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ON EXISTENCE OF A NASH EQUILIBRIUM POINT IN N-PERSON NONZERO SUM STOCHASTIC JUMP DIFFERENTIAL GAMES

by

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Abstract

Using the technique of Wan and Davis, we give an existence theorem for a Nash equilibrium point in N-person nonzero sum stochastic jump differential games. It is shown that if the Nash condition (generalized Isaacs condition) holds there is a Nash equilibrium point in feedback strategies. We extend the results to other solution concepts and discuss applications and extensions.

Key Words: Nash equilibrium, differential games, jump processes.

1. Introduction

Using the technique of [10], we give an existence theorem for a Nash equilibrium point in N-person nonzero sum stochastic jump differential games. It is shown that if the Nash condition (see Definition 3.2) holds, there is a Nash equilibrium point in feedback strategies. We extend the results to other solution concepts and discuss applications and extensions.

In [8] the existence of Nash equilibrium points in stochastic differential games is looked at by using the technique of [1]. It is an essential point of this technique that analogues of the time derivation of the gradient of the value function are constructed using a martingale method. Consequently, we can obtain the optimal value directly by optimizing the Hamiltonian at each point. We will here give parallel results for stochastic jump differential games, using the technique of [10].

Keeping the notation close to that in [10], we consider a jump process x_t specified under a basic probability measure P to which corresponds a pair of entities (Λ,λ) called the local description of the process; Λ determines the rate of occurrence of jumps while λ determines their positions. By using an indexed pair of Radon-Nikodym derivatives (α^u,β^u), we achieve control of x_t through mutually absolutely continuous transformation of the local descriptions from (Λ,λ) to (Λ^u,λ^u). Neither the jump process, nor the controls need to be Markovian.

The player i, $i = 1,\dots,N$, chooses a feedback control $u_i(t,x)$ over the finite interval $[0,T_f]$. Together, these controls determine $u(t,x) = (u_1(t,x),\dots,u_N(t,x))$. Corresponding to this choice of control, player i incurs a cost of the form:

(1.1) $$J_i(u) = E_u[\int_0^{T_f} c_i(s,u_s,\omega)d\Lambda_s + G_{if}] \quad \text{(see Definition 2.2)}$$

In Section 2 we give a mathematical formulation of the game and some related results. In Section 3 we prove the main theorem, whereas solution concepts other than Nash equilibrium are considered in Section 4. We devote Section 5 to applications and extensions.

2. Preliminaries

2.1 The Jump Process

The jump process $\{x_t\}$ is piecewise constant, takes values in a Blackwell space $(X,\mathcal{S})$ and has isolated discontinuities. If z_0 is a fixed element of X, we can define $(Y^j,\mathcal{Y}^j)$ as the copy of

$$(Y,\mathcal{Y}) = ((R^+ \times X) \cup \{(\infty,z_\infty)\},\ \sigma\{\underline{B}(R^+) \otimes \mathcal{S}, \{\infty,z_\infty\}\})$$

for $j = 1,2,\ldots$. In this case, the basic measurable space is $(\Omega,\mathcal{F}^o)$:

$$\Omega = \prod_{j=1}^{\infty} Y^j,\ \mathcal{F}^o = \sigma\{\prod_{j=1}^{\infty} \mathcal{Y}^j\}$$

We can now define $(S_j,Z_j)\colon \Omega \to Y^j$ as the coordinate mapping, such that $\{S_j\}$ are the "interarrival times" and $\{Z_j\}$ the "states" defining $\{x_j\}$. Further let $\omega_k = \Pi_{j=1}^k Y^j$ be the projection onto Ω_k. If finally $z_o(\omega) = z_o$ is another fixed element of X, we can let

$$T_k(\omega) = \Sigma_{j=1}^k S_j(\omega),\ T_\infty(\omega) = \lim_{k\to\infty} T_k(\omega)$$

such that the sample path of $\{x_t\}$ is

$$x_t(\omega) = \begin{cases} Z_j(\omega), & t \in [T_j(\omega), T_{j+1}(\omega)) \\ z_\infty, & t \geq T_\infty(\omega) \end{cases}$$

