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DYNAMIC AUCTIONS[1]

by

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Abstract

A dynamic trading game is examined in which two uninformed buyers engage in Bertrand-like competition to attempt to purchase a single object of uncertain quality from an informed seller. It is shown that there exists a unique perfect sequential equilibrium. The game is compared to an analogous bargaining game in which a single uninformed buyer makes offers to a single seller. Despite the fact that in the equilibrium of the competitive game, buyers compete away their surplus, it is shown that sellers can often gain a higher expected surplus in the bargaining game.

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Dynamic Auctions

Buyers often compete to buy a good of uncertain quality from an informed seller. When the object is of high quality a seller will require to be paid more to part with the object than if it were of lower quality. Buyers, if they knew the good was of higher quality, would also be willing to pay more. However, if they do not know the quality of the object they must find ways to ensure that they do not pay too high a price for a bad good. This paper examines a specific extensive form trading game in which time is used as a screening device by competing buyers. While it is a commonplace that, in conditions of complete information, a seller benefits from truly competitive behaviour by buyers, it is shown here that, in other environments, a seller may actually prefer to limit the degree of competition among purchasers of his object.

Many examples of economic problems fit into the framework of the paper. Banks compete for loans to customers of uncertain reliability, university faculties compete for graduate students of uncertain abilities, insurance companies compete for clients of uncertain riskiness. The important common features of these problems include the fact that a potential seller has private information which is relevant for the value of the trade both to himself and to the buyers. Furthermore, uninformed agents have similar preferences and actively compete with each other for the object, there is a potential for gains from trade, and informed agents can use delay to agreement as a credible signalling device.[2]

The strategic structure in this paper consists of T periods. In each period, two uninformed buyers offer prices simultaneously to an informed seller. The seller can choose which of the prices to accept or to reject them both. If he rejects and the period is less than T, he offers the object for sale in the following period. If no trade occurs by period T, the game ends with no trade.[3]

The formal model of the game is described in Section II. In the analogous two-person bargaining game, the use of the perfect Bayesian equilibrium concept guarantees uniqueness. Section III shows that there are many such equilibria in the three-person game. Other refinements such as the Cho-Kreps refinement of sequential equilibrium also yield many equilibrium outcomes. Often, these equilibria can be ruled out due to their dependence on what are argued to be incredible beliefs. In the next section, use is made of a concept introduced in Grossman and Perry (1986)(henceforth, GP), perfect sequential equilibrium. Theorem One characterizes the unique perfect sequential equilibrium path to a general specification of the game. The additional requirement of sequential consistency of beliefs imposes a constraint analogous to that of sequential rationality. A consequence is that some Pareto improving outcomes are no longer achievable.

Section V shows that, depending on the specification of the valuations, a seller may prefer to sell his good in an environment with only one buyer instead of many competing buyers even though these buyers compete away their surplus. Competition prevents the buyers from extracting surplus from the seller but it also limits the possibilities for efficient screening. In many situations, the screening cost overwhelms the seller's ability to extract surplus via competition.

A second question concerns the relative advantage of one- versus many-period games. One might think that since a many-period game allows more possibilities for trade, there will be cases in which the multi-period game is welfare improving over the one-period game. On the other hand, analyses from the literature on mechanism design have suggested that one-period mechanisms are often welfare maximizing subject to individual rationality and incentive compatibility. Section VI shows that in the class of games examined in this paper, a seller prefers a one-period game over a multi-period game.

Section II: The Model

Two identical buyers compete to buy a single good from a single seller. The trading mechanism consists of T+1 periods (the first period is t = 0) -- in each period both buyers submit price offers simultaneously, the seller observes the offers and decides which if any to accept. If he accepts, the game ends. The buyer whose price is accepted gains the good and pays the seller the amount of the accepted offer. If both offers are rejected and the period is t, less than T, the game continues and the same play is repeated. If the period is the last one, T, and a rejection occurs the game ends with no trade taking place. After any period t, it is assumed that all players know the full history of the game to period t.

Let D_B be the real line and D_S, the space of probability distributions over the finite set {0,1,2} where {0} represents no trade, {1} represents trade with buyer one (b1) and {2} trade with buyer two (b2). Let H_{t-1} represent the space of histories of pairs of rejected price offers up to and including period t-1 and let σ_{it} be a function mapping H_{t-1} into D_B. Buyer i's strategy choice is a sequence of such functions $\sigma_i = \{\sigma_{it}\}$ for t from 0 to T. Similarly let σ_{sqt} be a function mapping H_t into D_S. The strategy choice of a seller of type q is the sequence of functions $\sigma_{sq} = \{\sigma_{sqt}\}$ for t from 0 to T.

