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PARTIALLY-REVEALING RATIONAL EXPECTATIONS EQUILIBRIUM IN A COMPETITIVE ECONOMY

by

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Abstract

A class of pure exchange economies is considered in which some agents are informed of private information and others are uninformed. Existence of a _partially-revealing_ rational expectations equilibrium is proved--without restricting attention to particular functional forms for utility functions and probability distributions, without introducing "noise" into the model, and without departing from the standard definition of REE. The equilibrium is _robust_ in the sense that if the primitives of the economy are slightly perturbed, the economy continues to have a qualitatively-similar REE. With somewhat stronger assumptions placed on the economy, the equilibrium is also the _unique_ measurable REE.

1. Introduction

Private information influences public actions and so, conversely, public actions tell something about private information. Hence, when asymmetrically-informed economic agents interact in a competitive market, the common terms of trade reveal some of the participants' knowledge. This leads to the key insight which underlies the microeconomic rational expectations literature: an equilibrium concept which requires agents to exhaust all opportunities to improve their utilities or profits in a market environment should also permit them to take advantage of all information which is contained in price.

Such reasoning is absolutely essential in economic models of asymmetric information, but this reasoning must also be exercised with absolute caution. The rational expectations inference process makes possible circumstances where equilibrium prices reveal <u>all</u> existing knowledge to market participants with asymmetric private information. Such fully-revealing equilibria are valuable in the study of the aggregation of information by markets, and in the formal modeling of situations where we have reason to think that agents possess symmetric information at the time they make their market decisions. However, many real-world economic phenomena appear to hinge critically on the presence of <u>asymmetric information in equilibrium</u>. A fully-revealing rational expectations equilibrium suppresses any such effects; we need a well-developed theory of partially-revealing rational expectations equilibrium to further explore the consequences of asymmetric information in large markets.[1]

The theory of fully-revealing rational expectations equilibrium (REE) is largely complete. There generically exist fully-revealing REE's in competitive economies where the state space is finite (Radner, 1979).[2] Generic existence also holds when, for every agent, the number of dimensions of unknown information is less than the number of prices (Allen, 1981a and 1982). Nongeneric examples of nonexistence of equilibrium have also been provided in the "lower dimensional case" (e.g., Green, 1977; Kreps, 1977). There generically do not exist fully-revealing REE's in competitive economies where the number of dimensions of information unknown to an agent exceeds the number of prices (Jordan, 1983).[3] However, there do generically exist REE's where price "almost fully reveals" private information in the higher dimensional case (Jordan, 1982).

Unfortunately, results concerning <u>partially-revealing</u> rational expectations equilibria have been much more elusive. The following, fundamental question has long been an open problem:

> Is there an economy such that the existence of a partially-revealing REE is robust? To be precise, does there exist an economy possessing an REE with each of two properties: (a) agents possess (nontrivially) asymmetric information in equilibrium; and (b) if the primitives of the economy are slightly perturbed (possibly changing the functional forms), the economy continues to have a qualitatively-similar REE?

For that matter, mere examples of economies with partially-revealing REE's have been few and far between. A much simpler problem, thus, has also remained unanswered: Is there an example of an economy that does not utilize normally-distributed random variables but which, using the standard definition of REE, possesses a partially-revealing equilibrium?

The reason for difficulties in obtaining positive results in this literature is that partial revelation opens a Pandora's box of potential informational discontinuities which thwart traditional fixed-point arguments. We will now briefly review the literature on partially-revealing REE's.

Allen (1981b) constructed a class of economies with two dimensions of information but just one price. The full-information price function is monotone in each coordinate of information. Hence, each of the two agents, privately informed of just one coordinate, can infer the remaining coordinate. Price by itself is partially revealing, but price together with private information is fully revealing. Thus, we have symmetric information in equilibrium. Other articles have achieved asymmetric information, but only at the cost of altering the definition of REE or introducing special features into the model.

These articles can be broadly divided into two subsets. The first line preserves the general framework of the exchange economy, with general utility functions and many consumption goods, but slightly departs from a strict definition of REE. For example, in Allen (1983), Anderson and Sonnenschein (1982, 1985), and Ausubel (1984), agents form "irrational inferences" which come arbitrarily close to the rational expectations inference. In Allen (1985), markets do not clear, but they come arbitrarily close to fully clearing. The noise present in the above devices provides the requisite smoothness to enable proofs of existence via fixed point theorems.

The second line of literature preserves the strict definition of REE, but restricts attention to a special type of securities model where agents value a variety of "assets" only because they are convertible to a single consumption good at a future date. More importantly, utility functions are usually restricted to be exponential, and random variables are always required to be normally distributed. Diamond and Verrecchia (1981), Grossman (1977), Grossman and Stiglitz (1980), Hellwig (1980), Laffont (1985), and Verrecchia (1982) make one or both of these restrictions, and are then able to find closed-form solutions for partially-revealing noisy REE's.[4]

The current article affirmatively answers the open problems posed five paragraphs above. A class of pure exchange economies is constructed for which one can prove the existence of partially-revealing REE's, without imposing specific functional forms on the primitives. The REE price function is comparatively simple,[5] yet leaves agents with nontrivial asymmetric information in equilibrium. For a subclass of the constructed pure exchange economies, uniqueness of the equilibrium among measurable REE's is also demonstrated. Thus, this paper presents a class of economies which are potentially valuable for future modeling purposes.

However, the article makes no pretense of having "solved the rational expectations existence problem." Existence is established for a class of economies, and just that. The assumptions used here are much stronger than the standard conditions for showing existence of a competitive equilibrium in a symmetric information economy. Certainly, future research in this area will yield considerably more general results than are contained in the current paper.

This article consciously avoids "excess baggage" which is unnecessary for partial revelation. In particular, "noise" (i.e., symmetrically-incomplete information) is somewhat peripheral to a model of asymmetric information and is therefore banished from this paper. Instead, models are studied where every piece of information is known by at least one agent. Special functional forms, such as exponential utility or normal distributions, are also shunned. Indeed, this paper takes a "minimalist" approach to model-building: it uses the smallest state space (a pair of intervals), the smallest degree of uncertainty in equilibrium by uninformed agents (the state is one of a pair of points), and the smallest number of goods (two) which make an affirmative answer to the fundamental problem possible. Of course, analogous constructions can be used in larger models.

The paper is structured as follows. In Section 2, we specify the model and define REE. In Section 3, we solve for equilibria in a set of examples. In Section 4, we prove the main existence result for a class of economies where informed agents possess full information. (Existence and uniqueness of a solution to the fundamental differential equation is shown in the Appendix.) In Section 5, we discuss the robustness of partially-revealing REE. In Section 6, we extend existence to economies where informed agents have only partial private information. In Section 7, we characterize all REE's for a subclass of the economies. In Section 8, we prove the main uniqueness result for this subclass. We conclude in Section 9.

2. The Model: A Noiseless Pure Exchange Economy

We consider a pure exchange economy with asymmetric information but without noise. The state of the world is determined by: a continuous random variable, $\tilde{\beta}$, which is distributed on the unit interval $I \equiv [0,1]$; and a dichotomous random variable, $\tilde{\gamma}$, which takes on elements of $\Gamma = \{H,T\}$ ("heads" or "tails"). The realization, (β,γ), is payoff relevant to agents because it enters into their (state-dependent) utility functions. $(\tilde{\beta},\tilde{\gamma})$ is assumed to possess a joint density function. Let $h(\beta)$ and $t(\beta)$ denote the densities at (β,H) and (β,T), respectively; $h(\bullet)$ and $t(\bullet)$ are assumed to be continuous positive functions on I. We normalize $\int_0^1 [h(\beta) + t(\beta)]d\beta = 1$. Finally, probabilities are described using the triple $(I \times \Gamma, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ-field of subsets whose intersections with $I \times \{H\}$ and $I \times \{T\}$ (projected onto I) are each Borel measurable, and μ is Lebesgue measure.

Agents are divided into two classes, according to their private information. There are J^1 "informed" agents (whose signals, utilities, endowments and demands are subscripted by 1) and J^2 "uninformed" agents (subscripted by 2). Every agent ij ($i = 1,2$; $1 \le j \le J^i$) is exogenously conferred with a private signal $s_{ij}(\beta,\gamma)$ of the state of the world, before the time of trade. Informed agents receive one of three types of signals:

(a) $s_{1j}(\beta,\gamma) = (\beta,\gamma)$;

(b) $s_{1j}(\beta,\gamma) = \beta$;

or

(c) $s_{1j}(\beta,\gamma) = \gamma$.

Uninformed agents receive a null signal:

(d) $s_{2j}(\beta,\gamma) = 0$,

which confers no information, and is normally suppressed from our notation. In different sections of the paper, we will vary among (a), (b) and (c) the signals which informed agents receive. However, throughout, we assume that _some_ agent is informed of the true realization of $\tilde{\beta}$ and _some_ agent is informed of $\tilde{\gamma}$. That is what we mean when we say the economy is "_noiseless_":

if all agents in the economy could pool their information before the time of trade, they would possess complete information.

Unfortunately, agents lack the technology to acquire information directly from other agents (e.g., the possibility of markets for information is excluded). However, agents are permitted to observe equilibrium prices in the commodity market before they trade, and are able to draw appropriate inferences. Moreover, the probability density functions, $h(\cdot)$ and $t(\cdot)$, and the remaining specifications of the model, are common knowledge to all agents in the economy.

There are two commodities, denoted x and y, respectively. Prices are assumed to be nonnegative, and are normalized to sum to one. We usually only explicitly mention the price of the first good, and refer to this by p or ϕ. Agent ij's endowment is denoted by $(\bar{x}_{ij},\bar{y}_{ij}) \in \mathbb{R}^2_{++}$ and agent ij's consumption is denoted by $(x_{ij},y_{ij}) \in \mathbb{R}^2_{++}$, for all $i = 1,2$ and $1 \le j \le J^i$.[6]

Agents have (state-dependent) utility functions, $U_{ij}(x,y;\beta,\gamma)$: $\mathbb{R}^2_{++} \times I \times \Gamma \to \mathbb{R}$, which are assumed to satisfy:

Assumption (A1): U_{ij} are twice continuously differentiable in (x,y,β), for all i $(i = 1,2)$ and all j $(1 \le j \le J^i)$. For every fixed $(\beta,\gamma) \in (0,1) \times \Gamma$, $U_{ij}(\cdot,\cdot;\beta,\gamma)$ is strictly monotone and strictly concave in (x,y), and satisfies the boundary condition. Furthermore:

$(\partial U_{ij}/\partial x)(x,y;\beta,\gamma)$ is monotone increasing in β and

$$(\partial U_{ij}/\partial x)(x,y;0,\gamma) = 0,$$

and

$(\partial U_{ij}/\partial y)(x,y;\beta,\gamma)$ is monotone decreasing in β and

$$(\partial U_{ij}/\partial y)(x,y;1,\gamma) = 0,$$

for all $(x,y) \in \mathbb{R}^2_{++}$ and for all $\gamma \in \Gamma$.

The above conditions require that agents "increasingly enjoy good x (relative to good y)" as β increases.

In the face of incomplete information, agents maximize their expected utilities conditional on their private information and the market price, subject to the budget constraint. That is, agent ij solves:

$$\mathrm{Max}_{(x_{ij},y_{ij})} \; E[U_{ij}(x_{ij},y_{ij};\tilde{\beta},\tilde{\gamma}) \mid s_{ij}(\beta,\gamma),p(\beta,\gamma)]$$

$$\text{subject to } p(\beta,\gamma)x_{ij} + [1 - p(\beta,\gamma)]y_{ij}$$

$$\leq p(\beta,\gamma)\bar{x}_{ij} + [1 - p(\beta,\gamma)]\bar{y}_{ij}.$$

$E[U_{ij}|s_{ij},p]$ is well defined because $U_{ij}(x_{ij},y_{ij};\cdot,\gamma)$ is continuous, and hence bounded, on the compact set I. Furthermore, $E[U_{ij}|s_{ij},p]$ is continuously differentiable, strictly monotone, and strictly concave in (x,y) because each U_{ij} has these properties (except when $\beta = 0$ or 1). Hence, the maximization yields differentiable demand functions $x_{ij}(\cdot)$ for any (fixed) conditional beliefs by agent ij. Finally, all agents are assumed to be price takers.[7]

Let us now define our concept of competitive equilibrium. Let $p(\cdot,\cdot)$ denote a price function, which associates states of the world (β,γ) with prices $p(\beta,\gamma)$ of the first good. (The price of the second good is

$1 - p(\beta,\gamma)$.) Let $E_{ij}[\bullet|p(\beta,\gamma),s_{ij}(\beta,\gamma)]$ denote agent ij's (correct) conditional expectation operator, conditioning on the observations $p(\tilde{\beta},\tilde{\gamma}) = p(\beta,\gamma)$ and $s_{ij}(\tilde{\beta},\tilde{\gamma}) = s_{ij}(\beta,\gamma)$, and using the joint density functions $h(\bullet)$ and $t(\bullet)$.

<u>Definition 1</u>: A <u>rational expectations equilibrium (REE)</u> is a Borel-measurable price function $p: I \times \Gamma \rightarrow I$ and a vector of demand functions $x_{ij}: I \times \Gamma \rightarrow \mathbb{R}_+$ $(i = 1,2;\ i \leq j \leq J^i)$ such that:

(1) $x_{ij}(\beta,\gamma)$ maximizes $E_{ij}[U_{ij}(x_{ij},y_{ij};\tilde{\beta},\tilde{\gamma})|p(\beta,\gamma),s_{ij}(\beta,\gamma)]$

subject to:

$$p(\beta,\gamma)x_{ij} + [1 - p(\beta,\gamma)]y_{ij} \leq p(\beta,\gamma)\bar{x}_{ij} + [1 - p(\beta,\gamma)]\bar{y}_{ij},$$

for almost all $(\beta,\gamma) \in I \times \Gamma$, all $i = 1,2$, and all j $(1 \leq j \leq J^i)$,

and

(2) $$\Sigma_{i=1}^{2} \Sigma_{j=1}^{J^i} x_{ij}(\beta,\gamma) = \Sigma_{i=1}^{2} \Sigma_{j=1}^{J^i} \bar{x}_{ij},$$

for almost all $(\beta,\gamma) \in I \times \Gamma$.

