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# Similarity of Correlated Equilibria*

by

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## Abstract

A definition of correlated equilibrium for normal form games with general uncertainty is provided and its technical properties are studied.

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## 1. Introduction

A key part of a game equilibrium is the set of beliefs each player has about the actions of other players. This normally consists of a commonly held probability distribution over the possible actions of all players with respect to which each players' actions are optimal and consistent. A Nash equilibrium imposes the further restriction that the strategies of different players are independent. Aumann (1974, 1987) argued that players may rationally choose to correlate strategies through commonly observed signals which are not defined *a priori*. This possibility motivates the concept of a correlated equilibrium, in which the independence requirement of Nash equilibrium is dropped. No loss of rationality is involved in using correlated equilibrium; Aumann (1987) showed when the structure of the game is common knowledge, correlated equilibrium is equivalent to Bayesian rationality of players.

The purpose of this paper is to extend the correlated equilibrium concept to normal form games with uncertainty and to examine how equilibrium depends on the features of the game. This extension is warranted by the presence of two problems with the Nash equilibrium concept The first problem arises in the presence of uncertainty. In the standard model of normal form games with uncertainty, each player's information is revealed through the observation of her type, which is some random variable. Since a player's strategy depends on her observation and the types of different players may be correlated, one cannot expect strategies to be unconditionally independent. Harsanyi (1967) solved this problem when there are finitely many possible types by defining each player type to be a distinct player. A Nash equilibrium with respect to the redefined game is known as a Bayesian-Nash equilibrium, which requires conditional independence given the profile of player types. This requires defining all observable uncertainty in the game *a priori*. Some uncertainty affects the payoffs of players and should be included in the game. However, there is a great deal of other uncertainty in the environment, and additional uncertainty can always be introduced to a game. One solution is to include only those states of nature which are payoff relevant, but this draws a distinction between states on which payoffs depend only slightly and states which do not affect payoffs at all. For example, consider two games, one for which all payoffs depend slightly on the same commonly observed signal, and a second game for which the payoff functions are nonstochastic. If uncertainty is limited to payoff relevant states, then the resulting Bayesian-Nash equilibria in the two games are very different. In the former, equilibria for which the strategies of different players are perfectly correlated are admitted, while in the latter, players' strategies must be independent. Since these two games are empirically indistinguishable, one should not draw a significant distinction between them. Another problem with defining uncertainty *a priori* is that some other event (a "sunspot") can become relevant to the game merely because players believe it matters. Such beliefs cannot be excluded on grounds of Bayesian rationality. Since beliefs are endogenous, one cannot meaningfully discuss "exogenous information" without placing arbitrary limits on players' rationality. Extending the correlated equilibrium solution concept to games with asymmetric information via Harsanyi's construction solves these problems, since adding correlation opportunities to a game does not expand the set of correlated equilibria.

A second problem with the Bayesian-Nash equilibrium concept arises whenever the set of player types is uncountable. Because of measurability problems, Harsanyi's construction can no longer be used, so strategies must be defined as type dependent. With this formulation, however, the payoffs of players are not generally continuous with respect to strategies. As shown by Milgrom and Weber (1985), the problem is that correlation of strategies permitted in a Bayesian-Nash equilibrium is very fragile. One interpretation of the result is that spurious coordination of strategies creates discontinuities in the players' payoff functions with respect to strategies [Milgrom and Weber (1985), Cotter (1988)]. An alternative view, pursued in Sections 2 and 4 of this paper, is that is that the set of Bayesian-Nash strategies is not closed in the set of all strategies. Using this approach, the set of correlated strategies is the closed convex hull of the set of Bayesian-Nash strategies.

Though the bulk of game theory has focused on games with finite types, infinite games should not be ignored or viewed as pathological by the theory. There are many games for which a finite representation is unnatural or inconvenient, such as when differentiability or normality assumptions are useful in calculating an equilibrium. In addition, the inclusion of all possible hierarchies of beliefs about other players' beliefs [Mertens and Zamir (1985)] leads to a very large player type space. Restriction to finite type spaces also does not permit explicit comparisons of players' information structures unless those types are generated by a much larger underlying state space. When the state space is finite, the set of possible information structures about that state is not very interesting. A more interesting set of information structures could be obtained, for example, by studying Blackwell information matrices over a finite set of decision-relevant player types. Including all possible information matrices, even of fixed matrix dimensions, requires a considerably larger state space. If the underlying state space is taken to be uncountably infinite, then the extent to which the assumption of finite player types is restrictive becomes relevant. A common rationale for the study of games with finite types is the belief that any game can be approximated by a finite one. It would therefore be useful to know the extent to which the set of finite games is dense. Since this is a topological statement, a meaningful topology on the set of games is needed for which closeness of a pair of games implies closeness of their equilibria. The existence of such a topology for which the set of finite games is dense is an open question. Recent results of Cotter (1988, Section 3) cast doubt for the case of Bayesian-Nash equilibrium. However, in Section 5 a topology is constructed for which the correlated equilibrium correspondence is upperhemicontinuous and the set of finite games is dense.

These points are best explained in terms of a simplified model with finite action spaces and independently defined player types spaces, which is presented in Section 2. The general model is covered in Sections 3 and 4, while the continuity properties of correlated equilibria are discussed in Section 5.

## 2. An example

This model is a simplified version of the one presented in Section 3 and similar to the model studied by Milgrom and Weber (1985). There are I (finite) players with generic index $i \in I \equiv \{1,2,...,I\}$. Each player has the following characteristics, all of which are common knowledge:

*type space* $T_i$, a separable metric space. Let $T = \prod_{i \in I} T_i$ be the set of *joint types*.

*action space* $A_i = \{a_{i1},...,a_{iK_i}\}$, finite. Let $A = \prod_{i \in I} A_i$ be the set of *joint actions*.

*payoff function* $u_i: T \times A \to \mathbb{R}$, measurable

Player i's type $t_i$ describes all characteristics of that player. The information each player has about other players' types is generated by a common probability distribution $\mu$ on T, where player i observes $t_i \in T_i$.