Outcomes in the game can be represented by a triple (p,t,i) where p represents the price traded, t, the period trade takes place and i, the buyer who ultimately trades. Ex post payoffs if $t \leq T$ are represented by the utility function for buyer i,

$$U_i(p,t,i) = \delta^t(v - p);$$

for buyer j not equal to i,

$$U_j(p,t,i) = 0;$$

and for the seller,

$$U_s(p,t,i) = \delta^t(p - f), \qquad \delta \in (0,1).$$

The symbol v represents the (ex post) reservation value for the buyer and f the reservation value for the seller. If no trade takes place, players gain a net payoff of zero.

The game examined is a game of one-sided information. Uncertainty is parametrized by a variable q. Ex ante, it is common knowledge that q is distributed uniformly over the unit interval. <u>The seller alone observes the true realization of q.</u> Both the buyers' and the seller's reservation value depend non-trivially on the value of q in the following manner:

A1) $f(q) = q$, $v(q) = q + \epsilon$, $\epsilon > 0$.

Thus the seller's valuation is uniform over [0,1], ex ante, while the buyer's is always ϵ greater than the seller's valuation. The buyer and seller are risk-neutral; they attempt to maximize expected utility.

The game described here is a game of incomplete information -- the only subgame is the whole game The many Nash equilibria to this game are also subgame perfect equilibria. It will be shown later that many other commonly used refinements such as perfect Bayesian equilibrium (pbe) do not greatly reduce the number of equilibrium outcomes. However, many of the equilibria support outcomes on the basis of what will be argued are implausible out of equilibrium beliefs. This paper uses a modified version of the more stringent refinement concept, perfect sequential equilibrium (pse) introduced in Grossman and Perry (1986). Any pse is a pbe but the converse does not hold in general.[4]

A meta-strategy is defined as a strategy choice of a player after a given history and a given belief. (For the informed seller, beliefs are trivial.) Let K be the space of subsets of [0,1] and now redefine σ_{it} to be a function from the cross-product space H_{t-1} and K into D_B. Perfect sequential equilibrium uses the concept of a meta-strategy to specify actions after any history and for any given candidate belief.

For a meta-strategy to form part of a pse it must be sequentially rational. Sequential rationality is defined in GP only for two-person games, however. What follows provides a natural adaptation for the specific three person game examined here (a similar technique is also used in Madrigal and Tan (1986)). Fix a period, t, history h_{t-1}, and a belief (held commonly between buyers) k_{t-1}. Fix some updating rule $g(h_{t-1},(p_{1t},p_{2t}),k_{t-1})$ which specifies the beliefs that will be held following any history h_{t-1} and rejected price offers (p_{1t},p_{2t}) when beliefs were k_{t-1}. A meta-strategy characterizes the play of the subform for every pair of price offers (p_1,p_2) via p's effect on the strategies directly and through its indirect effect on the updating rule. Given the strategies of the seller types, a belief over the types of the seller and the strategy of the other buyer as well as his own strategy following the play of a given period t, a buyer, i, can determine his expected payoff at period t by varying his price offer p_{it}. A meta-strategy $(\sigma_1,\sigma_2,\sigma_{sq})$ is said to be sequentially rational at time t with respect to an updating rule g if, given h_{t-1}, k_{t-1} and the continuation strategies, the price offers determined by σ_1 and σ_2 are best responses to each other (or a 'paired best response'). A meta-strategy is required to be sequentially rational at every period t, for every history h_{t-1}, and belief k_{t-1}.

The definition of a pse requires, as well, the characterization of a credible updating rule. Let g_t be a function mapping KXH_{t-1} into K. The sequence of functions $\{g_t\}^T_{t=0}$ is an updating rule defined relative to a profile of meta-strategies σ. It is a **credible** updating rule if

i) (Non-increasing support) For all h_{t-1}, $h_t = (h_{t-1}, p_1, p_2)$, $g(h_t)$ lies in $g(h_{t-1})$.

ii) If Probability$(h_t|\sigma) > 0$, then $g(h_t)$ is given by Bayes' rule from the strategies, σ.

iii) (Consistency of beliefs) If Probability$(h_t|\sigma) = 0$ let $e(h_t,k_t)$ be the possibly random path of price offers according to the meta-strategies σ when the history is h_t and buyer beliefs are k_t. Let $U_q(e(h_t,k_t))$ be the expected utility of this path to a seller of type q. Let $\{(p',t')\}$ be the set of price offers rejected in period $t' \leq t$ in a given history h_t. Belief k_t is consistent if there does not exist another set k not equal to k_t such that

a) $U_{q'}(e) \geq \delta^{t'}(p' - f(q'))$ for all (p',t') in h_t and for all $q' \in k$.

b) $U_q(e) \leq \delta^{t'}(p' - f(q))$ for some (p',t') in h_t and for all $q \in k^c$,

The profile of meta-strategies σ and the updating rule g form a pse if σ is sequentially rational and g is credible. [5]

Section III: An Example

This section provides an example of the game described in this paper which helps to illustrate the nature of a pse in the model. It shows, as well, why the equilibrium refinement is a persuasive one in this context.