2.2 A Measure

To get a measure P on $(\Omega, \mathcal{F}^o)$, we assume that the $\{S_k\}$ are independent with survivor functions $F_t^k = P(S_k > t)$, as given by the functions Λ^k:

$$[0, d^k) \to R^+ (0 < d^k \leq \infty) \text{ satisfying}$$

(i) $\Lambda^k(o) = 0$, $\Lambda^k(\cdot)$ increasing and right continuous

(ii) $\Lambda^k(t) \uparrow \infty$ as $t \uparrow \infty$ if $d^k = \infty$

(iii) $\Delta\Lambda^k(s) = \Lambda^k(s) - \Lambda^k(s-) < 1$

(iv) There exist positive constants Θ_1, Θ_2, such that $\Lambda^k(t) \leq \Theta_2$ for $t \in [0, \Theta_1]$ and $k \in \mathcal{K}$ where $\mathcal{K}$ is an infinite subset of the integers 1,2,... .

Based on this:

$$F_t^k = \exp(-\Lambda^k(t) + \sum_{s \leq t} \Delta\Lambda^k(s)) \prod_{s \leq t} (1 - \Delta\Lambda^k(s))$$

where the countable set $\{s \leq t: \Delta\Lambda^k(s) \neq 0\}$ is referred to.

Remark: The T_k sequence is a Poisson process if $\Lambda^k(t) = t$, but the framework applies equally to discrete time models.

We further specify the functions $\lambda^k: \Omega_{k-1} \times R^+ \times \mathcal{S} \to [0,1]$ such that

(i) $\lambda^k(\cdot, \cdot, A)$ is measurable for each $A \in \mathcal{S}$.

(ii) $\lambda^k(\omega_{k-1}, t, \cdot)$ is a probability measure on $\mathcal{S}$ for each

$$(\omega_{k-1},t) \in \Omega_{k-1} x (0,d^k] \text{ such that } \lambda(\omega_{k-1},t,\{Z_{k-1}(\omega)\}) = 0.$$

P can thus be defined as

$$P(S_k > t,\ Z_k \in A | F^o_{T_{k-1}}) = -\int_{(t,\infty]} \lambda_k(\omega_{k-1},s,A)dF^k_s$$

From this we define $\mathcal{F}_t$ as $\mathcal{F}^0_t$ completed with all P-null sets of $\mathcal{F}^0$. So $\mathcal{F}_t$ is the completed σ-field on Ω generated by x_t up to time t.

2.3 Martingale

To characterize the fundamental family of martingales associated with $\{x_t\}$, we define

$$p(t,A) = \sum_j I_{(t\geq T_j)} I_{(Z_j \in A)}$$

$$\Lambda_t(\omega) = \Lambda^1(S_1) + \Lambda^2(S_2) + \ldots + \Lambda^{k-1}(S_{k-1}) + \Lambda^k(t - T_{k-1}),\ t \in (T_{k-1},T_k]$$

$$\lambda(t,A)(\omega) = \sum_{k=1}^{\infty} I_{(t\in(T_{k-1},T_k])} \lambda^k(\omega_{k-1};\ t - T_{k-1},A)$$

$$\tilde{p}(t,A) = \int_{(0,t]} \lambda(t,A)d\Lambda_t$$

$$q(t,A) = p(t,A) - \tilde{p}(t,A)$$

such that q(t,A) is a local martingale of $\mathcal{F}_t$.

2.4 Controls

Each player i, i = 1,2,...,N, can influence the jump process through a control $u^i(t,x)$ with values in a compact metric space U^i.

Definition 2.1: $u_i(\cdot)$ is in the class of admissible controls $\mathcal{U}_i$ if u_i is $\mathcal{F}_t$ predictable.

Remark: We could allow U_i to depend on $u^i \equiv (u_i,\ldots,u_{i-1}, u_{i+1},\ldots,u_n)$, but prefer the less general case for ease of notation.

Let us now define $U = \prod_{i=1}^{N} U_i$, $\mathcal{U} = \prod_{i=1}^{N} \mathcal{U}_i$, $\mathcal{U}^i = \prod_{j\neq 1}^{N} \mathcal{U}_j$ and the two functions $\alpha\colon R^+ \times U \times \Omega \to R^+$ and $\beta\colon R^+ \times X \times U \times \Omega \to R^+$. The measurable functions α and β satisfy:

(i) For each $(x,y) \in X \times U$, $\alpha(t,u,\omega)$ and $\beta(t,x,u,\omega)$ are $\mathcal{F}_t$ predictable.

