there is also a plurality-rule equilibrium μ in which

$$\mu(((1,0,0), u^A) = 0.49, \quad \mu((1,0,0), u^B) = 0.49, \quad \mu((0,0,1), u^C) = 0.02,$$

with

$$S_1(\mu) = 0.98, \quad S_2(\mu) = 0, \quad S_3(\mu) = 0.02, \quad \text{and} \quad W(\mu) = \{1\} .$$

This equilibrium is supported by $(q_{12}, q_{13}, q_{23}) = (0, 1, 0)$. (This election result would not be an equilibrium under approval voting, because the B-type voters would choose to give second approval votes to candidate 2; candidate 2's predicted score would then be higher than candidate 3's, invalidating the perception that a race between candidates 1 and 2 for first place is much less likely than a race between candidates 1 and 3.) Similarly, there is a plurality-rule equilibrium in which all A-type and B-type voters vote for candidate 2, who is then the only likely winner. However, candidate 3 is not a likely winner in any equilibrium under plurality rule.

Limits of Nash Equilibria

We return to Example 1, in which 60% of the voters are A-type or B-type "leftists." Recall that, under approval voting, there exists an equilibrium μ in which 2/3 of the leftists (of each type) vote only for their most-preferred candidate, and 1/3 of the leftists double-vote for candidates 1 and 2. That is,

$$\begin{aligned} \mu((1,0,0), u^A) &= 0.2, & \mu((1,1,0), u^A) &= 0.1, \\ \mu((0,1,0), u^B) &= 0.2, & \mu((1,1,0), u^B) &= 0.1, \\ \mu((0,0,1), u^C) &= 0.4. \end{aligned}$$

This equilibrium is supported by the pivot probabilities

$$q_{12} = 9/11 \text{ and } q_{13} = 1/11 = q_{23} ,$$

and $S_1(\mu) = S_2(\mu) = S_3(\mu) = 0.4$, so all three candidates have the same per-capita scores. The conditions of Theorem 2 are satisfied by q and μ , because all three candidates are likely winners.

If all three candidates have the same predicted scores, how is it that the conditional tie probabilities q_{ij} are not the same for all pairs of candidates? A naive interpretation of our model might suggest that all pivot probabilities should be perceived to be equal, if all candidates have the same predicted score. However, this interpretation overlooks the fact that our model is intended to be a limiting approximation to elections involving large finite electorates.

To see how such an equilibrium can arise as the limit of Nash equilibria for large finite electorates, consider a voting "game" in which there is a large number M of voters, of whom $0.3M$ are type u^A , $0.3M$ are type u^B , and $0.4M$ are type u^C ; these voters will select a candidate by approval voting. This voting game has a randomized Nash equilibrium in which each leftist (A-type or B-type) voter independently randomizes between voting only for his most-preferred candidate, with probability $1 - \rho$, and double-voting for both candidates 1 and 2, with probability ρ , while each C-type voter votes only for candidate 3. The parameter ρ depends on M according to the asymptotic formula

$$\rho \approx 1/3 + 0.9476/\sqrt{M}.$$

For example, suppose that $M = 1,000,000$. Then $\rho = 0.33428$, and the vote totals for candidates 1 and 2 are independent random variables, approximately normally distributed with mean 400,284 and standard deviation 258. The vote total for candidate 3 is precisely 400,000. At the Nash equilibrium, the probability that candidates 1 and 2 are tied ahead of candidate 3 is $p_{12} = 0.001027$, whereas the probability that candidates 1 and 3 are tied ahead of candidate 2 is $p_{13} = 0.000114$. Hence, $p_{12}/p_{13} = 9$. Similarly, $p_{12}/p_{23} = 9$. These pivot probabilities correspond to a normalized probability distribution q such that $q_{13} = q_{23} = 1/11$ and $q_{12} = 9/11$. That is, conditional on some pair of candidates being tied for first place, the probability that the tie is between candidates 1 and 2 is $9/11$. The candidates' predicted scores are all very close to 0.4 (actually, 0.400284, 0.400284, and 0.4), and the standard deviations of the scores are 0.000258, 0.000258, and 0.