Let $f(q) = q$, $v(q) = q+1/2$, $T = 2$ and $\delta \geq 3/4$. Here, the quality of the object from the buyers' points of view varies uniformly from 1/2 to 3/2 and it is always worth one-half more to the buyers than to the seller. Buyers submit bids in the first period. If the bids are rejected, they resubmit bids a second and final time. The following strategies form a perfect Bayesian equilibrium of the game but not a pse.[6]

Seller's strategies: (First period)

For $q \in [0,(1-\delta)/(1-\delta/2))$ accept $p \geq (1-\delta)q + \delta(1+q/2)$,

For $q \in [(1-\delta)/(1-\delta/2),\delta/(2-\delta))$ accept $p \geq 1$,

For $q \geq \delta/(2-\delta)$, accept $p \in [1,\delta(3/2)]$ and $p \geq (1-\delta)q + \delta(1+q/2)$.

(Second period): For all q, accept $p \geq q$.

Buyers' strategies: In period one, offer $p_{i1} = 1$, $i = 1,2$.

In period two, let $p^* = \max(p_{11},p_{21})$. Beliefs and strategies are:

a) $p^* \in [\delta,1)$, q is in $[(p^*-\delta)/(1-\delta/2),1]$ and offer $p_2 = .5(1+(p^*-\delta)/(1-\delta/2)) +.5$.

b) $p^* \in [1,\delta 3/2]$, q is in $[0,1]$, offer $p_2 = 1$.

c) $p^* \in (\delta 3/2, (1-\delta) + \delta 3/2]$, q is in $[q^*,1]$ where q^* satisfies $p^* = (1-\delta)q^* + \delta(1+q^*/2)$ and offer $p_2 = .5(q^*+1) + .5$.

The equilibrium path prescribes an offer of one in the first period by both buyers and acceptance by all sellers.

Note that if any price between one and $\delta 3/2$ is offered and rejected an `out of equilibrium message' has been sent by the seller. Perfect Bayesian equilibrium puts no restriction on how beliefs should be formed following such a rejection. Recent refinements have attempted to restrict the formation of such beliefs. One such is provided in Cho-Kreps (1987) and extended to general signalling games in Cho (1987) using the concept of `introspectively consistent' beliefs. If there is a group k such that the outcome from <u>any</u> best response based on <u>any</u> belief following an out-of-equilibrium message is dominated by the outcome resulting from following the equilibrium, introspectively consistent beliefs must put zero weight on that group. The beliefs characterized above satisfy this requirement. Suppose following a rejection of $p \in [1, \delta 3/2]$ both buyers believe $q = 1$ with probability one. A best response is for both to offer $p = 3/2$ in the following period. All types of sellers would, then, be better off and, so, there is no group k with the characteristic described above. Any belief is introspectively consistent. The pbe satisfies the criterion.[7] Notice that according to the pbe, the buyers do not believe $q = 1$ after a rejection and that if they held this belief their subsequent actions would render the seller's actions suboptimal.

In a perfect sequential equilibrium, a belief of [0,1] following the rejection of a price $p \in [1, \delta 3/2]$ is not consistent. Fix a first period price $p \in [1, \delta 3/2]$ and define k such that $p = (1-\delta)k + \delta(1+k/2)$. k is in $[(1-\delta)/(1-\delta/2), \delta/(2-\delta)]$ whenever p is in $[1, \delta 3/2]$. If both buyers had beliefs, in the second period, that q was in [k,1], the unique equilibrium response is a price, $p_2 = 1+k/2$. Furthermore, for all $q > k$, sellers prefer p_2 in the second period to p in the first period while for $q < k$, immediate acceptance is preferred. The existence of this set [k,1] renders the belief [0,1] incredible in pse language. Intuitively, the argument is the following. Suppose a price $p = 1$ is offered and rejected. The only way a type $q < k$ could have been better off so rejecting is if the buyers, in fact, believe that these very types would not have rejected. In the intuitive criterion pbe described first, buyers believe that very low-quality sellers might have rejected the price offer of one as well as high quality sellers. However, buyers have this belief only if they believe that the low-quality sellers expect that the buyers will **not** have this belief, that the buyers will (over-optimistically) believe that some of the low-types did accept and, so, will offer a higher price in the continuation game than the pbe actually prescribes. An additional order of introspection should lead the buyers to reject this belief on the assumption that all sellers could mimic the same reasoning process. A rejection of a price of one in the first period should lead to a concentration of beliefs on higher sellers and a higher price offer as a result. This, in turn, destroys the first pbe profile.

The unique pse path to the game is calculated by exploiting this argument in a natural way.

Seller's strategies: (First period) Accept $p \geq (1-\delta)q + \delta(1+q/2)$.

(Second Period) Accept $p \geq q$.

Buyers' strategies: In period one offer $p(q^*)$ such that

$$\int_0^{q^*} x + 1/2 - p(q^*)dx = 0,$$

and $p(q^*) = (1-\delta)q^* + \delta(1+q^*/2)$. Note that this implies $q^* = 0$ and no trade occurs in the first period.