In some economies, the REE price function reveals all relevant, unknown information to agents. Such a situation, and the associated demands, are now defined:

<u>Definition 2</u>: The <u>fully-informed demand function</u>, $x^*_{ij}: I \times I \times \Gamma \rightarrow \mathbb{R}_+$, of an agent is given by: $x^*_{ij}(\phi;\beta,\gamma) = \arg\max_{x_{ij}} \{U_{ij}(x_{ij},y_{ij};\beta,\gamma)\}$ subject to $\phi x_{ij} + (1 - \phi)y_{ij} \leq \phi\bar{x}_{ij} + (1 - \phi)\bar{y}_{ij} \equiv w_{ij}$. Also define $x^*_i = \Sigma_{j=1}^{J^i} x^*_{ij}$ and $w_i = \Sigma_{j=1}^{J^i} w_{ij}$.

If, in an REE, $x_{ij}[\beta,\gamma] = x^*_{ij}[p(\beta,\gamma);\beta,\gamma]$ almost everywhere for all i and j, we refer to this as a <u>fully-revealing rational expectations equilibrium</u>.

Thus, $x_1^*(\Phi;\beta,\gamma)$ denotes the aggregate demand of informed agents for the first good, when the market price is Φ and informed agents believe that the state of the world is (β,γ) with probability one, etc.

3. <u>A Set of Examples</u>

In this section, we will exhibit a set of pure exchange economies which satisfy the assumptions of Section 2 (and additional assumptions which will be stated in Section 4), and we will solve explicitly for a partially-revealing rational expectations equilibrium. We will also assert that for a particular subset of these economies, the constructed equilibrium is the unique REE, deferring the proof of uniqueness until Section 8.

Two intellectual debts should be acknowledged at this juncture. First, the class of examples below is related to examples analyzed in Radner (1979, pp. 661-665) and Allen (1981a, pp. 1175-1176) to discuss nonexistence of fully-revealing REE. The class of examples is also a somewhat more distant descendant of the nonexistence example of Kreps (1977). Second, the modeling device of two-to-one price functions was first utilized in Jordan (1982), although there it was used to construct REE's which were arbitrarily close to fully revealing.

Consider the following set of examples with one informed agent and one uninformed agent. The informed agent is privately informed of β and γ, i.e., $s_1(\beta,\gamma) = (\beta,\gamma)$. He possesses positive endowments $(\bar{x}_1,\bar{y}_1)$ of the two goods and uses the state-dependent utility function U_1 given by:

$$(3)\quad U_1(x,y;\beta,\gamma) = \begin{cases} \alpha(\beta)\log x + [1-\alpha(\beta)]\log y, & \text{if } \gamma = H, \\ \beta \log x + [1-\beta]\log y, & \text{if } \gamma = T. \end{cases}$$

The uninformed agent receives no private information, i.e., $s_2(\beta,\gamma) = 0$. He possesses positive endowment $(\bar{x}_2,\bar{y}_2)$ and uses the utility function U_2 given by:

$$(4)\quad U_2(x,y;\beta,\gamma) = \beta \log x + [1-\beta]\log y, \text{ for } \gamma = H,T.$$

$\alpha(\bullet)$ is assumed to be a strictly monotone, continuously differentiable function which satisfies $\alpha(0) = 0$, $\alpha(1) = 1$, and $\alpha(\beta) \neq \beta$ for all $\beta \in (0,1)$.

We immediately have the following proposition, which assures that these examples are interesting.

Proposition 1: There does not exist a fully-revealing REE of these economies.

Proof: Suppose, to the contrary, that there does exist one. As in Definition 2, let $w_i = p\bar{x}_i + (1-p)\bar{y}_i$ denote the wealth of agent i. The demand function of the informed agent is given by:

$$(5)\qquad x_1^*(\phi;\beta,\gamma) = \begin{cases} \alpha(\beta)w_1/\phi, & \text{if } \gamma = H, \\ \beta w_1/\phi, & \text{if } \gamma = T. \end{cases}$$

Meanwhile the fully-informed demand function of the uninformed agent is: $x_2^*(\phi;\beta,\gamma) = \beta w_2/\phi$, for γ = H or T. For any state (β,H), where $\beta \in (0,1)$, equilibrium requires $x_1^* + x_2^* \equiv \bar{x}_1 + \bar{x}_2$. Hence, $(1-p)/p = [(1-\alpha(\beta))\bar{x}_1 +$

$(1 - \beta)\bar{x}_2]/[\alpha(\beta)\bar{y}_1 + \beta\bar{y}_2]$ yields an explicit, unique value for $p(\beta,H)$.

Now define $\beta' = [w_1/(w_1 + w_2)]\alpha(\beta) + [w_2/(w_1 + w_2)]\beta$, where w_1 and w_2 are evaluated using $p(\beta,H)$. Observe that $\beta' \in (0,1)$ and $\beta' \neq \beta$. As above, it is easy to see that markets clear in state (β',T) if and only if $p(\beta',T) = p(\beta,H)$.

But then the uninformed agent displays different demands in states (β,H) and (β',T), despite observing equal prices, yielding a contradiction. [].

Knowing that there is no fully-revealing REE, we will now construct a "pairwise revealing" REE: the price function will be a two-to-one mapping from states of the world to prices. The uninformed agent, upon observing the equilibrium price, infers that (β,γ) is one of two possible states of the world; the informed agent obviously still uses his fully-informed demand function. Observe from (5) that $x_1^*(\phi;\beta,H) = x_1^*(\phi;\alpha(\beta),T)$ for all $\phi,\beta \in I$. Thus it is natural to seek a pairwise-revealing REE where $p(\beta,H) = p(\alpha(\beta),T)$ for all $\beta \in I$; then $x_1^*(p(\beta,H);\beta,H) = x_1^*(p(\alpha(\beta),T);\alpha(\beta),T)$ for all $\beta \in I$, and the uninformed agent cannot distinguish between (β,H) and $(\alpha(\beta),T)$ by either price or by his own demands. (In Section 7, we will establish conditions guaranteeing that any pairwise-revealing REE associates (β,H) and $(\alpha(\beta),T)$.)

Suppose an equilibrium of the posited form. We will next derive an expression for the probability which the uninformed agent attaches to heads, given that the state is either (β,H) or $(\alpha(\beta),T)$:

$$(6) \qquad \pi(\beta) \equiv \text{Probability } [(\tilde{\beta},\tilde{\gamma}) = (\beta,H) \mid p(\tilde{\beta},\tilde{\gamma}) = p(\beta,H)].$$

Let us say it was known that $\Phi_0 \le p(\tilde{\beta},\tilde{\gamma}) \le \Phi_1$, where $\Phi_0 = p(\beta_0,H)$ and $\Phi_1 = p(\beta_1,H)$. It could be inferred that $(\tilde{\beta},\tilde{\gamma}) \in \Omega_H(\beta_1) \equiv \{(\beta,H): \beta_0 \le \beta \le \beta_1\}$ or $(\tilde{\beta},\tilde{\gamma}) \in \Omega_T(\beta_1) \equiv \{(\beta,T): \alpha(\beta_0) \le \beta \le \alpha(\beta_1)\}$. Then:

$$\pi(\beta_0) = \lim_{\beta_1 \downarrow \beta_0} \frac{\text{Prob}[\Omega_H(\beta_1)]}{\text{Prob}[\Omega_H(\beta_1)] + \text{Prob}[\Omega_T(\beta_1)]} = \lim_{\beta_1 \downarrow \beta_0} \frac{\int_{\beta_0}^{\beta_1} h(\beta)d\beta}{\int_{\beta_0}^{\beta_1} h(\beta)d\beta + \int_{\alpha(\beta_0)}^{\alpha(\beta_1)} t(\beta)d\beta}.$$

Using l'Hopital's rule, differentiating the integrals with respect to β_1, and evaluating at β_0 gives:

$$(7) \qquad \pi(\beta) = \frac{h(\beta)}{h(\beta) + t(\alpha(\beta))\alpha'(\beta)}.$$

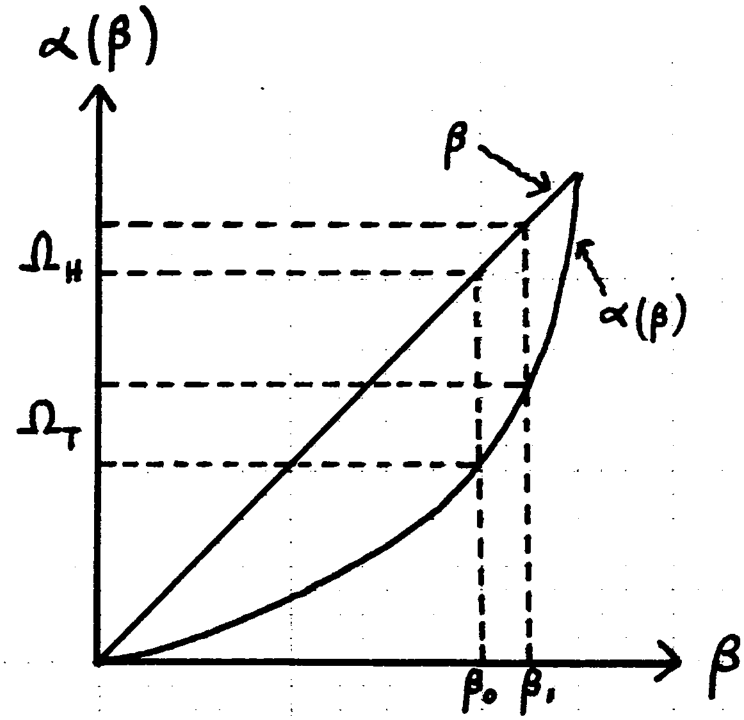
<u>Ceteris paribus</u>, $\pi(\beta)$ is inversely related to $\alpha'(\beta)$; this observation, and equation (7) generally, are also apparent from Figure 1.

(INSERT FIGURE 1 ABOUT HERE)

Let us now define generally the uninformed agent's demand function, given a market price Φ and given he believes that the state is either (β,H) or (β',T), attaching a probability π to the former state.

<u>Definition 3</u>: The <u>pairwise-informed demand function</u> is given by: $x_{2j}^{**}(\Phi;\beta,\beta',\pi) = \arg\max_{x_{2j}} \{\pi U_{2j}(x_{2j},y_{2j};\beta,H) + [1-\pi]U_{2j}(x_{2j},y_{2j};\beta',T)\}$ subject to $\Phi x_{2j} + (1-\Phi)y_{2j} \le \Phi\bar{x}_{2j} + (1-\Phi)\bar{y}_{2j} \equiv w_{2j}$. Also define $x_2^{**} = \Sigma_{j=1}^{J^2} x_{2j}^{**}$.

If, in an REE, uninformed agents' demands equal pairwise-informed



<u>Figure 1</u>

Inference by uninformed agents when $p(\beta,H) = p(\alpha(\beta),T)$.

(As $\alpha'(\beta)$ increases, the conditional probability of T increases.)

demand functions almost everywhere, we refer to this as a pairwise-revealing rational expectations equilibrium.

In this set of examples, $\beta' = \alpha(\beta)$ and π is given by (7); consequently, the uninformed agent's demand in state (β,γ) is given by:

(8) $$x_2^{**}(\phi;\beta,H) = (w_2/\phi)\{\pi(\beta)\beta + [1 - \pi(\beta)]\alpha(\beta)\} = x_2^{**}(\phi;\alpha(\beta),T).$$

Substituting (5) and (8) into $x_1^* + x_2^{**} = \bar{x}_1 + \bar{x}_2$ yields a transformed version of the pairwise-revealing price function $p^{**}(\beta,H)$:

(9) $$\Phi^{**}(\beta,H) \equiv \frac{1 - p^{**}(\beta,H)}{p^{**}(\beta,H)} =$$

$$= \frac{\bar{x}_1\{1 - \alpha(\beta)\} + \bar{x}_2\{1 - \pi(\beta)\beta - [1 - \pi(\beta)]\alpha(\beta)\}}{\bar{y}_1\{\alpha(\beta)\} + \bar{y}_2\{\pi(\beta)\beta + [1 - \pi(\beta)]\alpha(\beta)\}}.$$

Proposition 2: If $\Phi^{**}(\beta,H)$ is strictly monotone in β, then there exists a pairwise-revealing REE of this economy. In this event:

$$p^{**}(\beta,\gamma) = \begin{cases} 1/[1 + \Phi^{**}(\beta,H)], & \text{if } \gamma = H, \\ 1/[1 + \Phi^{**}(\alpha^{-1}(\beta),H)], & \text{if } \gamma = T. \end{cases}$$

Proof: If $\Phi^{**}(\bullet,H)$ is strictly monotone, then price reveals the state to be (β,H) or $(\alpha(\beta),T)$. The uninformed agent thus optimizes via $x_2^{**}(\bullet)$. []

Example 1: Let $\alpha(\beta) = \beta^n$, where $0 < n < \infty$ and $n \neq 1$. (If $n = 1$, there exists a fully-revealing REE.) Let $\bar{x}_1 = \bar{y}_1 = \bar{x}_2 = \bar{y}_2 = 1$, and let

$h(\beta) \equiv 1/2 \equiv t(\beta)$ for all $\beta \in I$. ($\tilde{\beta}$ and $\tilde{\gamma}$ are independent random variables, $\tilde{\beta}$ is uniformly distributed, and $\text{Prob}(\tilde{\gamma} = H) = 1/2 = \text{Prob}(\tilde{\gamma} = T)$.)