A (behavioral) *strategy* for player is a measurable function $s_i: T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions on $A_i$, i.e., the $(K_i\text{-}1)$-dimensional simplex. Note that this is equivalent to the definition of distributional strategy studied by Milgrom and Weber. Let $s_{ik_i}(t_i) = s_i(t_i)(\{a_{ik_i}\}) = \text{Prob}[a_{ik_i}|t_i]$. For any behavioral strategy $s_i$ an equivalent distributional strategy is given by a measure $\sigma_i$ on $T_i \times A_i$ where $\sigma_i(W_i \times \{a_{ik_i}\}) = \int_{W_i} s_{ik_i}(t_i)\mu_i(dt_i)$. Let $S_i$ be the set of strategies for player i, and $\bar{S} = \prod_{i \in I} S_i$. Define the induced payoff function $v_i: \bar{S} \to \mathbb{R}$ to be the expected payoff to player i when each player j uses the strategy $s_j \in S_j$:

$$v_i(s_1,...,s_I) \equiv \sum_{(k_1,...,k_I)} \int_T u_i(\mathbf{t}, a_{1k_1},...a_{Ik_I}) s_{1k_1}(t_i)...s_{Ik_I}(t_I)\mu(d\mathbf{t}).$$

A *Bayesian-Nash equilibrium* is a Nash equilibrium for the game defined by the payoff functions $\{v_1,...,v_I\}$ and the strategy sets $\{S_1,...,S_I\}$. Certainly the resulting strategies $s_1,...,s_I$ are not independent, but are independent given the profile of player types $\mathbf{t} \equiv$

$(t_1,...,t_I)$. Therefore any opportunities for correlation of strategies are limited to those provided by the type space T. This wreaks havoc with the usual methods for studying Bayesian-Nash equilibrium. Existence of a Bayesian-Nash equilibrium becomes an open question. To use the Fan-Glicksberg fixed-point theorem, each $S_i$ must be a compact metric space and each player's payoff function must be continuous with respect to $\overline{S}$ and quasiconcave with respect to each $S_i$. Each $v_i$ is linear with respect to $S_i$. To define $S_i$ as a compact metric space, give it the topology of weak convergence of random variables. A sequence $\{s_i^n\}$ converges to $s_i \in S_i$ if and only if for every measurable $W_i \subset T_i$ and $k_i$, $\text{Prob}^n[a_{ik_i} | t_i \in W_i] \equiv \int_{W_i} s^n_{ik_i}(t_i)\mu_i(dt_i)$ converges to $\text{Prob}[a_{ik_i} | t_i \in W_i] \equiv \int_{W_i} s_{ik_i}(t_i)\mu(dt_i)$. However, this does not imply convergence of $\text{Prob}^n[(a_{1k_1},...a_{Ik_I}) | (t_1,...,t_I) \in W_1 \times ... W_I] \equiv \int_{W_1 \times ... \times W_I} s^n_{1k_1}(t_1)...s^n_{Ik_I}(t_I)\mu(dt)$ for each $(k_1,...,k_I)$. Therefore $v_i$ is not jointly continuous in $\overline{S}$. The following example is due to Milgrom and Weber (1985).

*Example 2.1*: Suppose there are two players, with $A_1 = \{U,D\}$ and $A_2 = \{L,R\}$. Consider the following nonstochastic game:

| | L | R . |
|---|---|---|
| U | (2,2) | (0,0) |
| D | (0,0) | (1,1) |

Let $T_1 = T_2$ be the unit interval with its Borel sets and Lebesgue measure, and let $\mu$ be such that $t_1$ and $t_2$ are perfectly correlated. For each n let $s_1^n(t_1)$ be the point mass on U if $t_1 \bullet 2^n$ is odd, and the point mass on D otherwise. Let $s_2^n(t_2)$ be the point mass on L if $t_2 \bullet 2^n$ is odd, and the point mass on R otherwise. Then for each n, perfect correlation is achieved, so the expected payoff to each player is 3/2. However, for each i $\{s_i^n\}$ converges to the strategy which places probability 1/2 on each action for every $t_i$, which yields expected payoff of 3/4 to each player. $\Diamond$

Note that Example 2.1 is valid when $(t_i - 1/2)\varepsilon$ is added to each payoff, where $\varepsilon$ is an arbitrarily small number. Therefore restricting uncertainty to those states of nature that are payoff relevant does not help, and as argued in the Introduction, the modified game should not be considered significantly different from the original one.

The problem is that weak convergence of each component of the strategy pair $(s_1^n,s_2^n)$ does not define the degree of correlation between them, so some payoff relevant information is lost. The Nash equilibrium concept requires, however, that the pair be independently distributed for every n and **t**. This is how a sequence of pairs of perfectly correlated strategies can "converge" to an uncorrelated pair. Therefore the convergence concept should be modified so that the correlation between strategies is preserved. To do this, consider $(s_1,...,s_I)$ as a mapping from T into $\Delta(A)$, the set of probability distributions on A, and define joint weak convergence to be weak convergence of this mapping. A sequence of strategy vectors $\{(s_1^n,...s_I^n)\}$ converges to $(s_1,...,s_I)$ if and only if for every $(k_1,...,k_I)$ and measurable $W_i \subset T_i$ for $i \in I$,

$$\int_{W_1}...\int_{W_I} s_{1k_1}^n(t_1).\,.s_{Ik_I}^n(t_I)\mu(d\mathbf{t}) \rightarrow \int_{W_1}...\int_{W_I} s_{1k_1}(t_1)...s_{ik_I}(t_I)\mu(d\mathbf{t}).$$

Note that now $\bar{S}$ is not closed. The sequence $\{(s_1^n,s_2^n)\}$ in Example 2.1 converges jointly to the strategy which assigns, for all **t**, probability 1/2 to the pair (U,L) and probability 1/2 to the pair (D,R). This limit is not a Bayesian-Nash strategy since the strategies of players 1 and 2 are not independent conditional on **t**. It is a *correlated equilibrium* as defined by Aumann (1974, 1987). Such an equilibrium can be implemented through a correlation device such as a coin flip. Suppose both players observe the flip of a coin which is common knowledge. Both players believe that if the coin comes up heads, player 1 will play U and play 2 will play L, while if it comes up heads, player 1 will play D and player 2 will play R. Note that even without binding commitments or cooperation, both players find it optimal to follow those beliefs. Player 1 chooses U if the coin is heads and D if it is tails, and player 2 chooses L if the coin is heads and R if it is tails.