(Second period) For $p^* = (1-\delta)q^* + \delta(1+q^*/2)$, believe q is in $[q^*,1]$ and offer $p_2 = 1 + q^*/2$.

If $p^* < \delta$, believe q is in [0,1] and offer $p_2 = 1$.

If $p^* > (1-\delta) + \delta(3/2)$, any belief is consistent. Set it to some $[q^*,1]$ and offer $p_2 = 1 + q^*/2$.

Trade in the pse is delayed to the second period because buyers cannot `commit' to adopt incredible beliefs. Any higher first period price would lead to trade with positive probability but not by high enough quality sellers to justify that price for the buyers. Sellers are less willing to accept a given price in the first period because a consistent belief following a rejection would lead to a better price in the second period. Delay, which is a potent signalling device in bargaining models, in these models may simply force the wasting of resources without any corresponding information gains. The combined effects of sequential rationality, consistent beliefs and competitive behaviour are to impose ex post inefficiencies in the game.

Section IV: Characterization of the PSE Path

The above example suggests that the use of a refinement such as pse will be required to avoid inconsistencies in beliefs. Theorem One shows that, in games such as that described above, the pse paths can be described by a unique sequence of subintervals of [0,1] and acceptance functions.

Theorem One: Assume A1. There is a T* such that for all T - period games, T > T*, there exists a unique and identical pse path.

Proof; See Appendix One.

The proof of the theorem is a constructive proof. In particular, it is shown that seller types are partitioned into intervals, $\{q_i\}$, $q_i > q_{i+1}$, i = 0 to T*, $q_0 = 1$, and

$$q_{i+1} - q_i = 2\epsilon(1-\delta)(1-(-\delta)^{i+1})/(1+\delta). \qquad (1)$$

Along the pse path, the first offer by the buyers is accepted by sellers of type less than q in $(q_{T^*-1}, q_{T^*-2}]$, the next offer is accepted by some q in the next interval and so on. The proof also shows that in each period, buyers make an offer exactly equal to the expected value of the object to them conditional on the seller types who accept, that is, in each period, buyers gain an expected utility of zero.

For games with periods less than T* there is also a unique pse path -- its form, though, is typically different from that described in the theorem. The example in Section 3 is an illustration of equilibria in such games. Notice in that example that the pse path consisted of a period (the first period) in which there was no possibility of trade. The theorem along with equation (1) indicates that this phenomenon does not remain true if the number of periods in the game is large enough since $q_{i+1} - q_i$ is greater than zero. However, the possibility of inefficiencies remains as the next section shows. Observe that for any fixed ϵ, as δ goes to one, (1) also indicates that seller types are essentially perfectly separated.

Section V: The Inefficiency of Competitive Markets

This section addresses the following question. Is the ex ante utility of the seller always higher when there is more than one buyer competing against each other? It is shown, here, that the seller may prefer to restrict the competitive behaviour of his buyers if there is uncertainty about the quality of the object he is selling. In particular, he may prefer to face a single buyer who is allowed to make all the offers than many competing buyers.

The similarity of the game form and payoffs in this model with that of the one-sided offer models of non-cooperative bargaining theory makes it natural to examine the relative welfare consequences of two versus three person bargaining games. It might be expected that the more competitive game with two buyers will always be more efficient than the bargaining game, particularly since any pse in the competitive game generates zero surplus for the competing buyers. This expectation is reinforced by the observation that a bargaining problem where an uninformed buyer makes offers to an informed seller is formally equivalent to that of a monopsonist making price offers to an atomless market of informed sellers. In these situations, we might expect the monopsonist optimal pricing behaviour to be Pareto inefficient and, as well, to be detrimental to the sellers. The presence of informational asymmetries and the possibility of making offers very quickly makes both presumptions false, however, as the next example shows.

Consider a game of the type described in Section 2. Fix $\epsilon = 1/2$. Let T be arbitrarily large and finite. Now consider the behaviour of this game as δ comes close to one. Equation (1) shows that the partition intervals $\{q_i\}$ become arbitrarily close to each other. Thus, for example, the seller types who accept in the first period are types with valuations arbitrarily close to zero. Lemma 3 of the Theorem shows that buyers pay no more than the expected value of the object to them given the types of sellers who accept in any given period. Therefore the first price offered and accepted in the pse is a price that is arbitrarily close to one-half when δ is close to one. Similarly a seller of type q will expect to get a price arbitrarily close to $q + 1/2$. Sellers of type $q = 1/2$ get close to one and sellers of type one get a price close to one and one-half. Self-selection constraints require that no seller wish to mimic the acceptance of a seller of a different type.

Let t2 be the period in which sellers of type $q = 1/2$ accept. Therefore, for $q = 0$ to accept the equilibrium price of one-half, it must be the case that

$$1/2 > \delta^{t2}(1) \text{ or } 1/2 > \delta^{t2}.$$

Since sellers gain all the surplus in this game, this equation shows that with probability one-half, seller types incur a waiting cost of δ^{t2}. The total surplus that sellers can hope to gain in this game is less than $1/4 + (1/2)(1/2)(1/2) = 3/8$.