Proposition 3: There exist $\underline{n}$ and $\bar{n}$ $(0 < \underline{n} < 1 < \bar{n} < \infty)$ such that, in Example 1, if $\underline{n} \le n \le \bar{n}$, then there exists an REE with the price function:

$$(10)\quad p^{**}(\beta,\gamma) = \begin{cases} \dfrac{(1/2)\beta + (1/2)\beta^n + n\beta^{2n-1}}{1 + n\beta^{n-1}}, & \text{if } \gamma = H, \\[2ex] \dfrac{(1/2)\beta^{1/n} + (1/2)\beta + n\beta^{(2n-1)/n}}{1 + n\beta^{(n-1)/n}}, & \text{if } \gamma = T. \end{cases}$$

If $n < \underline{n}$ or $n > \bar{n}$, then p^{**} given by (10) is not strictly monotone in β.

Remark 1: Numerical calculations yield $\underline{n} \approx .107$ and $\bar{n} \approx 7.36$. If $\underline{n} \le n \le \bar{n}$ and $n \ne 1$, the price function of (10) is (strictly) partially revealing.

Proof of Proposition 3: (9) specializes to (10). The price function (10) yields a consistent REE if and only if $p^{**}(\beta,H)$ is strictly monotone in β. Differentiating (10) yields:

$$(11)\qquad (\partial p^{**}/\partial\beta)(\beta,H) = [1 + (-n^2 + 3n)\mu + (4n^2 - n)\mu^2 + 2n^3\mu^3]/[2(1 + n\mu)^2],$$

where $\mu \equiv \beta^{n-1}$. By inspection, $\underline{n} < 1/4$ and $\bar{n} > 3$; numerical calculations expand the interval where p^{**} is strictly monotone. []

(INSERT FIGURE 2 ABOUT HERE)

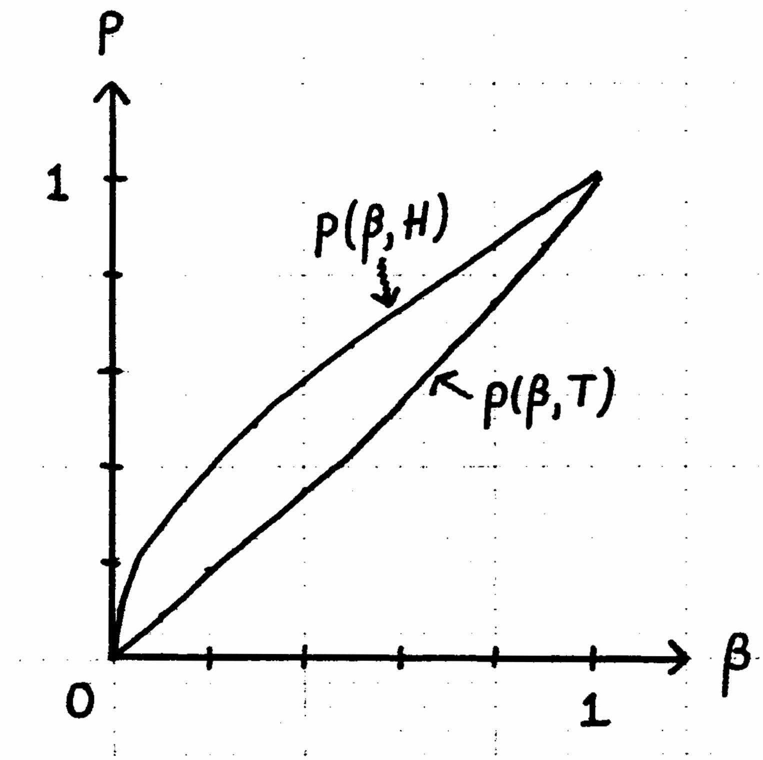


Figure 2

Graph of the REE price function in Example 1, when n = 1/2.

Uniqueness will not be seriously addressed until Section 8. However, let us foreshadow the results of that section with a special case.

Proposition 4: In the economies of Example 1, p** given by (10) is the only possible (measurable) REE price function. Hence:

(a) If $\underline{n} \leq n \leq \bar{n}$, then $p^{**}(\cdot)$ is the unique REE price function; and

(b) If $n < \underline{n}$ or $n > \bar{n}$, there does not exist any REE.

Proof: An application of Theorem 5 in Section 8.

4. Existence of REE When Informed Agents Possess Full Private Information

Thus far in the paper, we have shown that a set of examples satisfying the assumptions of Section 2 possess partially-revealing rational expectations equilibria. In this section, we will show by more general means that a much larger class of economies, which includes Example 1, possesses partially-revealing REE's as well. We will no longer be able to write explicit formulae for the price functions, but the existence proof suggests a way in which solutions can be numerically approximated.

Suppose that informed agents' utility functions satisfy Assumption (A1). We will first show that, in general, there exists a function $A(\beta,\Phi)$ which is analogous to the $\alpha(\beta)$ of the previous section: for any price Φ and state (β,H), there exists $\beta' = A(\beta,\Phi)$ such that aggregate full-information demands of informed agents (defined in Definition 2) are equal in states (β,H) and (β',T). We proceed as follows.

Definition 4: Given Assumption (A1), define the function A: $I \times \bar{I} \to I$ implicitly by:

$$(12) \qquad x_1^*(\phi;A(\beta,\phi),T) = x_1^*(\phi;\beta,H), \text{ for all } (\beta,\phi) \in I \times I,$$

and define A': $I \times \bar{I} \to I$ implicitly by:

$$(12') \qquad x_1^*(\phi;A'(\beta',\phi),H) = x_1^*(\phi;\beta',T), \text{ for all } (\beta',\phi) \in I \times \bar{I}.$$

Observe that it follows from Assumption (A1) that $A(\cdot,\cdot)$ and $A'(\cdot,\cdot)$ are well defined and monotone increasing in the first argument. Moreover, both functions are continuously differentiable on $(0,1) \times (0,1)$; $A_1(\beta,\phi) > 0$ and $A_1'(\beta,\phi) > 0$ whenever $\beta \in (0,1)$ and $\phi \in (0,1)$.

Now let $p(\cdot,\cdot)$ be any function mapping $I \times \Gamma$ into Γ which is differentiable almost everywhere in the first coordinate. Let $z(\beta) \equiv p(\beta,H)$ denote the function $p(\cdot)$ projected on the "heads branch," and let $w(\beta) \equiv p(\beta,T)$. We will call $p(\cdot)$ a candidate pairwise-revealing price function if $z(\cdot)$ and $w(\cdot)$ are each strictly monotone and the range of $z(\cdot)$ and $w(\cdot)$ are both $\bar{I}$. In this section, we will be able to write differential equations in z and w which any candidate pairwise-revealing price function must satisfy in order to be a rational expectations equilibrium. Under additional assumptions, we can further guarantee that there exist unique solutions to the differential equations.

Assume that $p(\cdot,\cdot)$ is a candidate pairwise-revealing price function. Observe that $z^{-1}(\cdot)$ and $w^{-1}(\cdot)$ are well-defined functions mapping $\bar{I} \to I$.

Let Φ be any price in $\bar{I}$. If $p(\bullet,\bullet)$ is an REE price function, then uninformed agents cannot distinguish between states $(z^{-1}(\Phi),H)$ and $(w^{-1}(\Phi),T)$; thus their demands are equal in the two states. Since total endowments are also equal in the two states, market clearance condition (2) implies: $x_1^*(\Phi;z^{-1}(\Phi),H) = x_1^*(\Phi;w^{-1}(\Phi),T)$. Using (12) and (12'), we obtain:

$$(13) \qquad z(\beta) = w[A(\beta,z(\beta))], \text{ for almost all } \beta \in I, \text{ and}$$

$$(13') \qquad w(\beta') = z[A'(\beta',w(\beta'))], \text{ for almost all } \beta' \in I.$$

Let $\dot{z}(\bullet)$ and $\dot{w}(\bullet)$ denote the derivatives of $z(\bullet)$ and $w(\bullet)$, respectively. Differentiating (13) and (13') yields:

$$(14) \qquad \begin{aligned} 1/\dot{w}(\beta') &= A_1(\beta,z(\beta))/\dot{z}(\beta) + A_2(\beta,z(\beta)) \\ 1/\dot{z}(\beta) &= A_1'(\beta',w(\beta'))/\dot{w}(\beta') + A_2'(\beta',w(\beta')), \end{aligned}$$

for almost all $\beta \in \bar{I}$ and $\beta' = w^{-1}(z(\beta))$.

Next, let us derive a general expression analogous to equation (7), where $\pi(\beta)$ (defined in (6)) denotes the probability attached by uninformed agents to heads, given that the state is known to be either (β,H) or (β',T) (since the observed price is $z(\beta) = w(\beta')$). Note that $w^{-1}(z(\beta))$ takes the role of $\alpha(\beta)$ in (7), so by identical reasoning we obtain:

$$(15) \qquad \pi(\beta) = \frac{h(\beta)}{h(\beta) + t(w^{-1}(z(\beta)))\dot{z}(\beta)/\dot{w}(w^{-1}(z(\beta)))} = \frac{h(\beta)/\dot{z}(\beta)}{h(\beta)/\dot{z}(\beta) + t(\beta')/\dot{w}(\beta')}.$$

The above work, in combination with the market clearance condition (2), will yield us a single differential equation in one variable. We have the choice whether to suppress all the w's, yielding a differential equation in z only, or to suppress all the z's, yielding a differential equation in w only. It is illuminating, and useful for the subsequent proof, to pursue both options. Define the function $\Pi(\bullet,\bullet,\bullet)$ on $I \times I \times [0,\infty]$ by:

$$(16)\qquad \Pi(\beta,z,v) = \frac{h(\beta)}{h(\beta) + t[A(\beta,z)]A_1(\beta,z) + t[A(\beta,z)]A_2(\beta,z)v},$$

and let D be the subset of $I \times I \times [0,\infty]$ where $0 < \Pi(\beta,z,v) < 1$. On the domain D, we can define an excess demand function F by:

$$(17)\qquad F(\beta,z,v) = x_1^*(z;\beta,H) + x_2^{**}(z;\beta,A(\beta,z),\Pi(\beta,z,v)) - \bar{x}_1 - \bar{x}_2.$$

Alternatively, we can write a probability function wholly in terms of w; define the function $\Pi'(\bullet,\bullet,\bullet)$ on $I \times I \times [0,\infty]$ by:

$$(16')\qquad \Pi'(\beta',w,u) = \frac{h[A'(\beta',w)]A_1'(\beta',w) + h[A'(\beta',w)]A_2'(\beta',w)u}{t(\beta') + h[A'(\beta',w)]A_1'(\beta',w) + h[A'(\beta',w)]A_2'(\beta',w)u},$$

and let D' be the subset of $I \times I \times [0,\infty]$ where $0 < \Pi'(\beta',w,u) < 1$. On the domain D', we can define an excess demand function G by:

$$(17')\qquad G(\beta',w,u) = x_1^*(w;\beta',T) + x_2^{**}(w;A'(\beta',w),\beta',\Pi'(\beta',w,u)) - \bar{x}_1 - \bar{x}_2.$$

Rational expectations equilibrium requires that excess demands equal zero in equilibrium; Assumption (A1) yields the boundary conditions $z(0) = 0 = w(0)$ and $z(1) = 1 = w(1)$. Together, these give a differential equation in z:

(18) $F(\beta, z(\beta), \dot{z}(\beta)) = 0$ for almost all $\beta \in I$, $z(0) = 0$ and $z(1) = 1$,

and an equivalent differential equation in w:

(18') $G(\beta', w(\beta'), \dot{w}(\beta')) = 0$ for almost all $\beta' \in I$, $w(0) = 0$ and $w(1) = 1$.

It is convenient to observe that every solution $z(\bullet)$ to (18) yields a solution $w(\bullet)$ to (18'), via $w^{-1}(\phi) \equiv A(z^{-1}(\phi), \phi)$; also, every solution $w(\bullet)$ to (18') yields a solution $z(\bullet)$ to (18), via $z^{-1}(\phi) \equiv A'(w^{-1}(\phi), \phi)$. For some subintervals of I, it will be simpler to examine (18); for others it will be simpler to look at (18').

To guarantee the existence and uniqueness of solutions to (18) and (18'), we make two additional assumptions:

<u>Assumption (A2)</u>: $F(\beta, z, v)$ is continuously differentiable almost everywhere on domain D, $F_1 \equiv \partial F/\partial \beta > 0$, and $F_2 \equiv \partial F/\partial z < 0$. Furthermore,[8] F is monotone increasing in β and monotone decreasing in z.

<u>Assumption (A3)</u>: $\partial x_2^{**}(\phi; \beta, A(\beta, \phi), \pi)/\partial \pi < 0$, for $\beta, \phi, \pi \in (0,1)$.

Roughly speaking, $F_1 > 0$ means that excess demand for the first good increases as the state of the world improves, and $F_2 < 0$ means that excess demand decreases as the price goes up. Since these are statements about F, they are only statements about the heads branch of state space. However, it is evident that they will equally apply to the tails branch; in fact, it is easy to show that (A2) implies that G is continuously differentiable, $G_1 > 0$ and $G_2 < 0$.