A correlation device which can always be used is $\bar{S}$. In this example, the limit strategy is a probability distribution over $S_1 \times S_2$, with probability 1/2 assigned to the behavioral strategy (U,L) and probability 1/2 to (D,R). To see that using $\bar{S}$ as the correlation device entails no loss of generality, let $R_i$ be the unit interval with player i's information about $Z_i$ given by some partition (possibly infinite) on $Z_i$. A correlation device, such as a coin flip, is a pair $(Z,\eta)$ where $\eta$ is a probability distribution on $Z = \prod_{i \in N} Z_i$. With respect to $(Z,\eta)$, a correlated strategy is a vector of player responses $\mathbf{g} = (g_1,\ldots,g_N)$ with $g_i:Z_i \rightarrow S_i$ measurable. Player i receives $z_i \in [0,1]$ then decides on a strategy (not action) $g_i(z_i)$. When she observes her type $t_i$ as well, the mixed strategy given by $g_i(z_i)(t_i)$ is chosen. Note that even if $z_i$ is uninformative, no information about $t_i$ is lost, and $z_i$ provides no additional information about $(t_1,\ldots,t_I)$.

Define a *correlated equilibrium* to be a correlation device $(Z,\eta)$ and a correlated strategy $\mathbf{g}$ such that for each i and any measurable function $g_i':Z_i \rightarrow S_{i,}$

$$\int_Z v_i(\mathbf{g}(\mathbf{z}))\eta(d\mathbf{z}) \geq \int_Z v_i(\mathbf{g}_{-i}(\mathbf{z}_{-i}),g_i'(z_i))\eta(d\mathbf{z}).$$

A correlated equilibrium is a Nash equilibrium with respect to the redefined game. This definition is not very convenient, since $(Z,\eta)$ is part of the equilibrium and the use of functions as strategies is mathematically unwieldy. Following Aumann (1987), assume without loss of generality that the information revealed by $z_i$ about $\mathbf{z}$ is precisely that revealed by $g_i$. Any correlated strategy defines a *correlated strategy distribution* (c.s.d.), which is a probability distribution $\upsilon$ on $\bar{S}$, where $\upsilon = \eta \circ \mathbf{g}^{-1}$. If $(\eta,\mathbf{g})$ is a correlated equilibrium, then $\upsilon$ is a *correlated equilibrium distribution* (c.e.d.), where for each i and any measurable function $\delta_i:S_i \rightarrow S_i$,

$$\int_{\bar{S}} v_i(s)\upsilon(ds) \geq \int_{\bar{S}} v_i(s_{-i},\delta_i(s_i))\upsilon(ds).$$

In a c.e.d., the strategy space $\bar{S}$ serves as a message space. Each player observes, not her own strategy, but a message that, if followed, will cause her to play the strategy given by that message. Since strategies rather than actions are suggested, the information each player has about **t** is unaffected.

Each c.s.d. $\upsilon$ defines a *joint strategy*, a measurable function $s{:}T \to \Delta(A)$, via the relationship

$$s_{k_1 \ldots k_I}(\mathbf{t}) \equiv \mathrm{Prob}[(a_{k_1},a_{k_2},\ldots,a_{k_I})|\mathbf{t}] = \int_{\bar{S}} s_{1k_1}(t_1)\ldots s_{Ik_I}(t_I)\upsilon(d\bar{s}).$$

Let S be the set of joint strategies. With respect to joint weak convergence, S is a compact metric space and $\bar{S} \subset S$. However, the following example shows that a joint strategy corresponds to many c.s.d.'s which are strategically different.

*Example 2.2*: Consider the previous matching game except that $T_1$ and $T_2$ are singletons. Identify $S_i$ with the unit interval, where $s_i \in S_i$ is the probability of choosing heads by player i. On $\bar{S}$, the unit square, let $\upsilon$ be Lebesgue measure and $\upsilon'$ be normalized Lebesgue measure on the region defined by the triangle with vertices (0,0), (0,1/2), and (1/2,0) and the trapezoid with vertices (1,0), (1,1/2), (1/2,1), (0,1). Both $\upsilon$ and $\upsilon'$ generate the joint distribution placing probability 1/4 on each of (U,L), (U,R), (D,L), (D,R). However, under $\upsilon$, each player has a uniform prior about the other player's strategy, while under $\upsilon'$, each player has some information. $\Diamond$

Therefore the description of a c.s.d., rather than its joint strategy, is necessary for studying correlated equilibria.

A topology on the space of c.s.d.'s will be needed. Since $\bar{S}$ is a compact metric space, it is tempting to use weak convergence of measures. This approach is not appropriate, however, since if $\bar{S}$ is given individual weak convergence then $\Delta(\bar{S})$ is a compact metric space but $v_i$ is not continuous. A stronger topology on $\Delta(\bar{S})$ is needed for $v_i$ to be continuous, but then $\Delta(\bar{S})$ cannot be compact. A solution is to define equivalence classes of c.s.d.'s that are strategically equivalent. Give $\Delta(\bar{S})$ the weakest topology that makes the following functions continuous for every $(k_1,\ldots,k_I)$:

$$\upsilon \to \int_{\bar{S}} \Big[\int_W s_{1k_1}(t_1)\ldots s_{Ik_I}(t_I)\mu(dt)\Big]\upsilon(d\bar{s}) \ \forall\ W \subset T$$

$$\upsilon \to \int_{\bar{S}} \Big[\gamma_i(s_i)\int_{W_{-i}} \bar{s}_{-ik_{-i}}(t_{-i})\mu_{-i}(dt_{-i})\Big]\upsilon(d\bar{s}) \ \forall \text{ continuous } \gamma_i{:}S_i \to \mathbb{R} \text{ and } W_{-i} \subset T_{-i}$$

To interpret these conditions, let $\{\upsilon^\alpha\}$ be a convergent net (generalized sequence) of c.s.d.'s. in this topology. The first condition implies that the corresponding net of joint strategies converges. The second statement says that if player i deviates by choosing a mixed strategy that depends on the $s_i$ received but not on $t_i$, such that $\text{Prob}[a_{ik_i}|s_i]$ depends continuously on $s_i$ for each $k_i$, then the resulting net of joint strategies converges.