Now consider the total surplus available when, instead of a competitive game the seller is in a bargaining game in which a single buyer makes offers in each period. This game is examined in Vincent (1988) and has a unique pbe (and pse) path. As δ goes to one, an argument based on the Gul-Sonnenschein-Wilson proof of the Coase conjecture shows that the first price offer is arbitrarily close to the last price offer of one. That is, the game ends arbitrarily quickly and the total surplus gained by the sellers is close to one-half. They gain almost all the surplus in the game.[8]

The bargaining game dominates the dynamic auction game from the seller's point of view because of the informational asymmetry. In the bargaining game, the single buyer is willing to take advantage of the fact that he alone is purchasing the object to take the risks represented by offering a price close to one early on even though it will be accepted by sellers of objects which are worth strictly less than one to him. He is willing to do this because a rejection tells him that he can gain a positive surplus from higher quality types in the continuation game. This gain compensates him (only just) for the risk taken earlier on. The absence of another buyer allows him to appropriate to himself the information contained in a rejection to go after the high-quality seller types. A consequence is that the game ends quickly with little efficiency cost, a gain which the seller shares in. In the competitive game, buyers compete away any gains from information and are forced, as a result, to use less efficient ways of screening out low-quality sellers. When offers are made arbitrarily quickly, sellers are arbitrarily finely screened and this screening imposes a substantial cost on the participants.

Section VI: One Versus Many Periods

Another welfare question concerns the relative advantage of one- versus multi-period games. A seller often cannot commit himself not to attempt to trade again if no trade occurs in the first round of bidding. In other cases, though, a seller may be able to choose the type of trading institution among a class of mechanisms. It is well-known that, for an uninformed buyer facing an informed seller, the mechanism which allows the buyer to maximize his expected surplus can be generated by a single-period take-it-or-leave-it game (Samuelson (1984)). Somewhat less can be said about expected profit maximizing mechanisms for an informed seller facing one or more buyers. Suppose a seller can choose among multi-period games of the sort described in Section I with freedom to specify both $T \geq 1$ and δ (by specifying the length of time between offers). Furthermore, although he will be informed when he plays in the game, when he chooses the mechanism, he does not know his private information. What can be said about the type of mechanism he will prefer?[9]

Clearly, two countervailing effects are at play. A simple one-period game allows the seller to avoid the costs of the information transmission process that may occur in multi-period games. On the other hand, one-period games often force the abandonment of substantial gains from trade. Consider a model of the form $v(q) = q + \epsilon$, $f(q) = q$. With a low ϵ, it can be shown that in a one-period game, trade only occurs when q is less than 2ϵ, thus generating welfare losses of $\epsilon(1-2\epsilon)$. Nevertheless, an adaptation of a result by Samuelson shows that the seller will always prefer a one-period game.[10] The costs of satisfying self-selection constraints in a dynamic game will always outweigh the additional potential gains from trade that a multi-period game may make possible.

Section VI: Conclusion

This paper analyzes equilibrium behaviour in a dynamic auction game. In conventional models of static auctions, information transmission only occurs -- if at all -- through the process of bidding. When it is the seller who enjoys private information, then, this information cannot be transmitted in the play of the game. In the dynamic game examined in this paper, not only can the seller's information be transmitted but sometimes it must be transmitted, potentially at costs to efficiency. The example in Section III illustrates this point where the opportunity for buyers to gain information forces trade in equilibrium to occur only in the last period.

Dynamic auctions are also compared to dynamic bargaining games in this paper. In finite period models of complete information, a seller always prefers to face many buyers who are making offers (where he wins all the surplus) than to face just one (where he loses all the surplus). When the seller is privately informed about the quality of the good, however, this preference may be reversed. As Section V illustrates, the screening mechanism which allows buyers to gain necessary information about the object they are purchasing can be more costly in a competitive game than in a bilateral monopoly game. This cost is borne, in part, by the seller, and may lead him to prefer to deal with only one buyer.

This game has obvious similarities to the non-cooperative bargaining games in which an uninformed buyer makes offers in each period to an informed seller (see Sobel and Takahashi, Fudenberg, Levine and Tirole, Gul,Sonnenschein and Wilson, and Vincent(1986)). As a result, the problem can be considered to be a step towards extending non-cooperative bargaining theory to a more general, many-person strategic environment.

The model can also be interpreted as a multi-period first-price auction game or, equivalently, a dynamic Bertrand game. Much is known about equilibria in static auctions. This paper can be seen as an examination of auction theory in a dynamic context. In a static auction, if a seller refuses all outstanding bids, no trade occurs and the game ends. It is important to explore more precisely what occurs when it is common knowledge that gains from trade still exist and the seller is not able to commit ex ante to refuse to trade. If he is expected to put the object up for trade again, equilibrium strategies should be dynamically consistent. As well, the possibilities for the transmission of information now exist and must be examined.