Notice that uncertainty about a candidate's vote total may be large relative to the size of an individual's vote (the standard deviation of 258 is much bigger than 1), even though the uncertainty about the per-capita vote totals is very small. Thus, when the electorate is large, near-equality of the

predicted scores for different candidates does not necessarily mean that there is a large probability of a tie between them.

We do not offer here an equivalence theorem between the set of limits of Nash equilibria of large voting games, as the number of voters goes to infinity, and the set of voting equilibria, as defined here. Such a result would depend on assumptions about how the large finite voting games are specified. For example, the number of voters of each type may be either deterministic or random, and we may or may not admit the possibility of a vanishingly small fraction of the electorate that has preferences outside of the given type set T . We suspect that, for many specifications, the set of limits of Nash equilibria may turn out to be somewhat larger than the set of ordered voting equilibria. It seems reasonable to expect that a limit of Nash equilibria should be in $R(q)$ for some pivot-probability vector q , but it may be that this vector q is not the limit of perceptions satisfying the ordering condition. Future research should consider other (weaker) restrictions on perceptions. However, we believe that the ordering condition has some intuitive appeal, and that voters would be much less likely to act in accordance with Nash equilibria which violated that condition, should such Nash equilibria exist.⁷

A Candidate-Positioning Game

In the preceding analysis, we assumed that the voters' preferences over the candidates were exogenously given. However, real voters' preferences over candidates typically depend on the candidates' positions on the policy choices that confront the government, and candidates may choose their positions (at least, in part) with the objective of becoming likely winners. Changes in the electoral system may change the way that candidates position themselves on the issues. To investigate this question, we consider a candidate positioning game in the tradition of Hotelling 1929 and Downs 1957.

Suppose that the policy space is $[0,100]$, the set of all real numbers between 0 and 100 (representing, say, the percentage tariff to be imposed on imported goods), and suppose that there are three candidates, numbered 1, 2, and 3. Each candidate must publicly choose a point in policy space that represents the policy he promises to implement if elected (say, the tariff that he will impose). Each voter has a utility function that is defined on the policy space, and his utilities for the candidates are determined by the positions that the candidates take. To be specific, we suppose that each voter has a most-preferred point θ between 0 and 100, and his utility for candidate i , if the candidate takes position x , is $u_i(x) = -(x - \theta)^2$. The voters' ideal points are uniformly distributed over the set $\{0,1,2,\dots,100\}$, with $1/101$

of the voters having each ideal point in this set. If x_i denotes the position chosen by candidate i , then the set of utility types will be

$$T = \{(-(x_1-\theta)^2, -(x_2-\theta)^2, -(x_3-\theta)^2) \mid \theta \in \{0,1,2,\dots,100\}\},$$

with $f(u) = 1/101$ for each vector u in T .

Under any given electoral system V , for any vector $x = (x_1, x_2, x_3)$, we let $E(x)$ denote the set of voting equilibria for the electoral situation that would exist after the candidates chose the positions listed in x . A pair (x, μ) is a *positional equilibrium* under V if and only if it satisfies the following three conditions:

- (a) μ is in $E(x)$;
- (b) for each candidate i in $W(\mu)$, for every position y_i there exists a voting equilibrium η in $E(x_{-i}, y_i)$ such that either $W(\mu) \subseteq W(\eta)$ or $i \notin W(\eta)$;
- (c) for each candidate j who is not in $W(\mu)$, for every position y_j there exists a voting equilibrium η in $E(x_{-j}, y_j)$ such that $j \notin W(\eta)$.

(Here, (x_{-j}, y_j) denotes the vector that arises from x when the j -th component is changed from x_j to y_j .) Condition (a) asserts that μ is a voting equilibrium in the election which results when the candidates choose the positions indicated by x . Condition (b) asserts that a change of position by a likely winner can lead to either a new equilibrium in which no other likely winners are eliminated, or a new equilibrium in which he himself is eliminated. Condition (c) asserts that a change in position by a candidate who is not a likely winner can lead to a new equilibrium in which he is still not a likely winner.⁸

In the context of this definition, we can examine how different electoral systems differ in their implications for candidate positioning. The following two theorems assert that plurality rule and approval voting have very different implications: A candidate at almost any position can win under plurality rule, while only candidates at the median position can win under approval voting.