Let Y be the corresponding set of equivalence classes with the quotient topology. The following results are proven in Section 4.

(1) Y is a compact metric space

(2) For any $\upsilon$, $\upsilon'$ in the same equivalence class of Y, for each i and measurable $\delta_i{:}S_i \to S_i$, $\int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) = \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon'(\bar{s})$, so strategic opportunities are well-defined.

(3) the set of c.e.d.'s is nonempty, compact, and convex.

These results apply to a more general model which is presented below.

## 3. The general model

This model of normal form games is the same one used by Cotter (1988). It is a generalization of the model used by Radner and Rosenthal (1982) in their study of purifications of Bayesian-Nash equilibria, and parallels the use of independently defined player type spaces by Milgrom and Weber (1985) and the previous section. The main advantage of this model is the definition of players' information as an explicit parameter of a metric space, which is used in Section 5. There are I (finite) players with generic index $i \in I \equiv \{1,2,\ldots,I\}$. Uncertainty is given by a probability space $(\Omega,\boldsymbol{F},\mu)$ where $\Omega$ is a set of possible states of nature, $\boldsymbol{F}$ a countably generated $\sigma$-field of measurable subsets of $\Omega$, and $\mu$ a probability distribution on $(\Omega,\boldsymbol{F})$, all of which are common knowledge. Define the space of information fields or devices $\boldsymbol{F}^*$ as in Boylan (1971) and Cotter (1986) to be the set of sub-$\sigma$-fields of $\boldsymbol{F}$ (measurable partitions of $\Omega$), identifying those partitions which differ only by events of probability zero. Each player has the following characteristics:

*action space* $A_i$, a compact metric space. Let $A = \prod_{i\in I} A_i$ be the joint action space.

*information field* $\boldsymbol{G}_i \in \boldsymbol{F}^*$.

*payoff function* $u_i:\Omega \times A \to \mathbb{R}$.

Each player's payoff function is assumed to be continuous and integrable. To state this precisely, let C(A) be the Banach space of real-valued continuous functions on A with the supremum norm. For example, a payoff function $u_i:\Omega \to C(A)$ is a random vector taking values in C(A). Some results on Banach space-valued random vectors are used which can be found in, say, Diestel and Uhl (1977, Chapter 2). Let E be a Banach space with norm $\|\cdot\|$ and dual $E^*$. A random vector $f:\Omega \to E$ is said to be *measurable* if there exists a sequence $\{f^n\}$ of simple random vectors $f^n = \sum e_j^n \chi_{F_j^n}$, with $e_j^n \in E$, $F_j^n \in \boldsymbol{F}$ for each n and j, and $\chi$ the indicator function, such that for a.e. $\omega$, $\lim\|f^n - f\| \to 0$. Pettis' Measurability Theorem [Diestel and Uhl (1977, Theorem 2.1)] states that f is measurable if and only if it has essentially separable range (i.e., maps an event of probability one into a separable subset of E) and for every $x^* \in E^*$, $x^*(f(\omega))$ is scalar measurable. Note that the former requirement is automatic if E = C(A) since it is separable. A measurable f is said to be *Bochner integrable* if $\|f(\omega)\|$ is integrable. The Bochner integral in E is defined from any sequence of simple functions $\{f^n\}$ and is written $\int_\Omega f(\omega)\mu(d\omega)$. Define the Banach space

$L(E)$ to be the space of Bochner integrable random vectors with norm $|||f||| = \int_\Omega \|f(\omega)\|\mu(d\omega)$. The continuity and integrability assumptions on $u_i$ can be stated as follows.

*Assumption 3.1*: For each i, $u_i \in L(C(A))$.

All structural aspects of the game, as well as the strategies of all players, are assumed to be common knowledge. In this game, each player observes the outcome of the partition $\boldsymbol{G}_i$, including $E[u_i|\boldsymbol{G}_i](\omega)$.

A behavioral strategy, used by Radner and Rosenthal (1982) and Cotter (1988) is a function, consistent with the player's information, which maps the state space into the set of probability measures over that player's action space. Let $M(A_i)$ be the set of regular finite Borel measures on $A_i$ with the variation norm, and $\Delta(A_i)$ the set of regular probability measures on $A_i$. A behavioral strategy is defined to be a function $s_i: \Omega \to \Delta(A_i)$ which is, in some sense, consistent with that player's information. However, since $M(A_i)$ is not separable, to require $s_i$ to be strongly $\boldsymbol{G}_i$-measurable would be extremely restrictive.[1] For example, suppose $A_i$ and $\Omega$ are uncountable. If $\mu$ is not purely atomic then any behavioral strategy which is injective and maps into point masses is not measurable since it does not have separable range. However, a weaker version of measurability will suffice for this model. Note that $M(A_i)$ is the dual of $C(A_i)$ with the usual duality $\langle c_i,\eta_i\rangle = \int_{A_i} c(a_i)\eta_i(da_i)$. Let $s_i:\Omega \to M(A_i)$ be a random vector. Then $s_i$ is said to be *weak* $\boldsymbol{G}_i$-measurable* if for every $c_i \in C(A_i)$, the mapping $\omega \to \langle c_i,s_i(\omega)\rangle$ is scalar $\boldsymbol{G}_i$-measurable. A behavioral strategy for player i is defined to be a weak* $\boldsymbol{G}_i$-measurable random vector with values in $\Delta(A_i)$.