Appendix One

Theorem One: Let $f(q) = q$, $v(q) = q + \epsilon$, $\epsilon > 0$, $0 < \delta < 1$. There is a T^* such that for all T - period games, $T > T^*$, there exists a unique and identical pse path.

Proof; The proof of the theorem is preceded by four Lemmas. Define $p_t = (p_{1t}, p_{2t})$ $p^*_t = \max(p_{1t}, p_{2t})$, $p^-_t = \min(p_{1t}, p_{2t})$.

Lemma 1.1: Fix a pbe to the game. Suppose at any period t (with history h_t) and beliefs by both buyers that q is in $[q_t, 1]$, price p_t is offered. If `reject p^*_t' is a weak best response for $q' < 1$, then it is a strictly best response for all $q'' > q'$.

Proof: Reject is a weak best response for q' implies

$$p_t - f(q') \leq \max(0, E(\delta^{t'}(p_{t'} - f(q')))) \quad (1)$$

where E(.) is the expected value of the possibly random path $\{p_{t'}\}_{t'>t}$ determined by the pbe when history (h_{t-1}, p_t) has occurred. $q'' > q'$ and (1) imply

$$p_t - f(q'') < \max(0, E(\delta^{t'}(p_{t'} - f(q'')))).$$ ||

Corollary: In any pse, in any period t with beliefs $[q_t, 1]$, if $p^*_t \leq (1 - \delta)f(1) + \delta v(1)$ is rejected, then the credible updating rule must set q in $[q_{t+1}, 1]$ (or $(q_{t+1}, 1]$), $q_{t+1} \geq q_t$. If $p^*_t > (1 - \delta)f(1) + \delta v(1)$, no restriction can be placed on beliefs. ||

A consequence of the corollary is that for any history (with $p_t \leq (1 - \delta)f(1) + \delta v(1)$) the updating rule $g_t(h_t)$ can be characterized by q_{t+1}, the bottom end of the support. This convention is used throughout the proof.

Lemma 1.2: Let $T > 1$. Define $q_1 = 1-2\epsilon(1-\delta)$. For all $t < T-1$, if q is in $(q_1, 1]$, q follows a strategy

Accept $p_t \geq P_t(q) = (1-\delta)q + \delta((1+q)/2 + \epsilon)$, Reject otherwise.

Proof: Note that $(1+q_1)/2 + \epsilon = 1 + \delta\epsilon$. Since $v(1) = 1+\epsilon$ is the highest price ever offered with any belief, all sellers accept $p \geq 1+\delta\epsilon$. Also observe that q_1 is just indifferent between accepting $P_t(q_1)$ now and $1+\delta\epsilon$ next period.

Suppose $p_t \in (P_t(q_1), 1+\delta\epsilon]$ is offered. Let $q_t = P^{-1}{}_t(p_t)$. It is straightforward to show that following a rejection of p_t, a belief that q is in $(q_t,1]$ is a credible belief. To see that it is the only credible belief suppose that some other belief (k,1] is possible. First observe that if $k \in (q_1,1]$, $k = q_t$ is the only belief that is consistent with Bayes' rule and optimizing behaviour.

On the other hand, if $k \leq q_1$, the belief $(q_t,1]$ renders (k,1] 'incredible'. Believing $(q_t,1]$ leads to a final offer $p_{t+1} = .5((q_t+1) + \epsilon \leq 1+\delta\epsilon$ in the next period. Since

$$p_t - q \geq \delta(p_{t+1} - q) \text{ for all } q \leq q_t, \text{ and}$$

$$p_t - q < \delta(p_{t+1} - q) \text{ for all } q > q_t,$$

exactly those q in (k,q_t) would, in fact, prefer accepting p_t and those in $(q_t,1]$ would prefer rejecting p_t.

Since for any history h_t with $t < T-1$, a rejection of p_t in the relevant range leads to a belief $(P^{-1}{}_t(p_t),1]$, and a subsequent offer of $.5(q_t+1) + \epsilon$, it must be the case that sellers in $(q_1,1]$ use the strategy P_t and sellers less than q_1 must always accept $p \geq P_t(q_1)$. ||

Partition the interval [0,1] into T* subintervals $\{q_i\}_0^{T*}$ in the following manner:

$q_0 = 1$, $q_1 = 1 - 2\epsilon(1-\delta)$, $P(q_1) = (1-\delta)q + \delta(.5(1+q_1)+\epsilon)$,

$q_{i+1} = q_i - 2(\epsilon - (P_t(q_i)-q_i))$, (a) where

$P_t(q_i)-q_i = \delta(P_t(q_{i-1})-q_i)$ (b).