Assumption (A3) means that, given the true state is either (β,H) or $(A(\beta,\phi),T)$, uninformed agents demand more of the first good as they believe the true state is more likely (β,H). The following results can be established equally well if (A3) is replaced by:

Assumption (A3'): $\partial x_2^{**}(\phi;\beta,A(\beta,\phi),\pi)/\partial\pi > 0$, for $\beta,\phi,\pi \in (0,1)$.

We are now ready to prove:

Theorem 1: Given Assumptions (A1), (A2) and (A3), there exist unique solutions z: $I \to I$ and w: $I \to I$ to differential equations (18) and (18'). Furthermore, $z(\cdot)$ and $w(\cdot)$ are strictly monotone increasing, continuous, and piecewise continuously differentiable.

Proof: See the Appendix.

Finally, let us define a condition on informed agents' information:

Full Private Information (FPI): For all j $(1 \leq j \leq J^1)$, $s_{1j}(\beta,\gamma) = (\beta,\gamma)$.

Theorem 1 immediately implies our first existence theorem on REE's.

Theorem 2: Given an economy satisfying assumptions (A1), (A2) and (A3) in which informed agents possess full private information, there exists a unique pairwise-revealing rational expectations equilibrium.

Proof: Using Theorem 1, define the price function:

$$(19)\qquad p(\beta,\gamma) = \begin{cases} z(\beta), & \text{if } \gamma = H, \\ w(\beta), & \text{if } \gamma = T. \end{cases}$$

Theorem 1 assures that $z^{-1}(\bullet)$ and $w^{-1}(\bullet)$ are well-defined functions on I, so uninformed agents can perform almost everywhere the calculations set forth earlier in this section. We conclude that the equilibrium conditions (1) and (2) are satisfied almost everywhere. []

5. Notes on the Robustness of Partially-Revealing REE

Theorem 2, above, proved the existence of rational expectations equilibrium under Assumptions (A1), (A2) and (A3). In this section, we will briefly clarify what Theorem 2 tells us about one of our original motivating questions: Do there exist economies where the existence of partially-revealing REE is robust?

First, Example 1 demonstrates that the theorem is nonvacuous. Observe that Example 1, with $\underline{n} < n < 1$, satisfies (A1), (A2) and (A3). Similarly, with $1 < n < \bar{n}$, Example 1 satisfies (A1), (A2) and (A3').

Second, Theorem 2, while literally only guaranteeing the existence of "pairwise-revealing" REE's, typically provides us with <u>strictly partially-revealing</u> (almost everywhere) REE's. Whenever, in equation (19), $z(\beta) \neq w(\beta)$ almost everywhere for $\beta \in I$, uninformed agents fail with probability one to attain full information. Moreover, this property--$z(\beta) \neq w(\beta)$ a.e.--may be thought of as a generic property of economies satisfying (A1), (A2) and (A3). For example, when $\underline{n} \leq n \leq \bar{n}$ and $n \neq 1$ in Example 1, $z(\beta) \neq w(\beta)$ except at $\beta = 0$ and $\beta = 1$. Moreover, the failure to attain full information is meaningful: for $\beta \in (0,1)$, uninformed agents would alter their demands if they were provided with full information. Meanwhile, the case $n = 1$ is the nongeneric case of full revelation.

Third, Theorem 2 further provides us with a class of economies where existence of partially-revealing REE is <u>robust</u>. Select any economy (e.g., Example 1 with $\underline{n} < n < 1$) satisfying (A1), (A2) and (A3). Now perturb the utility functions, endowments and probability distributions in such a way that (A1), (A2) and (A3) continue to be satisfied: this is possible because the inequalities required by (A2) and (A3) are <u>strict</u>, so we are never situated at the boundary of the permissible region of economies.[9] (Importantly, the perturbation argument enables us to depart from any specific functional form in the primitives of the economy.) Then Theorem 2 equally establishes existence of a pairwise-revealing REE in the perturbed economy. As noted in the previous paragraph, this REE is typically partially revealing (almost everywhere).

Fourth and last, Theorem 2 only establishes existence and uniqueness within the set of "pairwise-revealing REE's." This obviously also establishes existence within the superset of all REE's, but does not establish uniqueness within the superset. Given Theorem 2, it remains a logical possibility that there exist other REE's in which the set of states associated with a given price has a cardinality other than two. We postpone this defect in the results until Sections 7 and 8: there we make additional assumptions above and beyond (A1), (A2) and (A3), but we are then able to establish uniqueness within the set of all REE's.

6. Existence of REE When Informed Agents Possess Partial Private Information

The standard technique for showing the existence of fully-revealing rational expectations equilibrium is to consider the "artificial economy where each trader has all the economy's information" (Grossman, 1978). One produces an equilibrium price function of this artificial economy and then demonstrates that every trader can infer full information from price in this equilibrium. Finally, one reinterprets this price function as an REE of the actual economy. In this section, we use an analogous method to construct partially-revealing rational expectations equilibria in our model, when the informed agents are conferred with only partial private information.

Our technique here is merely to consider the artificial economy where informed agents have full private information. In Theorem 2, we proved the existence of rational expectations equilibrium for the artificial economy, provided that certain conditions were met. It will be easy to see that this equilibrium can be reinterpreted as an REE of the actual economy with partial private information.

Suppose a private information structure where every informed agent possesses <u>some</u> private information, and every piece of information is known by <u>some</u> informed agent. That is, when the true state is (β,γ), every informed agent privately knows at least that $\tilde{\beta} = \beta$ or $\tilde{\gamma} = \gamma$; and β and γ are each privately known by at least one informed agent. In the notation of previous sections, we define:

<u>Partial Private Information (PPI)</u>: (a) For all j $(1 \leq j \leq J^1)$, $s_{1j}(\beta,\gamma) = \beta$ or $s_{1j}(\beta,\gamma) = \gamma$ or $s_{1j}(\beta,\gamma) = (\beta,\gamma)$; (b) there exists j $(1 \leq j \leq j^1)$ such that $s_{1j}(\beta,\gamma) = \beta$ or $s_{1j}(\beta,\gamma) = (\beta,\gamma)$; (c) there exists j $(1 \leq j \leq J^1)$ such that $s_{1j}(\beta,\gamma) = \gamma$ or $s_{1j}(\beta,\gamma) = (\beta,\gamma)$.

We immediately have:

<u>Theorem 3</u>: Consider any economy which satisfies (A1), (A2) and (A3), and in which informed agents possess partial private information (PPI). Let $p(\bullet,\bullet)$ be given by (19), for the economy which is identical except that informed agents possess full private information (FPI).

If $p(\beta,H) \neq p(\beta,T)$ for almost every $\beta \in I$, then $p(\bullet,\bullet)$ from the FPI economy is also a strictly partially-revealing REE price function for the PPI economy.

<u>Proof</u>: Suppose that agent 1j $(1 \leq j \leq J^1)$ of the PPI economy is privately informed of $\tilde{\beta}$. Since $p(\beta,H) \neq p(\beta,T)$ a.e., he can infer $\tilde{\gamma}$ in almost all states of the world. Suppose that agent 1k $(1 \leq k \leq J^1)$ of the PPI economy is privately informed of $\tilde{\gamma}$. Since Theorem 1 assured that $z(\bullet)$ and $w(\bullet)$ of (19) are strictly monotone, he can infer $\tilde{\beta}$ in all states of the world. []

Observe that we do not use parts (b) and (c) of the Partial Private

Information assumption in our proof. However, if $\tilde{\beta}$ or $\tilde{\gamma}$ is not privately known by any agent, the constructed REE is somewhat nonsensical--the information appeared in the price function from nowhere! Also recall that "$p(\beta,H) \neq p(\beta,T)$ a.e." may be thought of as a generic property of economies satisfying (A1), (A2), (A3) and (FPI). In particular, it was satisfied by Example 1 for $n \neq 1$, suggesting a second example:

Example 2: Modify Example 1 solely by replacing the one (fully) informed agent with two (strictly partially) informed agents (subscripted "11" and "12"). The first informed agent receives a private signal $s_{11}(\beta,\gamma) = \beta$ and a positive endowment $(\bar{x}_{11},\bar{y}_{11})$. The second informed agent receives a private signal $s_{12}(\beta,\gamma) = \gamma$ and a positive endowment $(\bar{x}_{12},\bar{y}_{12})$, where $\bar{x}_{11} + \bar{x}_{12} = 1 = \bar{y}_{11} + \bar{y}_{12}$. Each uses the utility function (3) with $\alpha(\beta) = \beta^n$.

Proposition 5: For the same $\underline{n}$ and $\bar{n}$ of Proposition 3 and $\underline{n} \leq n \leq \bar{n}$, each economy of Example 2 has an REE with the price function (10).

7. Characterization of REE When the Pairing is Independent of Price

In this and the next section, we will impose an intermediate restriction on economies which allows us to characterize all equilibria and sometimes establish uniqueness. It is "intermediate" in the sense that it is a stronger restriction than Assumptions (A1)-(A3), but a weaker restriction than the functional forms (3) and (4) which were imposed on utility functions in the economies of Section 3.

As we recall, Assumption (A1) led to the derivation of a function

$A(\beta,\phi)$ such that for any $\beta \in I$ and $\phi \in I$, informed agents had identical aggregate demands in states (β,H) and $(A(\beta,\phi),T)$, when the price was ϕ. The examples, however, satisfied the stronger property that $A(\beta,\phi)$ only depended on β; we could then replace $A(\beta,\phi)$ by a function, denoted $\alpha(\beta)$, of the single variable β. We make this property the basis for a new assumption:

Price-Independent Pairing (PIP)

There exists a strictly monotone C^1 function α: $I \rightarrow I$ such that $\alpha(0) = 0$, $\alpha(1) = 1$, $\alpha(\beta) \neq \beta$ for almost every $\beta \in (0,1)$, and:

$$x_1^*(\phi;\beta,H) = x_1^*(\phi;\alpha(\beta),T), \text{ for all } \phi \in I \text{ and } \beta \in I. \tag{20}$$

In addition, it is assumed that for fixed $\phi \in I$, $x_1^*(\phi;\beta,H)$ is a continuous and strictly monotone function of β.

Throughout this and the next section we also revert to assuming Full Private Information (FPI). We now proceed to characterize rational expectations equilibrium under these two assumptions.

Lemma 1: Assume (PIP) and (FPI). Let $p(\bullet,\bullet)$ be any REE price function. Then for almost all $\beta_1,\beta_2 \in I$ and $\gamma \in \Gamma$, $p(\beta_1,\gamma) = p(\beta_2,\gamma)$ only if $\beta_1 = \beta_2$.

Proof: Suppose $p(\beta_1,\gamma) = p(\beta_2,\gamma) \equiv \phi$. Observe that uninformed agents cannot distinguish between the two states (β_1,γ) and (β_2,γ); therefore $x_2(\phi;\beta_1,\gamma) = x_2(\phi;\beta_2,\gamma)$. By market clearance equation (2) and the fact that aggregate endowments are equal in the two states, informed agents must also have equal demands. Using (FPI), this implies $x_1^*(\phi;\beta_1,\gamma) = x_1^*(\phi;\beta_2,\gamma)$, contradicting (PIP) unless $\beta_1 = \beta_2$. []

Lemma 2: Assume (PIP) and (FPI). Let $p(\bullet,\bullet)$ be any REE price function. Then for almost all $\beta_1,\beta_2 \in I$, $p(\beta_1,H) = p(\beta_2,T)$ only if $\beta_2 = \alpha(\beta_1)$.

Proof: Suppose $p(\beta_1,H) = p(\beta_2,T) \equiv \phi$. As in the proof of Lemma 1, $x_1^*(\phi;\beta_1,H) = x_1^*(\phi;\beta_2,T)$, contradicting (PIP) unless $\beta_2 = \alpha(\beta_1)$. []

Lemmas 1 and 2 suggest that not more than two states can be associated with any price. We make this precise in Lemma 3. For every $\phi \in$ Range p, let $\#p^{-1}(\phi)$ denote the number of elements of $p^{-1}(\phi)$. We also introduce:

Notation 1: For any $(\beta,\gamma) \in I \times \Gamma$, define:

$$\text{twin}(\beta,\gamma) = \begin{cases} (\alpha(\beta),T), & \text{if } \gamma = H, \\ (\alpha^{-1}(\beta),H), & \text{if } \gamma = T. \end{cases}$$

Lemma 3: Assume (PIP) and (FPI). Let $p(\bullet,\bullet)$ be any price function. Then, for almost every $\phi \in$ Range p, $\#p^{-1}(\phi) \leq 2$.

If $\#p^{-1}(\phi) = 2$, there exists $\beta \in I$ such that $p(\beta,H) = \phi = p(\text{twin}(\beta,H))$. If $\#p^{-1}(\phi) = 1$, define $(\beta,\gamma) = p^{-1}(\phi)$ and $\phi' \equiv p(\text{twin}(\beta,\gamma))$. Then $\#p^{-1}(\phi')$ also equals 1.

Proof: Let $p(\beta_1,\gamma_1) = p(\beta_2,\gamma_2) = p(\beta_3,\gamma_3) = \phi$, where $(\beta_1,\gamma_1) \neq (\beta_2,\gamma_2) \neq$

(β_3,γ_3). By Lemma 1, $\gamma_1 \neq \gamma_2 \neq \gamma_3$. Since Γ has only two elements, $\gamma_1 = \gamma_3$, and again by Lemma 1, $\beta_1 = \beta_3$. We conclude that $\#p^{-1}(\phi) \leq 2$.