(ii) There exists positive constants C_1, C_2, C_3, δ' such that

$$C_1 \leq \alpha(\cdot) \leq \min\{C_2, \frac{1-\delta'}{\Delta\Lambda_t}\}$$

$$C_1 \leq \beta(\cdot) \leq C_3$$

for all $(t,x,u,\omega) \in R^+ \times X \times U \times \Omega$.

(iii) $\int_X \beta(t,x,u,\omega)\, \lambda(dx,t,\omega) = 1$ for all $(t,u,\omega) \in R_+ \times U \times \Omega$.

Remark: The conditions (ii) are unpleasantly strong and obtaining our results without them is an important goal of future research. Given the results of [5], it seems that his very different techniques may give existence without (ii). For now, we need these conditions to assure mutual absolute continuity of all solution measures.

We further define $\alpha^u(t,\omega) = \alpha(t,u(t,\omega),\omega)$, $\beta^u(t,\omega) = \beta(t,x,u(t,\omega),\omega)$.

There now exists for each $k \in Z^+$ and $u \in \mathcal{U}$, functions $\alpha_k^u\colon \Omega_{k-1} \times R^+ \to R$ and $\beta_k^u\colon \Omega_{k-1} \times X \times R^+ \to R$ such that

$$\alpha^u(t,\omega) = \sum_k \alpha_k^u(\omega_{k-1}, t - T_{k-1}(\omega)) I_{(t \in (T_{k-1},T_k])}$$

$$\beta^u(t,x,\omega) = \sum_k \beta_k^u(\omega_{k-1}, x, t - T_{k-1}(\omega)) I_{(t \in (T_{k-1},T_k])}$$

For a given $u \in \mathcal{U}$, a measure P_u is defined on $(\Omega, \mathcal{F}_\infty)$ by the Radon-Nikodym derivative of its restriction to $\mathcal{F}_{T_M}$, $M = 1,2,\ldots$ as

$$\frac{dP_u}{dP} \Big| \mathcal{F}_{T_M} = \prod_{k=1}^{M} L_k(\omega)$$

where

$$L_k(\omega_k) = \alpha_K^u(\omega_{k-1},S_k)\, \beta_k^u(\omega_{k-1},Z_k,S_k)\, \exp\Big[-\int_{(0,S_k]} (\alpha_k^u(\omega_{k-1},s) - 1)$$

$$d\Lambda^{kc}(s)\Big]\, \pi_{S_k}^{k-}\, I(S_k \leqslant d^k)$$

$$\Lambda^{kc}(s) = \Lambda^k(s) - \sum_{y \leqslant s} \Delta\Lambda^k(y)$$

and

$$\pi_t^k = \prod_{s \leqslant t} (1 - \alpha_k^u(\omega_{k-1},s)\Delta\Lambda^k(s))(1 - \Delta\Lambda^k(s))^{-1}$$

2.5 Costs

We now define the cost structure of the game, which takes place over the finite interval $[0,T_f]$.

Definition 2.2: For each player i, $i = 1,2,\ldots,N$, the cost rate is function $c_i\colon [0,T_f] \times U \times \Omega \to R^+$ such that

(i) $c_i(t,u,\omega)$ is an $\mathcal{F}_t$-predictable function of (t,ω) for each $u \in U$.

(ii) There is a positive constant C_4, such that $c_i(t,u,\omega) \leq C_4$ for all $(t,u,\omega) \in [0,T_f] \times U \times \Omega$.

The terminal costs G_{if} are nonnegative $\mathcal{F}_{T_f}$-measurable random variables also bounded by C_4.

Given this, the cost to player i corresponding to $u \in \mathcal{U}$ is

$$J_i(u) = E_u[\int_{(0,T_f]} c_i(s,u_s,\omega)d\Lambda(s,\omega) + G_{if}(\omega)]$$

Remark: As pointed out by in [10], this formulation includes cases where the integral part of the cost function is of the form

$$E_u \int_{(0,T_f]\times X} \kappa(x,s,u_s)\, \tilde{p}^u(ds,dx)$$

where $\tilde{p}(t,A)$ is the compensation for $p(t,A)$ under P_u.

2.6 Value Functions

From the above, the value function for player i given $u^i(\cdot)$ is the process $\{W_{it}\}$ given by

$$W_{it}(u^i) = \bigwedge_{u_i \in \mathcal{U}_i} E_u[\int_{(t,T_f]} c_i(s,u_s)d\Lambda_s + G_{if}|\mathcal{F}_t]$$

It further satisfies the "principle of optimality" ([10], Theorem 4.1):

Theorem 2.1: For any $u_i \in \mathcal{U}_i$, the process $M^u_{it}(u_i) = \int_{(0,t]} c_i(s,u_s)d\Lambda s + W_{it}(u^i)$ is an $(\mathcal{F}_1,P_u)$-submartingale. It is a martingale iff u_i is optimal, given u^i.