Let $S_i$ be the set of strategies for player i, and $\overline{S} = \Pi S_i$ the set of *Nash behavioral strategies*. Note that for any Nash strategy $\overline{s} = (s_1,\ldots,s_I)$ and a.e. $\omega$, $(s_1(\omega),\ldots,s_I(\omega))$

[1]I am indebted to Maxwell Stinchcombe for reminding me of this fact.

defines a probability measure on A such that the marginals on each $A_i$ are independent. By allowing correlation on states of nature not defined *a priori*, a larger set of player strategies can be defined. Such a strategy must be consistent with each player's information as well as the information of any group of players. For any subset of players $J \subset I$, let $A_J = \prod_{j\in J} A_j$ and $\boldsymbol{G}_J = \bigvee_{j\in J} \boldsymbol{G}_j$. For any random vector $s:\Omega \to \Delta(A)$, let $s_J:\Omega \to \Delta(A_J)$ be defined by $s_J(\omega)$ = the marginal of $s(\omega)$ on $A_J$. Let $S = \{s:\Omega \to \Delta(A) \mid s_J \text{ is } \boldsymbol{G}_J\text{-measurable}\}$ be the set of *joint behavioral strategies*. Clearly $\overline{S} \subset S$, and both are contained in the the dual of $L(C(A))$, denoted $L(C(A))^*$. Since $M(A)$ is not separable unless A is countable, $L(C(A))^* \neq L_\infty(M(A))$, the space of Bochner-integrable random vectors with values in $M(A)$ which are a.e. bounded. However,

*Theorem 3.2*: S is a weak$^*$-compact subset of $L(C(A))^*$ with the duality $\langle f,s\rangle = \int_\Omega\int_A f(\omega)(a)s(\omega)(da)\mu(d\omega)$, and $S_i$ is a weak$^*$-compact subset of $L(C(A_i))^*$.

*Proof*: Identical to the proof of Theorem 2.3 of Cotter (1988). $\Diamond$

Example 2.1 shows that as a subset of $L(C(A))^*$, $\overline{S}$ is not a closed subset of S. Therefore $\overline{S}$ has two different topologies. With respect to the product topology generated by each $S_i$, $\overline{S}$ is a compact metric space, which is distinct from weak$^*$ convergence in $L(C(A))^*$.

Each player's payoff function $u_i$ induces a payoff function with respect to strategies $v_i:S \to \mathbb{R}$ with $v_i(s) = \langle u_i,s\rangle$. Note that $v_i$ is continuous with respect to joint convergence, but not independent convergence.

A theory of integration for strategies will be needed. For every measure $\upsilon$ on $\overline{S}$ (or S for that matter) there exists an integral in S, denoted $\int_F s\,d\upsilon$, such that for every $f \in L(C(A))$, $\langle f,\int_F s\,d\upsilon\rangle = \int_F \langle f,s\rangle\upsilon(ds)$ [Rudin (1973, Theorem 3.26)]. In addition, for every $s_i \in S_i$, there exists an integral in $\Delta(A_i)$ known as the *Gelfand integral* and denoted $\oint_\Omega s_i(\omega)\mu(d\omega)$, such that for every $c \in C(A_i)$, $\langle c, \oint_\Omega s_i(\omega)\mu(d\omega)\rangle =$ $\int_\Omega \langle c,s(\omega)\rangle\mu(d\omega)$. Conditional expectations for strategies can be defined using the Gelfand integral. For each $s_i \in S_i$ and every $\boldsymbol{H} \in \boldsymbol{F}^*$, there exists a unique strategy $E[s_i|\boldsymbol{H}]$ $\in S_i$ such that for every $c \in C(A_i)$, $\langle c,E[s_i|\boldsymbol{H}]\rangle = E[\langle c,s_i\rangle|\boldsymbol{H}]$. See Cotter (1988) for details.

## 4. Correlated strategies and equilibria

As argued in Section 2, a correlated strategy distribution is a probability measure on $\overline{S}$.

*Definition 4.1*: A *correlated strategy distribution* (c.s.d.) is a Borel probability measure $\upsilon$ on S which places probability one on $\overline{S}$. A *correlated equilibrium distribution* (c.e.d.) is a c.s.d. $\upsilon$ such that for each i and every measurable $\delta_i : S_i \to S_i$, $\int_{\overline{S}} v_i(s_i,\overline{s}_{-i})\upsilon(d\overline{s}) \geq \int_{\overline{S}} v_i(\delta_i(s_i),\overline{s}_{-i})\upsilon(d\overline{s})$.

Note that by Choquet's theorem, every joint strategy corresponds to some c.s.d., though as shown in Example 2.2, the representation is not unique.

The next question is whether a useful topology on the space of c.s.d.'s can be found. It was shown in Section 2 that this cannot be done for the entire space $\Delta(\overline{S})$, so a set of equivalence classes in $\Delta(\overline{S})$ is required.

*Definition 4.2*: Give $\Delta(\overline{S})$ the weakest topology such that the mapping from $\Delta(\overline{S})$ to S defined by $\upsilon \to \int_{\overline{S}} \overline{s}\upsilon(d\overline{s})$ is continuous, and for each i and every continuous function $\gamma_i : S_i \to \Delta(A_i)$, the mapping from $\Delta(\overline{S})$ to S defined by $\upsilon \to \int_{\overline{S}} [\gamma_i(s_i) \times \overline{s}_{-i}]\upsilon(d\overline{s})$ is continuous. Define Y to be the set of equivalence classes of c.s.d.'s generated by these mappings with the quotient topology. Identify $\upsilon \in \Delta(\overline{S})$ with its equivalence class in Y.