Suppose $\#p^{-1}(\phi) = 1$ and $\#p^{-1}(\phi') = 2$. Then there exists $(\beta',\gamma') \in I \times \Gamma$ such that $(\beta',\gamma') \neq \text{twin}(\beta,\gamma)$ and $p(\beta',\gamma') = \phi'$. Lemma 1 implies $\gamma' = \gamma$. Lemma 2 then implies that $(\beta',\gamma') = \text{twin}(\text{twin}(\beta,\gamma)) = (\beta,\gamma)$, since twin composed with itself is the identity. Therefore, $\phi' = p(\text{twin}(\beta,\gamma)) = \phi$, contradicting $\#p^{-1}(\phi) = 1$. We conclude that $\#p^{-1}(\phi')$ also equals 1. []

<u>Definition 5</u>: We define the <u>fully-revealing price correspondence</u> by:

$$p^*(\beta,\gamma) = \{\phi \in I: x_1^*(\phi;\beta,\gamma) + x_2^*(\phi;\beta,\gamma) = \bar{x}_1 + \bar{x}_2\},$$

and the <u>pairwise-revealing price correspondence</u> by:

$$p^{**}(\beta,\gamma) = \{\phi \in I: x_1^*(\phi;\beta,\gamma) + x_2^{**}(\phi;\beta,\gamma) = \bar{x}_1 + \bar{x}_2\},$$

where x_i^* was defined in Definition 2. $x_2^{**}(\phi;\beta,H)$ is shorthand for $x_2^{**}(\phi;\beta,\alpha(\beta),\pi(\beta))$ of Definition 3, using equation (7), and $x_2^{**}(\phi;\beta,T) = x_2^{**}(\phi;\text{twin}(\beta,T))$.

Observe that, under (A1) and the assumption that α is C^1 (contained in PIP), p^* and p^{**} can be shown to be nonempty-valued for all $(\beta,\gamma) \in I \times \Gamma$. Alternate conditions may also be used to assure nonemptiness of these price correspondences. We now state and prove the characterization theorem:

Theorem 4 (Characterization of REE): Let informed agents possess full private information, and make the price-independent pairing assumption. Then Borel measurable $p(\bullet,\bullet)$ is a rational expectations equilibrium price function if and only if:

(i) $\#p^{-1}(p(\beta,\gamma)) \leq 2$ for almost every $(\beta,\gamma) \in I \times \Gamma$;

(ii) $\#p^{-1}(p(\beta,\gamma)) = 1$ implies $p(\beta,\gamma) \in p^*(\beta,\gamma)$, a.e.;

and

(iii) $\#p^{-1}(p(\beta,\gamma)) = 2$ implies $p(\beta,\gamma) = p(\text{twin}(\beta,\gamma))$ and $p(\beta,\gamma) \in p^{**}(\beta,\gamma)$, a.e. (Lebesgue measure) in $I \times \Gamma$.

Proof: Let $p(\bullet,\bullet)$ be any (Borel measurable) REE price function. Let $(\beta,\gamma) \in I \times \Gamma$ and define $\phi = p(\beta,\gamma)$. By Lemma 3, $\#p^{-1}(\phi) \leq 2$ almost everywhere. Suppose that $\#p^{-1}(\phi) = 1$. Then an uninformed agent can infer that the state of the world is (β,γ), and so all agents demand $x^*_{ij}(\phi;\beta,\gamma)$. In order for market clearance equation (2) to be satisfied, we must have $\phi \in p^*(\beta,\gamma)$ a.e.

Suppose that $\#p^{-1}(\phi) = 2$. Without loss of generality, let $\gamma = H$. By Lemma 3, uninformed agents infer that the state of the world is either (β,H) or $(\alpha(\beta),T)$. We shall now argue that uninformed agents assign the conditional probability on (β,H) given by equation (7).

Define $\Omega_H = \{(\beta,H): p(\beta,H) = p(\alpha(\beta),T)\}$ and define $\Omega_T = \{(\alpha(\beta),T): (\beta,H) \in \Omega_H\} = \{(\alpha(\beta),T): p(\beta,H) = p(\alpha(\beta),T)\}$. Recall that p and α were defined to be Borel measurable. Hence, $f(\beta,H) \equiv p(\beta,H) - p(\alpha(\beta),T)$ is Borel measurable; consequently, Ω_H $(= f^{-1}(0))$ and Ω_T are Borel sets.

Using the change-of-variables formula (see Rudin, 1974, pp. 185-186):

$$\text{Probability } (\Omega_T) = \int_{\Omega_T} t(\beta)d\beta = \int_{\Omega_H} t(\alpha(\beta))\alpha'(\beta)d\beta,$$

whereas Probability $(\Omega_H) = \int_{\Omega_H} h(\beta)d\beta$. Consequently, the conditional probability of (β,H), given (β,H) or $(\alpha(\beta),T)$, equals the right side of equation (7), for almost every $(\beta,H) \in \Omega_H$.

Hence, the uninformed agent demands $x^{**}_{2j}(p(\beta,\gamma);\beta,\gamma)$ for almost every (β,γ) such that $\#p^{-1}(p(\beta,\gamma)) = 2$. Again, the informed agent (by FPI) demands $x^{*}_{1j}(p(\beta,\gamma);\beta,\gamma)$. In order for market clearance (2) to be satisfied, we must have $p(\beta,\gamma) \in p^{**}(\beta,\gamma)$ almost everywhere.

Conversely, let $p(\beta,\gamma)$ be any Borel measurable function satisfying (i), (ii) and (iii). Assign $x_{1j}(\beta,\gamma) = x^{*}_{1j}[p(\beta,\gamma);\beta,\gamma]$, for all $(\beta,\gamma) \in I \times \Gamma$ and all j $(1 \le j \le J^1)$. Let $R = \{(\beta,\gamma)\colon \#p^{-1}(p(\beta,\gamma)) = 1\}$. Assign $x_{2j}(\beta,\gamma) = x^{*}_{2j}[p(\beta,\gamma);\beta,\gamma]$, for all $(\beta,\gamma) \in R$, and $x_{2j}(\beta,\gamma) = x^{**}_{2j}[p(\beta,\gamma);\beta,\gamma]$, for all $(\beta,\gamma) \notin R$ (but $\in I \times \Gamma$), for all j $(1 \le j \le J^2)$. Then p coupled with $\{x_{ij}\}$ constitutes an REE. []

8. Uniqueness of REE When the Pairing is Independent of Price

Multiplicity of equilibrium always provides us with conceptual difficulties in general equilibrium and game theory. Most obvious is the predictive question of which equilibrium agents play; more subtle is the issue of how agents coordinate on a single equilibrium when multiple possibilities are available. These problems become especially acute when our equilibrium notion requires agents to form inferences from the equilibrium outcome. For example, if there exist two different REE price functions, p_1 and p_2, observation of a market price ϕ leads to two possible inferences $(p_1^{-1}(\phi)$ and $p_2^{-1}(\phi))$ which are typically different.

Theorem 2 guaranteed the existence of a rational expectations equilibrium, and Theorem 4 characterized all REE's under the additional assumption of price-independent pairings. Unfortunately, neither theorem ensured uniqueness of REE. In this section, we tighten our assumptions further and obtain a uniqueness theorem. First, we require that the correspondences p^* and p^{**} of Definition 5 are single-valued, in order to eliminate an obvious source of nonuniqueness apparent in the statement of Theorem 4. However, a second possibility for nonuniqueness remains in our characterization of REE--for every $\beta \in I$, the set of price(s) associated with states (β,H) and $(\alpha(\beta),T)$ may be <u>either</u> the singleton $\{p^{**}(\beta,H)\}$ <u>or</u> the doubleton $\{p^*(\beta,H),\ p^*(\alpha(\beta),T)\}$. We eradicate the second potential source of nonuniqueness with a novel argument. Suppose that it can be shown that the doubleton "<u>takes up more space</u>" than the singleton, for all sets of β's. If the range of p^* is contained in the range of p^{**}, then p^{**} is the <u>unique</u> REE price function, since values from p^* "cannot fit" in the price space. We have:

<u>Theorem 5</u> (Uniqueness of REE): Assume (A1), (PIP) and (FPI). Further suppose that: p^* and p^{**} are single-valued, piecewise absolutely continuous (in β) functions; $p^{**}(\bullet,H)$ is one-to-one on I; range $p^* \subset$ range p^{**}; and:

$$(21)\qquad |(\partial p^*/\partial\beta)(\beta,H)| + |(\partial p^*/\partial\beta)(\alpha(\beta),T)|\ \alpha'(\beta) > |(\partial p^{**}/\partial\beta)(\beta,H)|,$$

for almost every (Lebesgue measure) $\beta \in I$.

Then $p(\bullet,\bullet)$ is a (Borel measurable) REE price function if and only if $p = p^{**}$ for almost every (Lebesgue measure) $(\beta,\gamma) \in I \times \Gamma$.

<u>Remark 2</u>: Some of the hypotheses of Theorem 5 are stronger than necessary, in order to facilitate readability. For example, we do not really need p^* to be single valued; it is sufficient for inequality (21) to hold almost everywhere on each branch of correspondence p^*.

<u>Proof of Theorem 5</u>: First observe that if $p^*(\bullet,H)$ and $p^{**}(\bullet,H)$ are absolutely continuous on $[b_1,b_2]$ (where $b_1 < b_2$), then their partial derivatives with respect to β are defined almost everywhere on that interval and $p^*(b,H) = p^*(b_1,H) + \int_{b_1}^{b} (\partial p^*(\beta,H)/\partial\beta)d\beta$ for all $b \in [b_1,b_2]$, etc. Analogous statements hold for $p^*(\bullet,T)$. Consequently, for any Lebesgue measurable subset $B \subset I$:

$$(22)\qquad m\{p^*(\beta,H):\ \beta \in B\} = \int_B |\partial p^*(\beta,H)/\partial\beta|d\beta$$

$$(23)\qquad m\{p^{**}(\beta,H):\ \beta \in B\} = \int_B |\partial p^{**}(\beta,H)/\partial\beta|d\beta,$$

and, by the change-of-variables formula (see Rudin, 1974, pp. 185-186):

$$(24)\qquad m\{p^*(\alpha(\beta),T):\ \beta \in B\} = \int_B |\partial p^*(\alpha(\beta),T)/\partial\alpha(\beta)|\alpha'(\beta)d\beta.$$

Furthermore, if p^* is one-to-one on $\{(\beta,H):\ \beta \in B\} \cup \{(\alpha(\beta),T):\ \beta \in B\}$, then (21), (22), (23) and (24) imply:

$$(25)\qquad m(\{p^*(\beta,H):\ \beta \in B\} \cup \{p^*(\alpha(\beta),T):\ \beta \in B\}) > m\{p^{**}(\beta,H):\ \beta \in B\},$$

provided $m(B) > 0$.

Now let $p(\cdot,\cdot)$ be any (Borel measurable) rational expectations equilibrium price function. Define $R_H = \{\beta \in I: p(\beta,H) \neq p^{**}(\beta,H)\}$ and $R_T = \{\beta \in I: p(\beta,T) \neq p^{**}(\beta,T)\}$. Observe that R_H and R_T are measurable sets since p^{**} is a measurable function. We will now demonstrate that $m(R_H) + m(R_T) = 0$, proving the "only if" part of the theorem.

Suppose not. Theorem 4 assures us that $p(\beta,\gamma) = p^{**}(\beta,\gamma)$ or $p(\beta,\gamma) = p^{*}(\beta,\gamma)$ almost everywhere, and if $p(\beta,H) = p^{*}(\beta,H)$ then $p(\alpha(\beta),T) = p^{*}(\alpha(\beta),T)$ almost everywhere. Hence, there exist measurable $S_H \subset R_H$ and $S_T \subset R_T$ such that $m(S_H) = m(R_H)$, $m(S_T) = m(R_T)$, $S_T = \alpha(S_H)$, $p(\beta,H) = p^{*}(\beta,H)$ for all $\beta \in S_H$, and $p(\beta,T) = p^{*}(\beta,T)$ for all $\beta \in S_T$. Define $Q_H^{*} = \{p^{*}(\beta,H): \beta \in S_H\}$ and $Q_T^{*} = \{p^{*}(\beta,T): \beta \in S_T\}$. Observe, by Theorem 4, that $m(Q_H^{*} \cap Q_T^{*}) = 0$, since at almost every (β,γ) corresponding to a point q in the intersection, $p(\beta,\gamma) = p^{**}(\beta,\gamma)$ because $\#p^{-1}(q) = 2$. Also define $Q^{**} = \{p^{**}(\beta,H): \beta \in S_H\}$; inequality (25) implies:

$$m(Q_H^{*} \cup Q_T^{*}) > m(Q^{**}), \text{ whenever } m(S_H) \neq 0 \neq m(S_T). \tag{26}$$

Finally, define $Q = (Q_H^{*} \cup Q_T^{*}) \sim Q^{**} \equiv \{q \in Q_H^{*} \cup Q_T^{*}: q \notin Q^{**}\}$. Since range $p^{*} \subset$ range p^{**}, we have $Q \subset (\text{range } p^{**}) \sim Q^{**}$. But then, for every $q \in Q$, there exists $\beta_1 \in S_H \cup S_T$, $\beta_2 \in \sim R_H$, $\beta_3 \in \sim R_T$, and $\gamma_1 \in \Gamma$ such that: $p(\beta_1,\gamma_1) = p(\beta_2,H) = p(\beta_3,T) = q$, i.e., $\#p^{-1}(q) = 3$ for all $q \in Q$. By (26), $m(Q) > 0$ unless $m(S_H) = 0 = m(S_T)$, which we must conclude by part (i) of Theorem 4. Thus, we have proved that p is an REE price function only if $p = p^{**}$ almost everywhere.