Theorem 3. Under plurality rule, for any number z strictly between 0 and 100, there exists a positional equilibrium such that one candidate takes the position z and is the only likely winner.

Theorem 4. Under approval voting, in any positional equilibrium all likely winners take the position 50.

Proof of Theorem 3. The definition of a positional equilibrium allows us to deter candidates from changing their positions by selecting among the resulting voting equilibria. The key to the proof is the existence of many voting equilibria under plurality rule.

To simplify the proof, we consider here only the case where $0 < z \leq 50$. (An analogous argument can be used to cover the case where z is between 50 and 100.) Let $x_1 = z$, $x_2 = 100 - z/2$, and $x_3 = 50$. Then there exists a voting equilibrium μ in which all voters with ideal points between 0 and $50 + z/4$ vote for candidate 1, all voters with ideal points between $50 + z/4$ and 100 vote for candidate 2, and no voters vote for candidate 3. This voting equilibrium is supported by letting $q_{12} = 1$, and $q_{13} = q_{23} = 0$. Perceiving the only serious race to be between candidates 1 and 2, each voter votes for his more-preferred of these two candidates. Because candidate 1 is closer to the median than candidate 2, candidate 1 will have the highest predicted score and thus will be the only likely winner. The perception that candidate 3 is not a serious contender is justified at equilibrium because candidate 3 gets no votes.

To verify that (x, μ) is a positional equilibrium, we must show that candidate 2 cannot gain by choosing a position closer to the median. If candidate 2 "relocates" between z and $100 - z$, then a majority of the voters will prefer candidate 2 to candidate 1. But this does not mean that candidate 2 will necessarily win the election, because there will be other voting equilibria supported by $q_{13} = 1$ in the new situation. We can deter candidate 2 from shifting his position by specifying that, if candidate 2 moves to any position outside of the interval $[100 - z/2, 100]$, all voters will behave according to the voting equilibrium in which only the race between candidates 1 and 3 is taken seriously. (In the real political arena, this switch in public perceptions could result from some political pundit calling candidate 2 an "opportunist" if he chooses a position short of $100 - z/2$.)

Given our specifications, candidate 1 is the unique likely winner (i.e., condition (b) is satisfied). Candidate 2 cannot become a likely winner by changing his position. And candidate 3 cannot make himself a likely winner by moving away from 50, if the voters continue to focus on the equilibrium of the resulting situation in which only the race between candidates 1 and 2 is taken seriously (i.e., condition (c) is satisfied).⁹ Q.E.D.

In the proof of Theorem 3, we found a voting equilibrium in which candidate 3 chose a position at the median point 50 and candidates 1 and 2 chose other positions away from 50, but the only serious race was perceived to be between candidates 1 and 2. Under plurality rule, a voter who voted for candidate 3 would have to forfeit any chance of influencing the serious race between candidates 1 and 2, and so the perception that candidate 3 was not a serious contender destroyed his ability to get votes.

Under approval voting, however, the median candidate cannot be so easily defeated by such a perception, because a voter can vote for candidate 3 without forfeiting his ability to also vote for the candidate that he prefers among candidates 1 and 2. In fact, as we show in the proof of the Lemma below, a candidate at 50 will always get approval votes from more than half the voters under any election result that can be justified by positive pivot probabilities.

Theorem 4 is directly implied by the Lemma, which asserts that, under approval voting, if any candidate's takes the position 50 then, in any voting equilibrium, all likely winners must be positioned at 50. Consequently, if no candidate were positioned at 50, then any candidate could make himself the unique likely winner by moving to 50. Therefore a positional equilibrium must have at least one candidate at 50, and it then follows from the Lemma that all likely winners will be at 50, as Theorem 4 asserts.