Note that this is equivalent to the weakest topology such that $\upsilon \to \int_{\bar{S}} \bar{s}\upsilon(d\bar{s})$ is continuous and for every function $F_i: S_i \to S_i$ of the form $F_i = \sum_{l=1}^{L} \gamma_i^l \chi_{H^l}$ with $\gamma_i^l: S_i \to \Delta(A_i)$ continuous for each $l$ and $H^l \in \boldsymbol{F}$, the mapping from Y to S defined by $\upsilon \to \int_{\bar{S}} (F_i(s_i) \times \bar{s}_{-i})\upsilon(d\bar{s})$ is continuous.

*Theorem 4.3*: Y is a compact metric space.

*Proof*: For each i choose $\{\gamma_i^k | k = 1,....\}$ a countable dense subset of $C(S_i, \Delta(A_i))$, and $\{H^l | l = 1,....\}$ a countable dense subset of $\boldsymbol{F}$ with respect to the symmetric difference metric $\rho(H,H') = \mu(H \cap H'^c) + \mu(H^c \cap H')$. Define $\Phi: Y \to S^\infty$ (with the product topology) by $\Phi(\upsilon) = (\int_{\bar{S}} \bar{s}\upsilon(d\bar{s}), \int_{\bar{S}} [(\sum_{l \in L} \gamma_i^k(s_i)\chi_{H^l}) \times \bar{s}_{-i}]\upsilon(d\bar{s}))_{i \in I, k=1,...,\infty, L}$ where L runs over all finite subsets of the positive integers. By the definition of the topology, $\Phi$ is continuous, and by the identification of equivalence classes, $\Phi$ is injective. To show $\Phi^{-1}$ is continuous, let $\{\upsilon^n\}$ be a sequence in Y such that for k, L, and $f \in L(C(A))$, $\{\int_{\bar{S}} \langle f, (\sum_{l \in L} \gamma_i^k(s_i)\chi_{H^l}) \times \bar{s}_{-i} \rangle \upsilon^n(d\bar{s})\}$ converges to $\{\int_{\bar{S}} \langle f, (\sum_{l \in L} \gamma_i^k(s_i)\chi_{H^l}) \times \bar{s}_{-i} \rangle \upsilon(d\bar{s})\}$. Let $\bar{F}_i = \sum_{l \in L} \bar{\gamma}_i^l \chi_{\bar{H}^l}$. For each $\bar{\gamma}_i^l$ choose $\gamma_i^l$ such that for every $s_i$ and $B \subset A_i$, $|\bar{\gamma}_i^l(B) - \gamma_i^l(B)| < \varepsilon/2L$, and $H^l$ such that $\mu(\bar{H}^l \,\Delta\, H^l) < \varepsilon/2L$ and $\{H^1,...,H^L\}$ is a disjoint partition of $\Omega$. Repeated use of the triangle inequality shows that for every $\bar{s} \in \bar{S}$, $|\langle f, E[\bar{F}_i(s_i)|\boldsymbol{H}] \times \bar{s}_{-i} \rangle - \langle f, \bar{F}_i(s_i) \times \bar{s}_{-i} \rangle| < \varepsilon$, uniformly in $\upsilon$.

Therefore $\Phi$ is a homeomorphism of Y onto a subset of $S^\infty$, so Y is a separable metric space. To show Y is compact, let $\{\upsilon^n\}$ be a sequence in Y. By a diagonalization argument, there exists a subsequence such that for every L and k, $\{\int_{\bar{S}} [(\sum_{l \in L} \gamma_i^k(s_i)\chi_{H^l}) \times \bar{s}_{-i}]\upsilon^n(d\bar{s})\}$ converges to $\int_{\bar{S}} [(\sum_{l \in L} \gamma_i^k(s_i)\chi_{H^l}) \times \bar{s}_{-i}]\upsilon(d\bar{s})$. One can find an equivalence class of measures on $\overline{S}$ taking on the desired values, since these mappings (using the duality between S and L(C(A)) define a subset of $C(\overline{S})$. Denoting a typical element of the equivalence class by $\upsilon$, the result is proven. ◊

The next problem is to show that the identification process in Definition 4.2 preserves c.s.d.'s that are strategically distinct, so that statements about correlated equilibria are well-defined on equivalence classes. The following version of Lusin's theorem will be of central importance:

*Lemma 4.4*: For any i, any $\upsilon \in \Delta(\overline{S})$, and any Borel measurable function $\delta_i:S_i \to S_i$, any probability measure $\upsilon$ on $\overline{S}$, and any $\varepsilon > 0$, there exists a continuous function $F_i:S_i \to S_i$ such that letting $U_i = \{F_i(s_i) \neq \delta_i(s_i)\}$, $U_i$ is open and $\upsilon_i(U_i) > 1 - \varepsilon$ where $\upsilon_i$ is the marginal on $S_i$.

*Proof*: Let $\{f_i^k \mid k=1,2,...\}$ be a countable dense subset of $L(C(A_i))$. For each k define $g_i^k:S_i \to [-1,1]$ by $g_i^k(s_i) = \langle f_i^k,\delta_i(s_i)\rangle/|||f_i^k|||$. For each k, there exists by Lusin's theorem a closed subset $V_i^k$ of $S_i$ with $\upsilon_i(V_i^k) > 1 - \varepsilon 2^{-k}$ such that on $V_i^k$, $g_i^k$ is continuous. Let $V_i = \bigcap_k U_i^k$. Then $V_i$ is closed, $\upsilon_i(V_i) > 1 - \varepsilon$, and $\delta$ is continuous on $V_i$.