Conversely, if $p^{**}(\cdot,H)$ is one-to-one on I, then in all states

$(\beta,\gamma) \in I \times \Gamma$, an uninformed agent infers from observing $p^{**}(\beta,\gamma)$ that $(\tilde{\beta},\tilde{\gamma}) = (\beta,\gamma)$ or twin(β,γ). p^{**} was specified in Definition 5 such that if informed agents demand x_1^* and uninformed agents demand x_2^{**}, markets clear. We conclude that if $p = p^{**}$ almost everywhere, then p is an REE price function. []

For an example of Theorem 5 in action, see Example 1 and Proposition 4 in Section 3.

9. Concluding Remarks

Actively pursued in the early 1980s, research on the partially-revealing rational expectations equilibrium has largely stalled in recent years. This article has attempted to provide some impetus for the resumption of work in the area, by establishing a positive answer to the open problem of whether there exist economies in which the existence of partially-revealing REE is robust. Admittedly, the assumptions required here for existence are considerably more restrictive than those we ideally use in general equilibrium theory. However, the present paper has successfully dropped the requirement of specific functional forms which had appeared in all previous articles which preserved the standard definition of REE. It has also successfully removed noise from the asymmetric information economy. Finally, the paper has in effect provided a recipe for constructing relatively simple models with which we may explore the implications of asymmetric information in competitive economies.

One area ripe for this type of analysis is the issue of insider trading. The general question of what ensues after an arbitrary number, n, of traders comes into private possession of inside information is quite complex. When n is small, almost any model is likely to lend itself to a large multiplicity of equilibria.

However, one tractable baseline case stands out: the case where $n = \infty$. There, informed traders may be "information takers" as well as "price takers," and then the (partially-revealing) rational expectations equilibrium, or the game equivalent, becomes an appropriate methodology for studying insider trading. I will pursue this analysis in future work.

One criticism which could be leveled at the REE literature, generally, is that price is <u>not</u> the only endogenous market variable observable by market participants, and hence should <u>not</u> be construed to be the only conduit (for transmitting information) from informed to uninformed agents. Another example of such an endogenous variable is quantity and, in fact, aggregate trading volume is typically well known to market participants. Hence, it is worth observing that in the current model the constructed outcome remains an equilibrium if rational expectations inference is permitted via the aggregate quantity $x_1(\phi;\beta,\gamma)$ demanded by informed agents, as well as through price $p(\beta,\gamma)$. The explanation of this pleasant fact is simple: in our construction, $x_1[p(\beta_1,\gamma_1);\beta_1,\gamma_1] = x_1[p(\beta_2,\gamma_2);\beta_2,\gamma_2]$ whenever $p(\beta_1,\gamma_1) = p(\beta_2,\gamma_2)$. Thus the aggregate quantity demanded by informed agents conveys no information not already transmitted by price.

A second criticism which has been leveled at this line of literature is the possible internal inconsistency of revealing prices and the costly acquisition of information. Grossman and Stiglitz (1980) observed that individuals would not invest positive resources into privately producing information if, in fact, market prices revealed that information costlessly to all agents. Thus, it is interesting to note that the "partial private information" model of Section 6 resolves the alleged paradox in a most satisfying way. Let us make the acquisition of information (knowledge of the true value β or γ) both costly and endogenous, and occur before the trading round. If agents expect the equilibrium of Theorem 3 to ensue, then we can immediately observe that no agent will acquire full private information (as knowledge of β, γ and $p(\beta,\gamma)$ is redundant). However, acquisition of β or γ (compared to no private information) improves attainable utility in the trading round.

It is fairly natural to think--since different individuals are situated differently in an economy--that one set of agents would be conveniently placed to learn β relatively inexpensively, a second set of agents could learn γ relatively inexpensively, and a third group of individuals would find it extremely costly (or not very useful) to acquire any private information. If information acquisition is endogenous, agents of the first type choose to learn β, agents of the second type learn γ, and the remaining individuals go without private information. Thus, we endogenously obtain the pre-trade information configuration which was assumed to be exogenous in Section 6, demonstrating that costly private information and rational expectations can be consistently incorporated into a fairly simple and robust model.

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Notes

1. A fully-revealing equilibrium also eliminates incentives for the private acquisition of information and, thus, may be viewed as internally inconsistent when the acquisition of information is costly. See Grossman and Stiglitz (1980) and the Conclusion.

2. Provided, of course, that standard conditions for the existence of competitive equilibrium are satisfied.

3. When the number of dimensions of unknown information equals the number of prices, it is possible to construct both examples of open sets of economies where existence occurs and examples of open sets of economies where nonexistence occurs (see Jordan and Radner, 1982; Radner, 1982, pp. 967-970; and infra).

4. Apart from the lack of generality in these restrictions, the particular functional forms preclude wealth effects (exponential utilities yield demands for risky assets which are independent of wealth) and create bankruptcy difficulties (when returns are at the tails of normal distributions). This is recognized, for example, by Grossman (1977, footnote 2) and Laffont (1985, footnote 9).

5. The price function possesses desirable continuity and monotonicity properties which make it credible that agents would succeed in carrying out the rational expectations inference exercise.

6. Informed agents' endowments could be made state-dependent, but that is not done in this paper. Uninformed agents' endowments should not be state-dependent, as observing one's own endowment would confer information.

7. The reader may prefer to use the terminology "representative agent" instead of "agent." One can then assume that the economy contains a continuum of identical nonatomic consumers of each type whose endowments integrate to $(\bar{x}_{ij}, \bar{y}_{ij})$. This alternative specification makes the price-taking assumption natural; inclusion of further details seems unnecessary.

8. This does not follow directly from the first sentence of (A2), as D is not necessarily a convex set.

9. In essence, we utilize a C^2 norm on utility functions, a difference norm on endowments, and a supremum norm on the probability density functions $h(\cdot)$ and $t(\cdot)$.

Appendix

Proof of Theorem 1

We will partition the set I into subintervals, demonstrating that there exist unique solutions to differential equations (18) and (18') on each subinterval and that the solutions piece together. Claims 1 through 4 will enable us to define the partition:

Claim 1: $\text{sign}\{F_3(\beta,z,v)\} = \text{sign}\{A_2(\beta,z)\}$, for all $(\beta,z,v) \in D$.

Proof of Claim 1: Let $\pi = \Pi(\beta,z,v)$. By (16): $\partial\pi/\partial v = -t[A(\beta,z)]A_2(\beta,z)\pi^2/h(\beta)$. Consequently, $\text{sign}\{\partial\pi/\partial v\} = -\text{sign}\{A_2(\beta,z)\}$. Differentiating (17), we obtain: $F_3(\beta,z,v) = (\partial x_2^{**}/\partial\pi)(\partial\pi/\partial v)$. Use of (A3) establishes the claim. []

Now define $Y(\beta) = \{(z,v)\colon (\beta,z,v) \in D \text{ and } F(\beta,z,v) = 0\}$ and $Z(\beta) = \{z\colon (z,v) \in Y(\beta) \text{ for some } v \in [0,\infty]\}$.

Claim 2: The function $z_0(\beta)$ defined implicitly by $F(\beta,z_0(\beta),0) = 0$ is well-defined and differentiable with positive derivative for all $\beta \in I$. Consequently, $Z(\beta)$ is nonempty for all $\beta \in I$. Furthermore, $Z(\beta) \subset (0,1)$ whenever $\beta \in (0,1)$.

Proof of Claim 2: By equation (16), $\pi(\beta,z,0) = h(\beta)/\{h(\beta) + t[A(\beta,z)]A_1(\beta,z)\}$ for all $\beta,z \in \bar{I}$. Since $A_1 > 0$ from (A1), and $h(\cdot)$ and $t(\cdot)$ are positive functions, we have $0 < \pi(\beta,z,0) < 1$, implying $(\beta,z,0) \in D$

for all $\beta, z \in \bar{I}$.

Using (A1), note that $F(\beta,0,0) \geq 0$ and $F(\beta,1,0) \leq 0$. Since the line segment from $(\beta,0,0)$ to $(\beta,1,0)$ is entirely contained in D and F is continuous in D, there exists $z_0(\beta) \in I$ such that $F(\beta,z_0(\beta),0) = 0$. (A2) requires that $F_1 > 0$ and $F_2 < 0$, so the implicit function theorem guarantees that $z_0(\bullet)$ is a function and $\partial z_0/\partial\beta = -F_1/F_2 > 0$.

When $\beta \in (0,1)$, (A1) implies $F(\beta,0,0) > 0 > F(\beta,1,0)$, completing the claim. []

Claim 3: Let $\beta \in I$. There exists $z \in Z(\beta)$ such that $A_2(\beta,z) = 0$ if and only if $Z(\beta)$ is the singleton $\{z_0(\beta)\}$.

Proof of Claim 3: For any $\beta \in I$, suppose there exists $z \in Z(\beta)$ such that $A_2(\beta,z) = 0$. Using (16), it follows that $(\beta,z,v) \in D$ for all $v \in [0,\infty]$. Since $z \in Z(\beta)$, we have $F(\beta,z,v) = 0$ for some v; by Claim 1, $F(\beta,z,v) = 0$ for all $v \in [0,\infty]$. But then, by the monotonicity condition in (A2), $F(\beta,z',v) \neq 0$ whenever $(\beta,z',v) \in D$ but $z' \neq z$, proving that $Z(\beta)$ is a singleton.

Conversely, suppose that $Z(\beta)$ is a singleton (and by Claim 2, $Z(\beta) = \{z_0(\beta)\}$), but $A_2[\beta,z_0(\beta)] \neq 0$. Then $F(\beta,z_0(\beta),0) = 0$ but, by Claim 1, $F(\beta,z_0(\beta),v) \neq 0$ for $v \neq 0$. Also note, by (16), that $0 < \Pi(\beta,z_0(\beta),0) < 1$. Since $\Pi(\bullet,\bullet,\bullet)$ is continuous whenever the denominator of (16) is nonzero, there exists $\varepsilon_1 > 0$ such that $0 < \pi(\beta,z,v) < 1$ whenever $z_0(\beta) - \varepsilon_1 < z < z_0(\beta) + \varepsilon_1$ and $0 \leq v < \varepsilon_1$. Since $A(\bullet,\bullet)$ is continuously differentiable (by (A1)), there exists $\varepsilon_2 > 0$ such that $A_2(\beta,z) \neq 0$ whenever $z_0(\beta) - \varepsilon_2 < z < z_0(\beta) + \varepsilon_2$. Let $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$ and define $S_\beta = \{(\beta,z,v)\colon\ z_0(\beta) - \varepsilon < z <$

$z_0(\beta) + \varepsilon$ and $0 \leq v < \varepsilon\}$. Observe that $F_2 \neq 0$ (by (A2)) and $F_3 \neq 0$ (by Claim 1) in S_β. By the implicit function theorem, there exists $(\beta, z', v') \in S_\beta$ such that $F(\beta, z', v') = 0$ but $z' \neq z_0(\beta)$. Then $z' \in Z(\beta)$, contradicting our hypothesis. []

<u>Claim 4</u>: Let $\beta \in I$ and $A_2(\beta, z_0(\beta)) \neq 0$. Then sign $\{A_2(\beta, z)\}$ = sign$\{A_2(\beta, z_0(\beta))\}$ for all $z \in Z(\beta)$.

<u>Proof of Claim 4</u>: Suppose not. Then there exist $(z_1, v_1) \in Y(\beta)$ and $(z_2, v_2) \in Y(\beta)$ such that $A_2(\beta, z_1) > 0 > A_2(\beta, z_2)$. By (A1), $A(\bullet,\bullet)$ is continuously differentiable, so there exists z_3 between z_1 and z_2 such that $A_2(\beta, z_3) = 0$. Observe that for all $v \in [0,\infty]$, $(\beta, z_3, v) \in D$; by Claim 1, $F(\beta, z_3, v_1) = F(\beta, z_3, v_2)$. By the monotonicity assumption in (A2), $F(\beta, z_1, v_1) \neq F(\beta, z_2, v_2)$. But this contradicts $F(\beta, z_1, v_1) = 0 = F(\beta, z_2, v_2)$, which is required by $(z_1, v_1) \in Y(\beta)$ and $(z_2, v_2) \in Y(\beta)$. []

Next, for any $\beta' \in I$, define $Y'(\beta') = \{(w,u)$: $(\beta', w, u) \in D'$ and $G(\beta', w, u) = 0\}$ and $W(\beta') = \{w$: $(w,u) \in Y'(\beta')$ for some $u \in [0,\infty]\}$. The following claims can be proved by analogous arguments.

<u>Claim 1'</u>: sign$\{G_3(\beta', w, u)\}$ = $-$sign$\{A_2'(\beta', w)\}$ for all $(\beta', w, u) \in D'$.

<u>Claim 2'</u>: The function $w_0(\beta')$ defined implicitly by $G(\beta', w_0(\beta'), 0) = 0$ is well-defined and differentiable with positive derivative for all $\beta \in I$. Consequently, $W(\beta')$ is nonempty for all $\beta' \in I$.

<u>Claim 3'</u>: Let $\beta' \in I$. There exists $w \in W(\beta')$ such that $A_2'(\beta',w) = 0$ if and only if $W(\beta')$ is the singleton $\{w_0(\beta')\}$.

<u>Claim 4'</u>: Let $\beta' \in I$ and $A_2'(\beta',w_0(\beta')) \neq 0$. Then sign $\{A_2'(\beta',w)\}$ = sign$\{A_2'(\beta',w_0(\beta'))\}$ for all $w \in W(\beta')$.

We now prove Claim 5, which defines partitions of I into subintervals with convenient properties.