Lemma. Under approval voting, for any position vector x with at least one candidate positioned at 50, if $\mu \in E(x)$, then only candidates positioned at 50 can be likely winners in μ .

Proof of Lemma. If all candidates are positioned at 50, the lemma is trivially true. Therefore, in the following argument we assume that there is at least one candidate located at 50, and another who is located elsewhere.

Note that

$$\begin{aligned} G(p,v,u(\theta)) &= \sum_i v_i \sum_{j \neq i} p_{ij} (u_i(\theta) - u_j(\theta)) \\ &= \sum_i v_i \sum_{j \neq i} p_{ij} ((x_j)^2 - (x_i)^2 + 2(x_i - x_j)\theta) . \end{aligned}$$

A voter with ideal point θ will (at equilibrium) give an approval vote to any candidate i for whom the prospective rating

$$\sum_{j \neq i} p_{ij} (x_j^2 - x_i^2 + 2(x_i - x_j)\theta)$$

is positive, and will not give an approval vote to any candidate for whom the prospective rating is negative. Assume that all of the perceived pivot probabilities are positive. Then the prospective rating of candidate i will be positive if $x_i = \theta$. Since the prospective rating is linear in θ , the set of all voters who cast an approval vote for candidate i must therefore contain either all voters with ideal points less than or equal to x_i , or all voters with ideal points greater than or equal to x_i (depending on the sign of $\sum_{i \neq j} p_{ij} (x_i - x_j)$). In either case, a candidate located at 50 must have a predicted score (i.e., a per-capita vote total) of at least 51/101.

Now, take $\epsilon > 0$, and let p be a vector of positive pivot probabilities satisfying the ordering condition for ϵ . Let h be a candidate located at 50, and, of all candidates not positioned at 50, let i be the candidate positioned furthest from 50 with $S_i(\mu) \geq S_h(\mu)$. (If no such candidate exists, then we are done.) Decompose the prospective rating of candidate i for a voter with ideal point 50 into three terms:

$$\sum_{j \neq i} p_{ij}(u_i(50) - u_j(50)) =$$

$$\sum_{\substack{S_j(\mu) \geq S_h(\mu) \\ j \neq h}} p_{ij}(u_i(50) - u_j(50)) + p_{ih}(u_i(50) - u_h(50)) + \sum_{S_j(\mu) < S_h(\mu)} p_{ij}(u_i(50) - u_j(50)) .$$

The first term is non-positive, because of the extreme position of candidate i among all candidates with scores as high as that of candidate h . For any

$$\epsilon < (u_h(50) - u_i(50)) / (k \max_{j \neq i} |u_i(50) - u_j(50)|) ,$$

the sum of the latter two terms is negative, because $p_{ij}/p_{ih} < \epsilon$ for all candidates j for whom $S_j(\mu) < S_h(\mu)$. Thus, for all sufficiently small ϵ , voters with ideal point 50 will not vote for candidate i ; it follows that the set of ideal points of voters who vote for candidate i must be a subset of either $\{0, \dots, 49\}$ or $\{51, \dots, 100\}$. In either case, candidate i 's score cannot be more than 50/101. Hence, there can be no candidate positioned away from 50 with a score as high as that of candidate h , i.e., all likely winners must be positioned at 50.¹⁰ Q.E.D.

There is a large literature on candidate-positioning games similar to the game considered in this section (see Shepsle 1991). Most of this literature considers only plurality rule and assumes that voters vote "sincerely" (or "honestly" or "non-strategically") for their most-preferred candidate.¹¹ Feddersen, Sened, and Wright 1990 study plurality rule with strategic voters (and with entry costs for candidates), but they only consider pure-strategy equilibria of games with a fixed finite set of candidates and voters. Their result (that, at equilibrium, all candidates who enter the election take the median position and receive equal vote totals) would not hold if they expanded the set of equilibria to include randomized equilibria, in which candidates have some uncertainty about the voters' behavior. Palfrey 1989 also considers plurality rule with strategic voters, and remarks that the multiplicity of voting equilibria makes the candidate-positioning game indeterminate, as our Theorem 3 asserts. However, he does not attempt to analyze the different implications of strategic voting under approval voting. In Cox 1987; Cox 1990,

general comparisons of candidate positioning under different electoral systems are based on the assumption that voters vote sincerely. Cox 1985 obtains results for approval voting similar to our Theorem 4, but under somewhat different assumptions.