The next step is to show that $S_i$ is homeomorphic to a retract of $[-1,1]^\infty$. Let $\Psi:S_i \to [-1,1]^\infty$ be defined to be $\Psi(s) = (\langle f_i^1,s_i\rangle,\langle f_i^2,s_i\rangle,....)$. Certainly $\Psi$ is continuous and injective, so since $S_i$ is compact, $\Psi$ is a homeomorphism of $S_i$ onto $\Psi(S_i)$. Note that $\Psi(S_i)$ is a compact convex subset of $[-1,1]^\infty$. It remains only to be shown that $\Psi(S_i)$ is a retract of $[-1,1]^\infty$. Let $d(y,z) = \sum_{l=1}^{\infty} 2^{-l}|y_l-z_l|$ be a metric on $[-1,1]^\infty$. Define the correspondence $r:[-1,1]^\infty \to\to \Psi(S_i)$ by $r(z) = \text{argmin}\{d(y,z) \mid y \in \Psi(S_i)\}$. To show r is single valued, suppose for some $z \in [-1,1]^\infty$ there exist $y^1, y^2 \in \Psi(S_i)$ such that $d(y^1,z) = d(y^2,z) \geq d(y,z)$ for all $y \in \Psi(S_i)$. Since $\Psi(S_i)$ is convex, it must be the case that $d(\lambda y^1+(1-\lambda)y^2,z) \geq \lambda d(y^1,z) + (1-\lambda)d(y^2,z)$. However, $d(\lambda y^1+(1-\lambda)y^2,z) =$

$d(\lambda(y^1-z),(1-\lambda)(z-y^2)) \le d(\lambda(y^1-z),0) + d(0,(1-\lambda)(z-y^2)) = d(\lambda y^1,\lambda z) + d((1-\lambda)z,(1-\lambda)y^2)$ $= \lambda d(y^1,z) + (1-\lambda)d(y^2,z)$. Therefore $d(\lambda y^1+(1-\lambda)y^2,z) = \lambda d(y^1,z) + (1-\lambda)d(y^2,z)$, so 0 falls along the line from $y^1$-z to z-$y^2$, i.e., z is on the line determined by $y^1$ and $y^2$. This is possible only if $z = (1/2)(y^1+y^2)$, so $z \in \Psi(S_i)$, a contradiction. Therefore r is single valued and continuous, and r(y) = y for $y \in \Psi(S_i)$. Therefore $\Psi(S_i)$ is a retract of $\mathbb{R}^\infty$. This proves that $S_i$ has the universal extension property, so there exists a continuous $F_i:S_i \to S_i$ such that $F_i = \delta$ on $U_i$, completing the proof. ◊

Recall that the pointwise convergence metric on $\boldsymbol{F}^*$ [Cotter (1986)] is the weakest topology on $\boldsymbol{F}^*$ such that for every $h \in L(\mathbb{R})$, the mapping $\boldsymbol{H} \to \|E[h|\boldsymbol{H}]\|$ is continuous.

*Lemma 4.5*: Let $\{\boldsymbol{H}^n\}$ be a sequence in $\boldsymbol{F}^*$ which converges pointwise to $\boldsymbol{H} \in \boldsymbol{F}^*$. Then $\{E[s_i|\boldsymbol{H}^n]\}$ converges to $E[s_i|\boldsymbol{H}]$ uniformly in $s_i \in S_i$.

*Proof*: For $f_i \in L(C(A_i))$, $|\langle f_i,E[s_i|\boldsymbol{H}^n]\rangle - \langle f_i,E[s_i|\boldsymbol{H}]\rangle| = |\langle E[f_i|\boldsymbol{H}^n],s_i\rangle - \langle E[f_i|\boldsymbol{H}],s_i\rangle| \le \||E[f_i|\boldsymbol{H}^n]-E[f_i|\boldsymbol{H}]\||$ which can be made arbitrarily small for sufficiently large n, uniformly in $s_i$. ◊

The next result shows that the strategic opportunities of individuals are well-defined on equivalence classes of c.s.d.'s.

*Theorem 4.6*: For any elements $\upsilon$ and $\upsilon'$ in the same equivalence class of Y and for every i and Borel measurable $\delta_i:S_i \to S_i$,

$$\int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) = \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon'(d\bar{s}).$$

*Proof*: Given $\varepsilon > 0$ there exists by Lemma 4.4 a continuous $F_i:S_i \to S_i$ such that, letting $U_i = \{s_i \in S_i \mid F_i(s_i) \neq \delta_i(s_i)\}$, $\int_{U_i\times\bar{S}_{-i}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) < \varepsilon$ and $\int_{U_i\times\bar{S}_{-i}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon'(d\bar{s}) < \varepsilon$. Since the set of finite partitions of $\Omega$ is dense in $\boldsymbol{F}^*$, there exists a sequence $\{\boldsymbol{H}^n\}$ of finite partitions which converges pointwise to $\boldsymbol{H} \in \boldsymbol{F}^*$. For sufficiently large n, by Lemma 4.5 and dominated convergence,

$$\left|\int_{\bar{S}} v_i(F_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}^n],\bar{s}_{-i})\upsilon(d\bar{s})\right| < \varepsilon$$

$$\left|\int_{\bar{S}} v_i(F_i(s_i),\bar{s}_{-i})\upsilon'(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}^n],\bar{s}_{-i})\upsilon'(d\bar{s})\right| < \varepsilon.$$

Then

$$\left|\int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon'(d\bar{s})\right|$$

$$\leq \left|\int_{U_i\times\bar{S}_{-i}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) - \int_{U_i\times\bar{S}_{-i}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon'(d\bar{s})\right| + 2\varepsilon < 4\varepsilon.$$

Since $\varepsilon$ is arbitrary, the proof is complete.

*Theorem 4.7* The set of c.e.d.'s is compact.