<u>Claim 5</u>: There exist sequences $\{\beta_k\}_{k=m+1}^{n}$ and $\{\beta_k'\}_{k=m+1}^{n}$ (where m and n are extended even integers and m < n) which each partition the interval $\bar{I}$ and satisfy the following four properties. First: $0 = \beta_{m+1} \leq \beta_{m+2} < \beta_{m+3} \leq \beta_{m+4} < \ldots < \beta_{n-1} \leq \beta_n = 1$ and $0 = \beta_{m+1}' \leq \beta_{m+2}' < \beta_{m+3}' \leq \beta_{m+4}' < \ldots < \beta_{n-1}' \leq \beta_n' = 1$. Second, $Z(\beta_k) = W(\beta_k')$ for all k ($m + 1 \leq k \leq n$) and each is the singleton $\{z_0(\beta_k)\}$ (which we abbreviate by $\{z_k\}$). Third, for each <u>even</u> k ($m + 2 \leq k \leq n$), $F_3(\beta,z,v) = 0$ for all $(\beta,z,v) \in D$ such that $\beta \in [\beta_{k-1},\beta_k]$ and $z \in Z(\beta)$; and $G_3(\beta',w,u) = 0$ for all $(\beta',w,u) \in D'$ such that $\beta' \in [\beta_{k-1}',\beta_k']$ and $w \in W(\beta')$. Fourth, for each <u>odd</u> k ($m + 3 \leq k \leq n - 1$), one of two possibilities, (I) or (II), holds. Either:

(I) $F_3(\beta,z,v) > 0$ for all $(\beta,z,v) \in D$ such that $\beta \in (\beta_{k-1},\beta_k)$ and $z \in Z(\beta)$; and $G_3(\beta',w,u) > 0$ for all $(\beta',w,u) \in D'$ such that $\beta' \in (\beta_{k-1}',\beta_k')$ and $w \in W(\beta')$; or

(II) $F_3(\beta,z,v) < 0$ for all $(\beta,z,v) \in D$ such that $\beta \in (\beta_{k-1},\beta_k)$ and $z \in Z(\beta)$; and $G_3(\beta',w,u) < 0$ for all $(\beta',w,u) \in D'$ such that $\beta' \in (\beta_{k-1}',\beta_k')$ and $w \in W(\beta')$.

Proof of Claim 5: Consider the function $H(\beta) \equiv A_2(\beta, z_0(\beta))$, where $z_0(\bullet)$ was defined in Claim 2. Since $z_0(\bullet)$ is continuous and $A(\bullet,\bullet)$ is continuously differentiable, $H(\bullet)$ is also continuous. Consequently, $H^{-1}(0)$ is a (countable) union of disjoint closed intervals, any of which may be single points. Thus, we may write $H^{-1}(0) = \cup_k [\beta_{k-1}, \beta_k]$, where the union is taken over even k ($m + 2 \le k \le n$), and $\beta_i \le \beta_j$ whenever $i < j$.

For each even k ($m + 2 \le k \le n$) and all $\beta \in [\beta_{k-1}, \beta_k]$, observe that $A_2(\beta, z_0(\beta)) = 0$. Claims 1 and 3 assure that $Z(\beta)$ is the singleton $\{z_0(\beta)\}$ and $F_3(\beta, z_0(\beta), v) = 0$ for every $v \in [0,\infty]$.

For each odd k ($m + 3 \le k \le n - 1$), either $A_2(\beta, z_0(\beta)) > 0$ for all $\beta \in (\beta_{k-1}, \beta_k)$ (possibility I), or $A_2(\beta, z_0(\beta)) < 0$ for all $\beta \in (\beta_{k-1}, \beta_k)$ (possibility II). In possibility I, Claims 1 and 4 imply that $F_3(\beta, z, v) > 0$ for all $(\beta, z, v) \in D$ such that $\beta \in (\beta_{k-1}, \beta_k)$ and $z \in Z(\beta)$ (and similarly for possibility II).

Consider the identity $A'[A(\beta,z),z] = \beta$. Differentiating yields $A'_1[A(\beta,z),z] = 1/A_1(\beta,z)$ and $A'_2[A(\beta,z),z] = -A'_1[A(\beta,z),z]A_2(\beta,z)$. Let $\beta' = A[\beta, z_0(\beta)]$, and observe that $w_0(\beta') = z_0(\beta)$, where $w_0(\bullet)$ was defined in Claim 2'. Hence: $\text{sign}\{A'_2[\beta', w_0(\beta')]\} = \text{sign}\{A'_2[A(\beta, z_0(\beta)), z_0(\beta)]\} = -\text{sign}\{A_2[\beta, z_0(\beta)]\}$.

Define $\beta'_k = A(\beta_k, z_k)$ for all k ($m + 1 \le k \le n$). Clearly, then, $W(\beta'_k) = \{z_k\}$. Note that for any $\beta' \in (\beta'_{k-1}, \beta'_k)$, there exists $\beta \in (\beta_{k-1}, \beta_k)$ such that $\beta' = A[\beta, z_0(\beta)]$. By the previous paragraph, by the construction of $\{\beta_k\}_{k=m+1}^{n}$, and by Claims 1, 1', 4, and 4', we conclude that $\text{sign}\{G_3(\beta', w, u)\} = \text{sign}\{F_3(\beta, z, v)\}$ for all $(\beta', w, u) \in D'$ and $(\beta, z, v) \in D$ satisfying $\beta' \in (\beta'_{k-1}, \beta'_k)$, $\beta \in (\beta_{k-1}, \beta_k)$, $w \in W(\beta')$ and $z \in Z(\beta)$. Also, $G_3(\beta'_k, z_k, u) = 0 = F_3(\beta_k, z_k, v)$ for all k ($m + 1 \le k \le n$) and all $u, v \in [0,\infty]$, establishing the remaining parts of the claim. []

Remainder of the Proof: It remains to demonstrate that the differential equations (18) and (18') have unique solutions on each element of the partitions defined in Claim 5. For all even k ($m + 2 \leq k \leq n$), this fact is trivial: the function $z_0(\cdot)$ ($w_o(\cdot)$) is a solution to (18) ((18')) on the interval $[\beta_{k-1},\beta_k]$ ($[\beta'_{k-1},\beta'_k]$) since $F(\beta,z_0(\beta),v) = 0$ for all $v \geq 0$ ($G(\beta',w_0(\beta'),u) = 0$ for all $u \geq 0$) on this interval. This solution is unique since, by Claim 3 (3'), $Z(\beta)$ ($W(\beta)$) is a singleton on this interval.

The proof is more complicated for odd k ($m + 3 \leq k \leq n - 1$). Define:

$\mathcal{B}_I = \{\beta \in [\beta_{k-1},\beta_k]$: k is odd and possibility I holds for $(\beta_{k-1},\beta_k)\}$,

$\mathcal{B}'_{II} = \{\beta' \in [\beta'_{k-1},\beta'_k]$: k is odd and possibility II holds for $(\beta'_{k-1},\beta'_k)\}$,

and define $\mathcal{B}_{II}$ and $\mathcal{B}'_I$ analogously. For every $[\beta_{k-1},\beta_k] \subset \mathcal{B}_I$, we will now argue that there exists a unique solution $z(\cdot)$ to differential equation (18) on $[\beta_{k-1},\beta_k]$ subject to the boundary conditions $z(\beta_{k-1}) = z_{k-1}$ and $z(\beta_k) = z_k$. Since (as observed in the main text after equation (18')) any solution to (18) induces a solution to (18'), and vice-versa, this equally establishes the existence and uniqueness of a solution $w(\cdot)$ to (18') on $\mathcal{B}'_I$ subject to the analogous boundary conditions.

Similarly, for every $[\beta'_{k-1},\beta'_k] \subset \mathcal{B}'_{II}$, a thoroughly analogous argument will demonstrate existence and uniqueness of a solution $w(\cdot)$ to differential equation (18') on $[\beta'_{k-1},\beta'_k]$ subject to the boundary conditions $w(\beta'_{k-1}) = z_{k-1}$ and $w(\beta'_k) = z_k$, and hence existence and uniqueness of a solution $z(\cdot)$ to (18) on $\mathcal{B}_{II}$. We proceed by establishing some additional claims.

Claim 6: Let $\beta \in \mathcal{B}_I$. Then $Z(\beta)$ is a convex set. (Similarly, if $\beta' \in \mathcal{B}'_{II}$, then $W(\beta')$ is a convex set.)

Proof of Claim 6: Suppose $z_1, z_2 \in Z(\beta)$ and $z_1 < z_2$. By definition, there exist v_1, v_2 such that $(\beta, z_1, v_1), (\beta, z_2, v_2) \in D$ and $F(\beta, z_1, v_1) = 0 = F(\beta, z_2, v_2)$.

Let z_3 satisfy $z_1 < z_3 < z_2$. Since $\beta \in \mathcal{B}_I$, $A_2(\beta, z_3) > 0$, and so $(\beta, z_3, v) \in D$ for all $v \in [0,\infty]$. By the monotonicity condition in (A2), $F(\beta, z_3, v_1) < 0 < F(\beta, z_3, v_2)$; since F is continuous, there exists v_3 between v_1 and v_2 such that $F(\beta, z_3, v_3) = 0$. We conclude $z_3 \in Z(\beta)$, establishing convexity.

If $\beta' \in \mathcal{B}'_{II}$, Claim 1' implies that $A'_2(\beta', w) > 0$ for all $w \in W(\beta')$, and so the same argument demonstrates that $W(\beta')$ is convex. $\square$

For $\beta \in [\beta_{k-1}, \beta_k] \subset \mathcal{B}_I$, define $z_{max}(\beta) = \sup\{z \in Z(\beta)\}$ and $z_{min}(\beta) = \inf\{z \in Z(\beta)\}$. By Claim 6, $Z(\beta) = [z_{min}(\beta), z_{max}(\beta)]$; and by Claim 5, $z_{max}(\beta_j) = z_0(\beta_j) = z_{min}(\beta_j)$ for $j = k - 1, k$.

Claim 7: For all $\beta \in [\beta_{k-1}, \beta_k] \subset \mathcal{B}_I$, $z_{max}(\beta)$ and $z_{min}(\beta)$ satisfy $F(\beta, z_{max}(\beta), \infty) = 0$ and $F(\beta, z_{min}(\beta), 0) = 0$, respectively (and so $z_{min}(\beta) \equiv z_0(\beta)$). Furthermore, $z_{max}(\cdot)$ and $z_{min}(\cdot)$ are differentiable on (β_{k-1}, β_k). $\partial z_{max}/\partial\beta$ and $\partial z_{min}/\partial\beta$ each equal $-F_1/F_2$, evaluated at $(\beta, z_{max}(\beta), \infty)$ and $(\beta, z_{min}(\beta), 0)$, respectively.

Proof of Claim 7: Since F is continuous, there exist $v_{max}(\beta), v_{min}(\beta) \in$

$[0,\infty]$ such that $F(\beta, z_{max}(\beta), v_{max}(\beta)) = 0 = F(\beta, z_{min}(\beta), v_{min}(\beta))$. Since $F_2 < 0$ and $F_3 > 0$, $v_{max}(\beta) = \infty$ and $v_{min}(\beta) = 0$, or else $z_{max}(\beta)$ and $z_{min}(\beta)$ would not be extreme values. By the implicit function theorem, $z_{max}(\bullet)$ and $z_{min}(\bullet)$ are differentiable and their derivatives equal $-F_1/F_2$. []

Claim 8: Let $[\beta_{k-1}, \beta_k] \subset \mathcal{B}_I$, and consider the (open) domain X_k of (extended) phase space defined by $X_k = \{(\beta,z): \beta_{k-1} < \beta < \beta_k$ and $z_{min}(\beta) < z < z_{max}(\beta)\}$. Then the equation $F(\beta,z,v) = 0$ implicitly defines a continuously differentiable vector field $v(\beta,z)$ everywhere on X_k. Moreover, $\partial v/\partial\beta = -F_1/F_3 < 0$ and $\partial v/\partial z = -F_2/F_3 > 0$.

Proof of Claim 8: Since $F_3 \neq 0$ everywhere on X_k, the claim follows directly from the implicit function theorem. $[\beta_{k-1}, \beta_k] \subset \mathcal{B}_I$ implies $F_3 > 0$ on X_k and (A2) implies $F_1 > 0$ and $F_2 < 0$, signing the derivatives. []

Claim 9: For any $(\hat{\beta}, \hat{z}) \in X_k$ and for a sufficiently small neighborhood of $\hat{\beta}$, there exists a unique solution $\Phi(\bullet)$ to the differential equation $\dot{z} = v(\beta,z)$ satisfying the initial condition $\Phi(\hat{\beta}) = \hat{z}$.

For any $\varepsilon > 0$, define $X_k(\varepsilon) = \{(\beta,z): \beta_{k-1} + \varepsilon \le \beta \le \beta_k - \varepsilon$ and $z_{min}(\beta) + \varepsilon \le z \le z_{max}(\beta) - \varepsilon\}$, and let ε be sufficiently small that $(\hat{\beta}, \hat{z}) \in X_k(\varepsilon)$. Then the solution $\Phi(\bullet)$ can be extended forward and backward to the boundary of $X_k(\varepsilon)$.

Proof of Claim 9: The claim follows from the existence, uniqueness, and extension theorems of ordinary differential equations, and the fact that $X_k(\varepsilon)$ is compact. (See, for example, Arnold (1973), section 8, corollaries

7, 8 and 11.) []

Figures 3a and 3b depict the vector field on X_k, together with properties assured by the claims. The lens-shaped region, X_k, is convex in the z direction and has continuous boundaries with slopes strictly between zero and infinity. The vector field is horizontal at the lower boundary, vertical at the upper boundary, and monotone increasing in z. Finally, the vertical strips $(Z(\beta))$ collapse to single points only at (β_{k-1}, z_{k-1}) and (β_k, z_k), and the vector field possesses singularities at only those two points.