Advocates of a two-party system sometimes argue in favor of plurality rule over alternative voting systems, claiming that plurality rule discourages the entry of third parties. The results above, restated in a two-candidates-plus-potential-entrant setting, support this claim, but cast it in a rather unfavorable light. Under plurality rule, an election between two extremist candidates (emerging as the nominees of major parties after a season of primaries and conventions) can lead to the election of one of them, even if a more moderate candidate (who would defeat either of the others in a head-to-head race) enters the fray. Under approval voting, unless one of the major-party representatives takes a centrist position, an independent entrant can take that position and expect to win for certain. This does not mean that the use of approval voting would necessarily lead to disintegration of the two-party system. Rather, its use might discourage the major parties from nominating extremists in the first place.

Multiple Equilibria and the Focal-Point Effect

The candidate-positioning game provides a striking contrast between plurality rule and approval voting. Under plurality rule, we find that a multitude of voting equilibria supports a plethora of positional equilibria, and almost any policy outcome can occur in equilibrium. Under approval voting, the unique equilibrium policy outcome is at the median voter's ideal point. In our example, voters' preferences impose no constraint on the equilibrium policy outcome under plurality rule, whereas voters' preferences fully determine the equilibrium policy outcome under approval voting.

The multiplicity of voting equilibria for multi-candidate elections under plurality rule is quite general. Let i and j be any two candidates such that a majority of voters prefers candidate i to candidate j , and no voters are indifferent between i and j . Under plurality rule, there will always be a voting equilibrium in which everyone votes either for i or for j , and candidate i is the unique likely winner. Such an equilibrium is supported by the perception that the race between candidates i and j is the only serious race (that is, $q_{ij} = 1$), and therefore a vote for any other candidate would be wasted. Thus, any candidate i who is not a Condorcet loser can be the unique likely winner in a voting equilibrium under plurality rule. (A candidate is a *Condorcet loser* if a majority would vote against him in a two-candidate race against any one of the other candidates.) Furthermore, in Example 1 we found an equilibrium under plurality rule in which candidate 3 was the unique likely winner, even though he

was a Condorcet loser. Thus, in any electoral situation, the set of candidates who can be unique likely winners at equilibrium under plurality rule includes everyone who is not a Condorcet loser, and may include some Condorcet losers as well.

What is the political significance of the multiplicity of equilibria? The existence of a large set of equilibria in an electoral situation provides great political influence to political leaders and other individuals who have access to mass media. That is, the existence of multiple equilibria does not merely mean that outcomes are difficult to predict theoretically: The multiplicity of equilibria may also have substantive implications for the distribution of power in society.

Schelling 1960 argues that, in games with multiple Nash equilibria, anything that tends to focus the players' attention on any one equilibrium may lead each player to expect the others to act in accordance with that equilibrium. With such expectations, each player does best for himself by also acting in accordance with the equilibrium, and so the expectations are fulfilled. An equilibrium that realized because of a focusing of attention is said to be *focal*.

In an electoral setting, if the voters come to believe some given prediction about the aggregate distribution of votes in an election, then their individual voting optimization decisions can confirm this prediction (consistent with ordered perceptions of the relative likelihood of various close two-candidate races) if and only if the prediction corresponds to a voting equilibrium. If there are multiple voting equilibria, then there is more than one prediction about the outcome of the election which, if believed by the electorate, could be fulfilled by rational voter behavior. An individual who could focus the voters' expectations would then be in a position to pick any voting equilibrium and make a self-fulfilling prophecy.