Note that a) and b) imply

$(P_t(q_{i-1})-q_i) = 2(\epsilon-\delta(P_t(q_{i-2})-q_{i-1})$, so

$$q_i-q_{i+1} = 2\epsilon-\delta(2\epsilon-\delta(P_t(q_{i-2})-q_{i-1})))$$
$$= 2\epsilon(1-\delta)(1-(-\delta)^{i+1})/(1+\delta)$$
$$\geq 2\epsilon(1-\delta)^2.$$

Define $T^* = \min\{i \mid q_i \leq 0\}$.

Let $P:(q_i,1]$ to R be as in Lemma 2. Define $q^*(q):(q_2,q_1]$ to $(q_1,1]$ implicitly by

$$P(q^*(q)) = .5(q^*(q)+q)+\epsilon \quad \text{(ZP)}$$

Observe that $q^*(q_1) = 1$ and $q^*(q_1) = q_1$ by (a).

Given $P':(q_i,1]$ to R and $q^{*\prime}:(q_i,q_{i-1}]$ to $(q_{i-1},q_{i-2}]$ we can extend the domain of P' and q*' iteratively by the rules:

i) $q^*:(q_{i+1},q_i]$ to $(q_i,q_{i-1}]$ satisfies $q^*(q) \in (q_i,q_{i-1}]$ such that $.5(q^*(q)+q)+\epsilon = P'(q^*(q))$ and then

ii) $P : (q_{i+1},1]$ to R satisfies $P(q) = P'(q)$ if $q > q_i$ and

$P(q) = (1-\delta)q + \delta P'(q^*(q))$ for $q \in (q_{i+1},q_i]$.

P and q* are thus defined over [0,1]. Note that P(.) and q*(.) are continuous and strictly increasing. Note also that since the derivative of P(q) inside the interval $(q_1,1]$ is $1-\delta/2$, the derivative of P(q) (left or right derivative) is not less than $1-\delta/2$.

The next two lemmas rely on the following two induction hypotheses:

A1) Suppose that in any pse, σ, for any $q' \in (q_j,q_{j-1}]$, $j \leq i$, and for any history $h_{t'-1}$, $t' \leq T-j$, the pse strategy of q' is given by: If $p^*_{t'} \geq P(q')$, accept p*, else reject, and that all $q < q'$ accept $p \geq P(q)$.

A2) Suppose that for any pse σ, for any $t \leq T-j$, $j \leq i$, if buyers' beliefs are $(q_t,1]$, $q_t \in (q_j,q_{j-1}]$, then their unique paired best response price offer $(p,p) = (p_{1t},p_{2t})$ satisfies $p = .5(q^*(q_t)+q) + \epsilon$.

Lemma 1.3: For any $t \le T-i$, for any pse, σ, if buyers' beliefs are $(k,1]$, $k \in (q_{i+1},q_i]$, then (p,p), $p= P(q^*(k))$ is the unique paired best response price offer by the buyers.

Proof: Note that $q^*(k) \in (q_i,q_{i-1}]$. If (p_1,p_2) is a paired best response by the buyers then it must be the case that

i) $.5(P^{-1}(p^*)+k) + \epsilon - p^* = 0$ and for all $p' \ge p^-$,

ii) $.5(P^{-1}(p')+k) + \epsilon - p' \le 0$.

If i) did not hold, then since P^{-1} is continuous, the buyer gaining the lowest expected utility will increase his payoff by offering a slightly higher price. (Recall that by A2, the buyers' payoff in any continuation game is zero). If ii) did not hold, then since i) implies that the buyer offering the higher price is getting zero, he could have received more by bidding some p' above p^-. Now observe that

$$G(p) = (P^{-1}(p)-k)(.5(P^{-1}{}_t(p)+k)+\epsilon-p) = 0 \text{ at } p = P(q^*(k)). \text{ and that}$$

$$\frac{dG(p)}{dp} = \frac{dP^{-1}(p)}{dp}(.5(P^{-1}(p)+k)+\epsilon-p)+(P^{-1}(p)-k)(.5(\frac{dP^{-1}(p)}{dp}) -1).$$

is strictly negative at $p = P(q^*(k))$ since $dP^{-1}{}_t(p)/dp < 1/(1-\delta/2)$ and $.5(P^{-1}(P(q^*(k)))+k)+\epsilon-P(q^*(k)) = 0$ by (a). Therefore, for all $p > P(q(k))$, buyers gain a strictly negative expected value, and since there is a p'' just less that $P(q^*(k))$ which yields a positive surplus, the minimum and maximum paired best response prices is $P(q^*(k))$. ||

Lemma 1.4: For any pse, σ, any history h_{t-1}, $t \le T-i-1$, for any $q \in (q_{i+1},q_i]$, q follows the strategy

If $p^* \ge P(q)$ accept q, else, reject.