2.7 Hamiltonian

We can decompose into

$$(2.1) \qquad M^u_{it}(u_i) = W_{io}(u^i) + N^u_{it}(u^i) + a^u_{it}(u^i)$$

where $\{N^u_{it}(u^i)\}$ is an $\{\mathcal{F}_t, P_u\}$ martingale and $\{a^u_{it}(u^i)\}$ is a predictable increasing process with $a^u_{io}(u^i) = 0$. $N^u_{it}(u^i)$ can then be written as:

$$(2.2) \qquad N^u_{it}(u_i) = \int_{(0,t]\times X} g_i(s,x,u^i)q^u(ds,dx)$$

for some $g_i \in L_1^{loc}(p)$, where $q^u(t,A) = p(t,A) - \tilde{p}^u(t,A)$.

Based on this we define the "Hamiltonian":

$$H_i(t,u,\omega) = c_i(t,u,\omega) + \alpha(t,u,\omega) \int_X g_i(t,x,u^i(t,\omega),\omega)\ \lambda(dx,t,\omega).$$

The idea is now to minimize this in a pointwise fashion.

3. Existence of Nash Equilibrium

We start by giving the conventional definition:

Definition 3.1: $u^* = (u_1^*,\ldots,u_N^*)$ is a Nash equilibrium point if for each i:

$$J_i^*(u^{i*}) = J_i(u_i^*,u^{i*}) \leqslant J(v_i,u^{i*}) \text{ for all } v_i \in \mathcal{U}_i.$$

Remark: In [3], equilibria was looked at where $u_i(t,x)$ only can depend on $u^i(s,x)$, $s \leqslant t$. While this seems to be a natural requirement, it is not the conventional Nash concept.

Further, we define:

Definition 3.2: The Nash condition holds if for each i, there exists a function:

$$u_i^*: [0,T_f] \times X \times \Omega \times U^i \to U_i$$

such that for each $(t,x,\omega,v_i) \in [0,T_f] \times X \times \Omega \times U_i$:

$$H_i(t,u_i^*,u^{i*},\omega) \leq H_i(t,v_i,u^{i*},\omega)$$

Remark: Since u_i^* depends on both past and future values of u^{i*}, the Nash condition is quite complicated and less innocuous than it first appears. Perhaps the dynamic programming methods of [3] can improve on this.

Our main result is now:

Theorem 3.1: If the Nash condition holds, there is a Nash equilibrium point.

Proof: We prove that the controls u_i^* above are optimal in $\mathcal{U}_i$ given u^{i*} for all i, following the proof in [10], (Theorem 4.2).

By (2.1) and (2.2):

$$M_{it}^u(u^i) = J_i^* + \int_{(0,t]\times X} g_i(s,x,v^i)q^u(ds,dx) + a_{it}^u(u^i) \tag{3.1}$$

Letting * stand for u^*:

$$M_{it}^*(u^{i*}) = \int_0^t c_i(s,x,s,u^*)d\Lambda_s + W_i(t)$$

$$= M_{it}^u + \int_0^t (c_{is}^* - c_{is}^u)d\Lambda_s$$

$$= J_i^*(u^{i*}) + \int_{(0,t]\times X} dq^u + a_{it}^u + (c_{is}^* - c_{is}^u)d\Lambda_s$$

$$= J_i^*(u^{i*}) + \int_{(0,t]\times X} g_i dq^* + a_i^*(u^{i*})$$

where

$$a_i^*(u^{i*}) = a_{it}^u(u^{i*}) - \bar{a}_{it}^u(u^{i*}) = a_{it}^u(u^{i*})$$

$$- \int_0^t [c_{is}^u - c_{is}^* + \int_X g_i(\alpha^u\beta^u - \alpha^*\beta^*)d\Lambda_s.$$

By [10], (Theorem 4.1), if $a_{iT_s}^* = 0$ a.s. then u_i^* is optimal and u^* is a Nash equilibrium. Hold u^{i*} constant, this follows from the proof in [10], (p. 219). Since this holds for all $u^*{}_i(u^{i*})$, we see that u^* thus constructed is a Nash equilibrium. Q.E.D.