INSERT FIGURE 3 ABOUT HERE

We complete the proof for possibility I as follows. Let $\hat{\beta} \in (\beta_{k-1}, \beta_k)$ and let z^0 be any point in the interior of $Z(\hat{\beta})$. We take $\phi(\hat{\beta}) = z^0$ as our initial condition and, first, we project $(\hat{\beta}, z^0)$ forward. Let $\{\epsilon_n\}_{n=1}^{\infty} \downarrow 0$. By Claim 9, there exists $N > 0$ such that for every $n \geq N$, there exists a unique solution $\phi(\bullet)$ which can be extended forward to the boundary of $X_k(\epsilon_n)$. The sequence of forward exit points from $\{X_k(\epsilon_n)\}_{n=N}^{\infty}$ has a convergent subsequence, whose limit point will be referred to as "the forward exit point from X_k." Observe, as in Figure 3a, that if z^0 is sufficiently near $z_{max}(\hat{\beta})$, then the forward projection of $(\hat{\beta}, z^0)$ exits X_k on the upper boundary. Meanwhile, if z^0 is sufficiently near $z_{min}(\hat{\beta})$, then the forward projection of $(\hat{\beta}, z^0)$ exits X_k on the lower boundary. Since the vector field is continuous and monotone in z, the forward exit point from X_k moves continuously and monotonically along the boundary as z^0 is shifted.

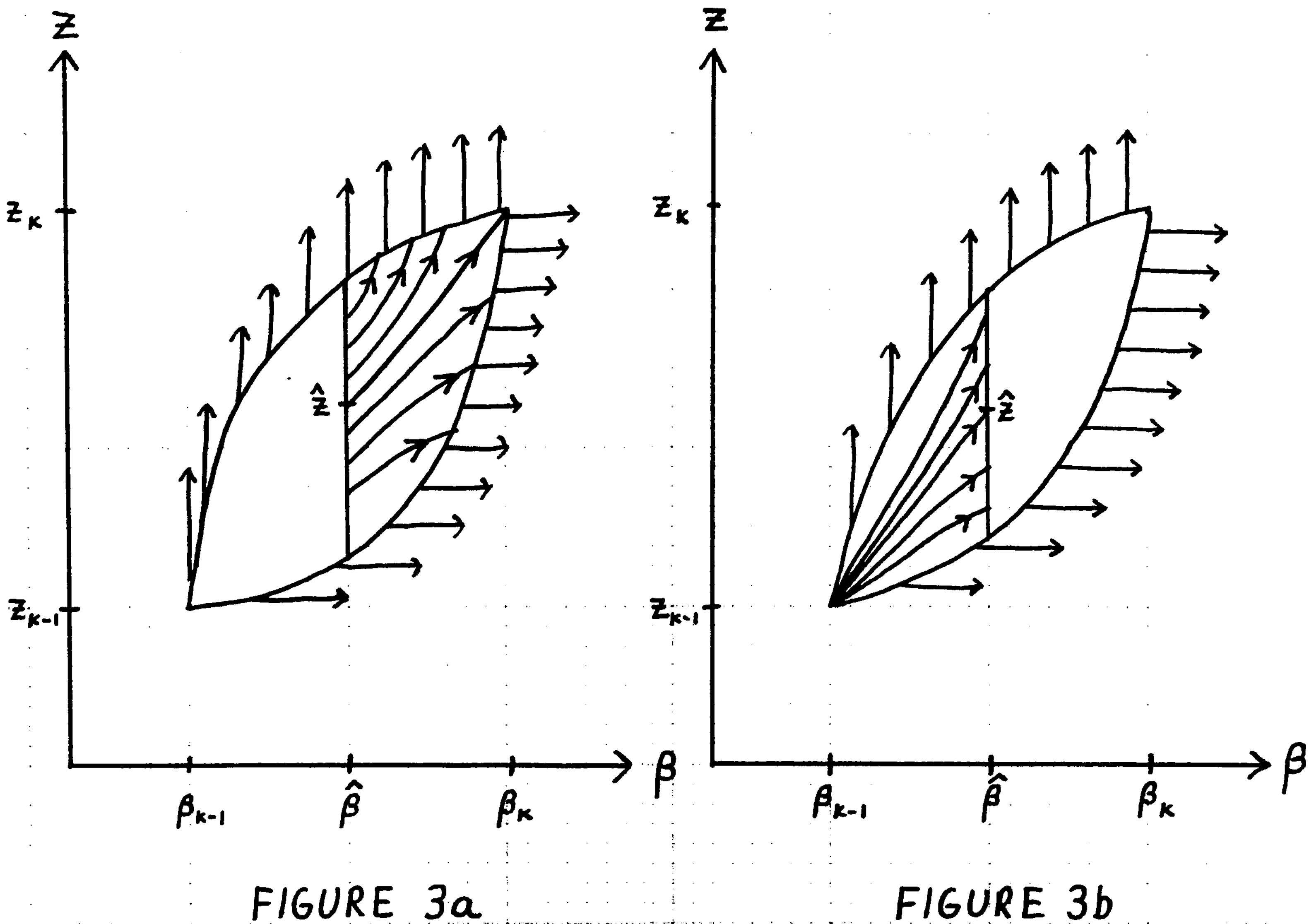

Figure 3

Forward (Figure 3a) and backward (Figure 3b) projections from points $(\hat{\beta}, z^0) \in X_k$, under possibility I.

Hence, there exists a unique $\hat{z}$ such that the forward projection of $(\hat{\beta},\hat{z})$ exits X_k at (β_k,z_k).

Now for any z^0 in the interior of $Z(\hat{\beta})$, we project $(\hat{\beta},z^0)$ backward. Analogously to before, we can define the "backward exit point from X_k." Observe, as in Figure 3b, that the backward projections of all $(\hat{\beta},z^0)$ exit X_k at (β_{k-1},z_{k-1}). Suppose not. Let (β^*,z^*) be the rightmost point where the backward projection of $(\hat{\beta},z^0)$ exits X_k. If (β^*,z^*) is contained on the upper (lower) boundary, $v(\beta^*,z^*) = \infty$ (0). But then, in either event, any forward projection from (β^*,z^*) goes outside X_k, and therefore must cross the boundary again before reaching $(\hat{\beta},z^0)$. Since $v(\bullet,\bullet) > 0$ everywhere in X_k, this contradicts our hypothesis that (β^*,z^*) is the rightmost point of exit, leading to the conclusion that all backward projections exit X_k at (β_{k-1},z_{k-1}).

We thus conclude, for arbitrary choice of $\hat{\beta} \in (\beta_{k-1},\beta_k)$, that there exists a unique $\hat{z}$ such that the forward projection of $(\hat{\beta},\hat{z})$ passes through (β_k,z_k) and the backward projection of $(\hat{\beta},\hat{z})$ passes through (β_{k-1},z_{k-1}). This establishes existence and uniqueness of a solution $z(\bullet)$ to (18) on the interval $[\beta_{k-1},\beta_k]$. Finally, $v(\bullet.\bullet)$ is well-defined, continuous and everywhere positive on X_k, so $z(\bullet)$ is continuously differentiable on (β_{k-1},β_k) and $\dot{z}(\bullet) > 0$, concluding possibility I.

Possibility II is treated in an analogous fashion. We will only highlight the differences. If $\beta' \in (\beta'_{k-1},\beta'_k) \subset \mathcal{B}'_{II}$, then $G_3 < 0$. As before, $G_1 > 0$ and $G_2 < 0$. Consequently, $w_{min}(\beta')$ is associated with $u = \infty$ and $w_{max}(\beta')$ is associated with $u = 0$. Define $X'_k = \{(\beta',w): \beta'_{k-1} < \beta' < \beta'_k$ and $w_{min}(\beta') < w < w_{max}(\beta')\}$. Again, we have a continuous, positive vector field on X'_k, but now it is monotone decreasing in w. The new situation is depicted in Figures 4a and 4b.

INSERT FIGURE 4 ABOUT HERE

As usual, we select $\hat{\beta}' \in (\beta'_{k-1}, \beta'_k)$ arbitrarily. Because the monotonicity of the vector field has been reversed, we now find that for all w^0 in the interior of $W(\hat{\beta}')$, the forward projection of $(\hat{\beta}', w^0)$ exits X'_k at (β'_k, z_k). However, if w^0 is sufficiently near $w_{max}(\hat{\beta}')$, the backward projection of $(\hat{\beta}', w^0)$ exits X'_k on the upper boundary; and if w^0 is sufficiently near $w_{min}(\hat{\beta}')$, the backward projection of $(\hat{\beta}', w^0)$ exits X'_k on the lower boundary. Again, by the continuity and monotonicity of the vector field, there exists a unique $\hat{w}$ such that the backward projection of $(\hat{\beta}', \hat{w})$ exits X'_k at (β'_{k-1}, z_{k-1}). Thus, we obtain existence and uniqueness of a solution $w(\bullet)$ to the differential equation (18') on the interval $[\beta'_{k-1}, \beta'_k]$, concluding our treatment of possibility II.

Our overall argument has established existence and uniqueness of solutions to differential equations (18) and (18') on every element of the partition. Moreover, for every k $(m + 2 \leq k \leq n)$, boundary conditions of $z(\beta_{k-1}) = z_{k-1} = w(\beta'_{k-1})$ and $z(\beta_k) = z_k = w(\beta'_k)$ were imposed, so the unique solutions on the elements of the partition properly piece together, proving Theorem 1. $\square$

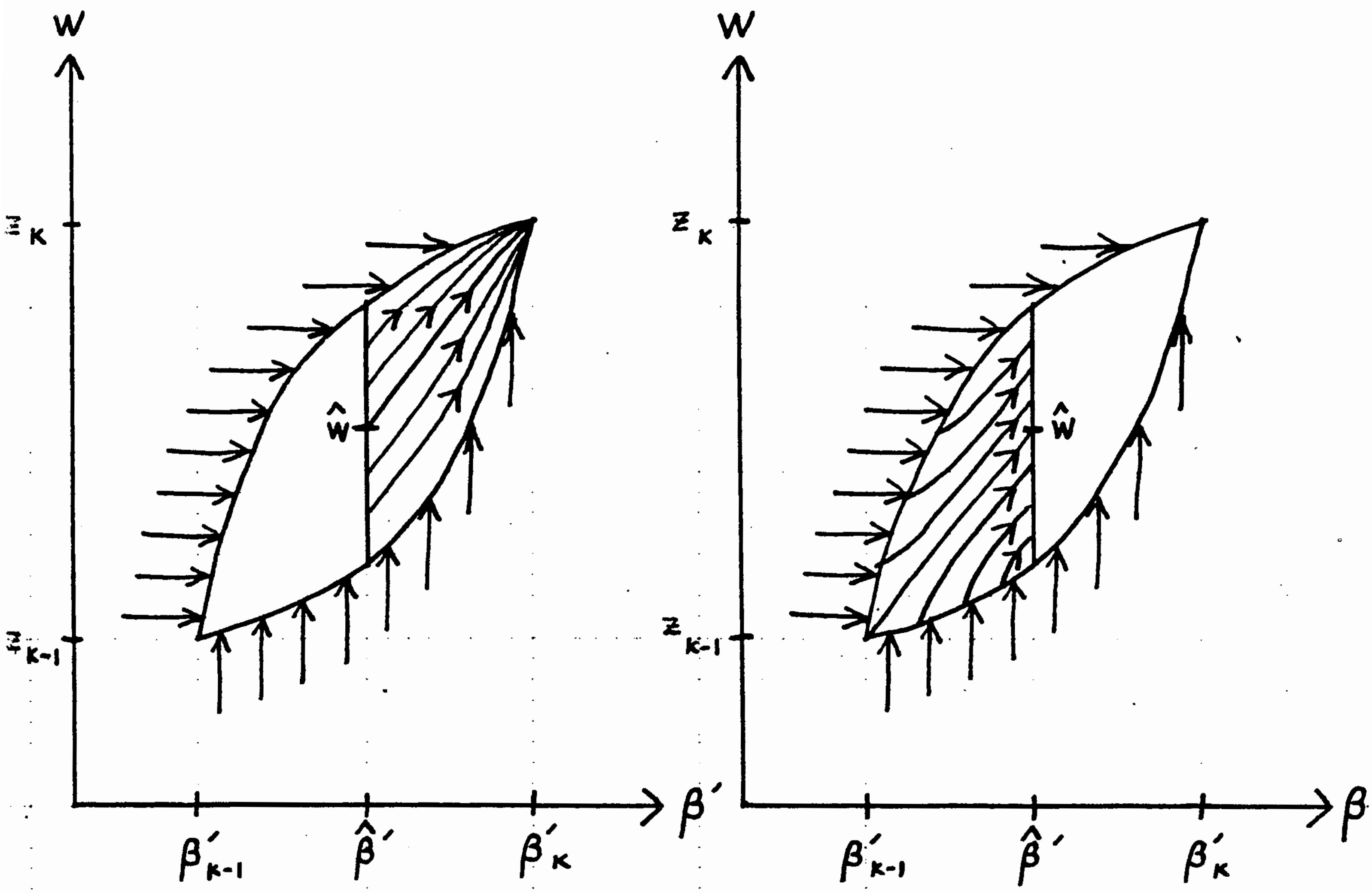

FIGURE 4a　　FIGURE 4b

<u>Figure 4</u>

Forward (Figure 4a) and backward (Figure 4b) projections from points $(\hat{\beta}', w^0) \in X'_k$, under possibility II.