*Proof*: Let $\{\upsilon^n\}$ be a sequence of c.e.d.'s converging to some $\upsilon \in Y$. Choose i and a measurable $\delta_i:S_i \to S_i$. By Lemma 4.4 there exists a continuous function $F_i:S_i \to S_i$ such that $V_i = \{s_i \in S_i \mid F_i(s_i) \neq \delta_i(s_i)\}$ is open and $\upsilon(V_i \times \bar{S}_{-i}) < \varepsilon$. As in the proof of Theorem 4.6 choose a finite partition $\boldsymbol{H} \subset \boldsymbol{G}_i$ such that

$$\left|\int_{\bar{S}} v_i(F_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}],\bar{s}_{-i})\upsilon(d\bar{s})\right| < \varepsilon.$$

Then

$$\int_{\bar{S}} v_i(\bar{s})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s})$$

$$= \int_{\bar{S}} v_i(\bar{s})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(F_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) + \int_{V_i\times\bar{S}_{-i}} \{v_i(F_i(s_i),\bar{s}_{-i}) - v_i(\delta_i(s_i),\bar{s}_{-i})\}\upsilon(d\bar{s})$$

$$\geq \int_{\bar{S}} v_i(\bar{s})\upsilon^n(d\bar{s}) - \int_{\bar{S}} v_i(F_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s}) - 2\varepsilon|||u_i|||$$

$$\geq \int_{\bar{S}} v_i(\bar{s})\upsilon^n(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}],\bar{s}_{-i})\upsilon^n(d\bar{s}) - 2\varepsilon(2 + |||u_i|||) \geq -2\varepsilon(2 + |||u_i|||)$$

since $\upsilon^n$ is a c.e.d.. Since $\varepsilon$ is arbitrary, $\upsilon$ is a c.e.d.$\Diamond$

In the next section it will be show that the set of c.e.d.'s is nonempty.

## 5. Continuity of game characteristics

As stated above, the usual assumption of finiteness of player types in Bayesian-Nash games cannot be justified on the belief that finite type games are dense in the set of all games with asymmetric information. When considering *correlated* equilibria, however, this hypothesis is justified. The next result contrasts with Example 3.2 of Cotter (1988).

*Theorem 5.1*: The correspondence mapping player characteristics [payoff functions and information fields] into the set of c.e.d.'s $\zeta:[L(C(A))\times \boldsymbol{F}^*]^I \to\to Y$ is upperhemicontinuous when $\boldsymbol{F}^*$ is given the pointwise convergence topology [Cotter (1986)].

*Proof*: For each i let $v_i^n \in L(C(A))$ and $\boldsymbol{G}_i^n \in \boldsymbol{F}^*$ converging to $v_i$ and $\boldsymbol{G}_i$ respectively. For each n let $\upsilon^n$ be a c.e.d. for the game $[(v_1^n,\boldsymbol{G}_1^n),\dots,(v_I^n,\boldsymbol{G}_I^n)]$. Choose i and a measurable $\delta_i:S_i \to S_i$ taking values on weak$^*$ $\boldsymbol{G}_i$-measurable strategies. Choose $\varepsilon$, $F_i$, and $\boldsymbol{H}$ as in the proof of Theorem 4.7. Following the proof of Theorem 4.7 yields

$$\int_{\bar{S}} v_i(\bar{s})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s})$$

$$\geq \int_{\bar{S}} v_i(\bar{s})\upsilon^n(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}],\bar{s}_{-i})\upsilon^n(d\bar{s}) - 2\varepsilon(1 + |||u_i|||)$$

for sufficiently large n. For such n choose a finite partition $\boldsymbol{H}^n \subset \boldsymbol{G}_i^n$ such that

$$\left|\int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}],\bar{s}_{-i})\upsilon^n(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}^n],\bar{s}_{-i})\upsilon^n(d\bar{s})\right| < \varepsilon.$$

Then

$$\int_{\bar{S}} v_i(\bar{s})\upsilon(d\bar{s}) - \int_{\bar{S}} v_i(\delta_i(s_i),\bar{s}_{-i})\upsilon(d\bar{s})$$

$$\geq \int_{\bar{S}} v_i(\bar{s})\upsilon^n(d\bar{s}) - \int_{\bar{S}} v_i(E[F_i(s_i)|\boldsymbol{H}^n],\bar{s}_{-i})\upsilon^n(d\bar{s}) -2\varepsilon(2 + |||u_i|||) \geq -2\varepsilon(2 + |||u_i|||)$$

completing the proof. ◊

If every information field $\boldsymbol{G}_i$ is a finite partition then $v_i$ is continuous on $\bar{S}$, so Theorem 3 of Hart and Schmeidler (1987) applies to show that a c.e.d. exists. Using Theorem 5.1 existence can be extended to all games.

*Corollary 5.2*: The correspondence $\zeta$ is nonempty-valued (i.e., a c.e.d. exists) and convex valued on $[L(C(A))\times\boldsymbol{F}^*]^I$.

## 6. Concluding remarks

These results suggest that at least for games with uncertainty, correlated equilibrium is a more natural solution concept than Bayesian-Nash equilibrium. In the latter case, correlation of strategies is restricted to an arbitrary space of uncertainty. Defining this space of uncertainty *a priori* is a dubious undertaking, since other states of nature (sunspots) can become decision relevant through the beliefs of players, which are endogenous, and the statement that a state of nature is payoff relevant is empirically meaningless. In addition, the type of correlation allowed in the Bayesian-Nash solution concept, which depends only on the underlying type space, is unstable. When similarity of strategies is defined to explicitly account for the degree of correlation between them, then the set of correlated strategies is required. This recognizes the fact that not all uncertainty in the game can be defined by the game theorist.

Perhaps many game theorists have avoided using the correlated equilibrium concept because it allows a large equilibrium set, which includes the convex hull of the set of Nash equilibria. This is not encouraging in light of results such as the folk theorem, which states that in an infinitely repeated game the set of Nash equilibrium payoffs consists of all payoffs that are individually rational. However, this equilibrium set is so large that expanding the solution  no harm is done by using correlated equilibria since that set cannot be larger. As with Nash equilibria, the solution is to refine the correlated equilibrium concept by ruling out unreasonable beliefs. Though there is no reason to rule out correlation on grounds of reasonableness, some correlated equilibria can be eliminated through, say, dominated strategy or justifiability arguments. Given the nice mathematical structure of correlated equilibrium (compactness and convexity) and the explicit use of beliefs, refinements may be easier to obtain than for Nash equilibrium.

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