Remark: It may well be possible to obtain this result under less restrictive assumptions if attention is confined to the Markovian case. In the context of control theory this is, of course, of considerable practical interest. However, in many modern applications of game theory, such restrictions are not natural.

4. Other Solution Concepts

It is easy to extend our main result to other solution concepts. Let us first give the conventional definitions:

Definition 4.1: $u^* \in \mathcal{U}$ is efficient if there is no $u \in \mathcal{U}$ such that $J_i(u) < J_i(u^*)$ for all $i = 1,2,\dots,N$.

Definition 4.2: $u^* \in \mathcal{U}$ is the core if there is no $S \subset \{1,2,\dots,N\}$ and no

$u \in \mathcal{U}$ such that $J_i(u^*_{\bar{S}}, u_S) < J_i(u^*)$ for all $i \in S$, where $u^*_{\bar{S}} = \{u^*_{\bar{S}}, i \in \bar{S}\}$, $u_S = \{u_i, i \in S\}$, and $\bar{S}$ is the complement of S.

<u>Theorem 4.1</u>: There is an efficient point if there exists a nonnegative $\lambda = (\lambda_1, \ldots, \lambda_N) \in R^N$, $\lambda \neq 0$ and a function u^*_i: $[0,T_f] \times \Omega \to U_i$ for all i such that for each $(t,\omega,u) \in [0,T_f] \times \Omega \times \mathcal{U}$:

$$\sum_{i=1}^{N} \lambda_i H_i(t,u^*,\omega) \leq \sum_{i=1}^{N} \lambda_i H_i(t,u,\omega)$$

<u>Proof</u>: Proceeding as in the proof of Theorem 3.1, we can prove that the u^* and the λ above satisfies for each

$$(t,\omega,u) \in [0,T_f] \times \Omega \times \mathcal{U}\text{:} \; \sum_{i=1}^{N} \lambda_i J_i(u^*) \leq \sum_{i=1}^{N} \lambda_i J_i(u).$$

So there is no u^*, such that $J_i(u^*) > J_i(u)$ for all i. Q.E.D.

<u>Theorem 4.2</u>: If the Nash condition holds and for each $S \subset \{1,2,\ldots,N\}$ there exist constant $\lambda^S_i \geq 0$, $i \in S$, not all zero, such that for each

$$(t,\omega,u) \in [0,T_f] \times \Omega \times \mathcal{U}\text{:} \; \sum_{i \in S} \lambda^S_i H_i(t,u^*,\omega) \leq \sum_{i \in S} \lambda^S_i H_i(t,u^*_{\bar{S}},u_S,\omega).$$

Then the core is nonempty.

<u>Proof</u>: Again proceeding as in the proof for Theorem 3.1, we can prove that the u^* and the λ^S above satisfy for each

$$(t,\omega,u) \in [0,T_f] \times \Omega \times \mathcal{U}\text{:} \; \sum_{i \in S} \lambda^S_i J_i(u^*) \leq \lambda^S_i J_i(u).$$

So there is no S and no u such that $(J_i(u^*_{\bar{S}}, y_S) < J_i(u^*)$ for all $i \in S$. Q.E.D.

5. Applications and Extensions

We will confine our discussion to social science applications. In this context, a prime example is matching games where players team up for longer or shorter periods and try to control the switching behavior of each other. The present paper is motivated by a model where firms use prices to influence the brand switching behavior of consumers [11]. Another class of examples, with very significant practical implications, can be found in R&D races between firms in cases where technology follows a jump process. An application on another level is the theory of incentive contracts in cases where a central player (a manager) looks for reward schemes which will induce other players (workers) to maximize the net output of a team production process of the jump category.

As is always the case, these and other examples pose the need for more powerful results. One promising avenue which might allow one to drop the conditions (ii) could be suggested by the technique of [5]. Alternatively, the deterministic piecewise Markov processes of [9] may be sufficiently general to help us in many applications. It should be quite easy to prove existence of equilibrium for games played with such processes. On the other end of the spectrum, the very general processes considered by [7] and [4], seems to have many potential applications (in fact, they were directly motivated by a variation of the brandswitching problem alluded to above). It should finally be pointed out that many applications suggest the desirability of a theory of competitive impulse control. Unfortunately, the author has been completely unsuccessful in his attempts to make progress in this direction. Perhaps the "Dynkin" games of [6] will be a good starting point.

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