Consider again our treatment of Example 1 under plurality rule. The majority block of A-type and B-type voters can avoid the worst possible outcome for themselves (the election of candidate 3) only by coordinating their focus on one of the two equilibria in which they all vote for the same candidate. Depending on the culture and traditions of the society in which the election takes place, almost any kind of public event could serve as a focal factor. For example, the equilibrium in which candidate 1 is the only likely winner might become focal if candidate 1 receives the endorsement of an important leader or organization, or if he wins a straw poll in a small (even unrepresentative) pre-election caucus that is widely reported in the news media, or if he bears a well-known family name.

A *focal arbiter* is an individual whose opinions and statements about the candidates command wide attention, giving him influence upon the outcome of the election by making focal an equilibrium in which a candidate he supports has a significant chance of winning (or one he condemns does not). Such

focal arbiters include party leaders, voters who participate in early pre-election polls (e.g., the voters in Iowa and New Hampshire in American presidential elections), and others with access to the mass media (including, of course, media personalities themselves, who select their own interpretations of which candidate is the "true" winner in early primary elections). To become a focal arbiter, an individual does not need to have particularly good judgement in evaluating candidates, or even to be perceived as having good judgement; it is only necessary that he be able to get his views prominently reported to the public. The power of focal arbiters is greatest in situations where there is a multitude of voting equilibria.

When analyzing the effects of various campaign events and activities on election outcomes, it is useful to distinguish between *focal manipulation* and *candidate presentation*. An event or activity serves the candidate-presentation function when it affects voters' preferences over candidates by conveying information to the voters about the candidates and their positions on the issues. Obviously, the outcome of an election can be affected substantially by publicity that changes the voters' understanding of the candidates' positions and qualifications. An endorsement from a widely respected public figure who knows a candidate may be considered as candidate presentation, if the endorsement helps to convince voters that this candidate has the ability to do a good job in office. McKelvey and Ordeshook 1985 have studied a model in which pre-election polls serve an implicit candidate-presentation function, by letting poorly-informed voters make inferences about candidates' positions from the preferences of well-informed voters. In contrast, pure focal manipulation includes any activity whose effect is to influence the voters' beliefs about who will win the election, and so direct voters' attention towards particular equilibria, without altering the voters' underlying utility vectors.

The existence of many voting equilibria in an electoral setting forces political campaigners not only to engage in effective candidate presentation, but also to obtain the support of focal arbiters and engage in various forms of direct focal manipulation. Such support and manipulation would be much less important if there were only one equilibrium that voters could rationally expect.

The great multiplicity of equilibria under plurality rule means that plurality rule is well designed to serve the interests of focal arbiters. The fact that anyone who is not a Condorcet loser can be elected with the help of the appropriate focal manipulations implies that the set of possible equilibrium outcomes under plurality rule may have very little dependence on the voters' preferences; consequently, focal manipulation may be much more important to a successful campaign than candidate presentation. The results in this paper suggest that the use of other electoral systems (such as approval voting or Borda-rule voting) may decrease the influence of focal arbiters as a class, by decreasing the range of voting equilibria.

Notes

1. For purposes of this paper, we restrict our attention to voting systems representable as "scoring rules." See Smith 1973 and Young 1975 for axiomatic perspectives on this restriction.
2. The vector $(0,0,0)$ represents an abstention, and is included for purposes of completeness. Since the analysis in this paper assumes costless voting, voters never prefer to abstain.
3. In assuming that a voter's payoff depends only on which candidate wins the election, we implicitly exclude the possibility that his utility payoff also depends on the vote totals of the various candidates (their "mandates") or on the ballot that he submits: Only the identity of the winner matters to the voters.
4. We assume that T is finite only for purposes of notational simplicity. Voters of the same type should *not* be viewed as forming a monolithic bloc: They may cast different ballots.
5. The voter population can be viewed either as fixed, with $f(u)$ representing the specific fraction of the population with utility type u , or as random, consisting of a large number of independent random draws of voters according to the probability distribution $f(\cdot)$. The results of the paper stand up to either interpretation.
6. To show that $Q^\epsilon(\mu)$ is nonempty for every election result μ , let $p_{ij} = \epsilon^n/k^2$, where n is the product of the number of candidates with predicted scores no less than candidate i 's, and the number of candidates with scores no less than j 's.
7. In Forsythe et al. 1990, the results of a series of three-candidate voting experiments based on Example 1 are presented. In elections preceded by polls or prior elections, outcomes accord well with our equilibrium analysis. In particular, the vote totals of the three candidates are typically much closer to one another under approval voting and Borda rule than under plurality rule, yet the "minority" candidate (candidate 3 in our example) wins far less frequently than either of the other two candidates.
8. The voters' actions depend on the perceived probabilities of ties between specific pairs of candidates. A candidate's choice of a position, however, should depend on his perception of the probability that he will win the election if located at that position. Our definition of a positional equilibrium is complicated by the fact that we cannot derive any simple relationship between the pivot probabilities, and the probability of any particular candidate winning. In an earlier version of this paper, we used the simple assumption that the pivot probability for any pair of candidates is perceived to be proportional to the product of the candidates' probabilities of winning the election (as estimated from pre-election polls), but we could not derive this relationship formally from any sensible stochastic model.

9. The placement of the two "serious" contenders at opposite ends of the political spectrum is not essential to the proof of Theorem 3. For any position $z \leq 50$, there is also (under plurality rule) a positional equilibrium (x, μ) in which $(x_1, x_2, x_3) = (z, z/2, 50)$. In the election result μ , only candidates 1 and 2 have positive predicted scores, and candidate 1 is the sole likely winner. At this positional equilibrium, any change in position by candidate 2 can lead to the election of candidate 3, and so candidate 2 not only might still lose, but also might see a position even further from his original one adopted.

10. Theorem 4 holds for any number of candidates $K \geq 2$, as the proof of the lemma demonstrates. In the three-candidate case, the result would still hold if we only required that equilibria be justifiable by perceptions which assign a much greater likelihood to the event of a tie involving the predicted top two vote-getters than to a tie between any other pair of candidates. With four (or more) candidates, the full force of the ordering condition is needed. Consider a four-candidate race involving candidates located at $(x_1, x_2, x_3, x_4) = (10, 40, 40, 50)$. This situation has an "unordered" voting equilibrium in which voters with ideal points between 0 and 25 vote only for candidate 1, voters between 25 and 30 cast approval votes only for candidates 2 and 3, and voters above 30 vote for candidates 2, 3, and 4. The two candidates located at 40 are the only likely winners, and candidate 1 has the lowest predicted score. This election result is justified by perceptions p for which p_{23} is much larger than $p_{12} = p_{13}$, with all other pivot probabilities much smaller. However, the ordering condition would require that p_{24} and p_{34} be much larger than p_{12} and p_{13} . (This example clarifies the concerns expressed in footnote 4 of Cox 1985, concerning his inability to rule out the existence of "non-centrist" equilibria using only arguments based on the elimination of dominated strategies. Cox's feeling that such equilibria are rare is confirmed, in our framework, by the impossibility of justifying such election results with perceptions that satisfy the ordering condition for all positive ϵ .)

11. One attempt to generalize the notion of "sincere voting" to general scoring rules has been to require that a voter's ballot assign higher scores to more-preferred candidates (i.e., if $u_i > u_j$, then $v_i \geq v_j$). This generalization is not fully determinate for a system such as approval voting. We offer here a stronger generalization: A *sincere ballot* is one which maximizes a voter's expected gain, given the perception that all ij -pivot probabilities are equal. With equal pivot probabilities, the expected gain function $G(v, p, u)$ is proportional to $\sum_{i \in K} v_i (u_i - \bar{u})$, where \bar{u} represents the mean of all candidate utilities (i.e., $\bar{u} = \sum_{i \in K} u_i / k$). A model based on sincere voting would replace our ordering condition with the simpler assumption of equal perceived pivot probabilities. (Weber 1977 presents a comparison of approval voting with other voting systems, based on this sincerity assumption.)

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