Proof: That the strategy described forms part of a pse strategy is straightforward to show. To see that it is the only pse strategy consider h_{t+1} after some $p_t \in (P(q_{i+1}, P(q_i))$ has been rejected. Let buyers' beliefs be $(k,1]$. A1) ensures $k < q_i$. If $k \in (q_{i+1}, q_i]$, then Lemma 3 shows that the unique next period pse price offer is $P(q^*(k))$. If $k > P^{-1}(p_t) = q_t$, by definition of $P(.)$, $p_t - k > \delta(P(q^*(k))-k)$ so there are types in the interval (q_t, k) who would have done better to accept p_t as well. Similarly for $k \in (q_{i+1}, q_t]$. For $k \le q_{i+1}$, an argument similar to the last part of Lemma 2 shows that the belief $(q_t, 1]$ renders this belief 'incredible'. ||

Proof of the theorem: Using Lemma 2 to validate A1 and A2 and iteratively applying Lemmas 3 and 4 implies that for any game of length $T \ge T^*$, the unique pse price offer in the first period is a price which (in the event of a rejection) would lead to next period beliefs $(q_{t+1}, 1]$, $q_{t+1} \in (q_{T^*-1}, q_{T^*-2}]$, the next in $(q_{T^*-2}, q_{T^*-3}]$, and so on. In each subsequent period, the assumptions of the hypotheses A1 and A2 hold and determine equilibrium behaviour in the continuation game. ||

Appendix Two

Theorem Two[11] : Let $B(\delta)$ be the ex ante expected utility to the sellers in a one-sided offer bargaining game and $A(\delta)$ be the corresponding utility in the auction game. There exists a $\delta^* < 1$ such that for all $\delta > \delta^*$, $B(\delta) > A(\delta)$.

Proof: Observe that from the proof in Theorem One, for the pse in the game with discount factor δ, the price, p, any q in $(q_{i+i}, q_i]$ receives is bounded by

$$q_{i+1} + 1/2 < p < q_i + 1/2 \text{ , or}$$

$$q_{i+1} - q + 1/2 < p - q < q_i - q + 1/2 \text{ , or}$$

$$p - q < (1-\delta)(1-(-\delta)^{i+1})/(1+\delta) + 1/2, \text{ and}$$

$$p - q > - (1-\delta)(1-(-\delta)^{i+1})/(1+\delta) + 1/2.$$

Thus for any $\lambda > 0$, there is a δ close enough to one such that

$$p - q \in (1/2-\lambda, 1/2+\lambda).$$

As δ goes to one, a seller of type zero receives a price close to 1/2, a seller of type q = 1/2 receives a price close to one. For this to occur along the equilibrium path, it must be the case that sellers of type q > 1/2 trade at a time at least t* such that $1/2 > \delta^{t^*}(1-\lambda) - \lambda$, or $\delta^{t^*} < (1/2+\lambda)/(1-\lambda)$.

Since in each period, sellers of the type who trade in that period gain all the surplus available, the total ex ante surplus gained by the sellers is just the total surplus discounted by the time of trade. That is, if the game lasts for at most n periods,

$$\begin{aligned} A(\delta) &= 1/2(q_1 + \delta(q_2-q_1)+\delta^2(q_3-q_1)+\ldots +\delta^n(1-q_{n-1})) \\ &< 1/2(1/2 + \delta^{t^*}(1/2)) < 1/2(1/2+(1/2+\lambda)/((1-\lambda)2)) \\ &= 3/8 \text{ (approximately).} \end{aligned}$$

Now consider the bargaining game. It is shown that as δ approaches one, the first period price offer of the single buyer approaches one. The argument is essentially the same as in Gul/Sonnenschein, Wilson, Theorem Three, the proof of the Coase Conjecture. Note that the bargaining game has a unique pbe (which is also the unique pse path) and that sellers' strategies can be denoted by an acceptance function $P(q)$ which is continuous, strictly increasing and independent of the history of the game.[12]

Lemma 2.1: For every $\lambda > 0$ there is a δ^* such that for all $\delta > \delta^*$, the first period price in the bargaining game is above $1-\lambda$.

Proof: The proof follows the lines of Gul/Sonnenschein/Wilson and can be provided by the author upon request. ||

If as δ goes to one, the first period price goes to one, then bargaining ends almost instantaneously and the buyer gains an expected surplus of almost zero. Thus the sellers gain virtually all the surplus in the game which, since it is arbitrarily efficient, is arbitrarily close to 1/2. As δ goes to one, $B(\delta)$ goes to 1/2 and exceeds $A(\delta) < 3/8$. ||

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1. This paper is a revised version of the third chapter of my thesis, defended in Princeton University, November, 1987.

2. Just after finishing the revised version of this paper, I received a connected paper, developed independently by Georg Noldeke and Eric van Damme, "Signalling in a Dynamic Labour Market". Their model has a similar extensive form game but different specifications of payoffs and uncertainty and is motivated by the desire to examine the screening properties of a slightly different equilibrium concept than that used here.

3. In typical bargaining models attention is focussed on infinite horizon games. With three-person games, however, it is difficult to ensure uniqueness of equilibrium outcomes in infinite horizon extensive forms even with complete information. Many outcomes other than the competitive outcome can be supported as equilibria (see Vincent(1987)). The purpose of this paper is to observe the effects of competitive behaviour and, so, a finite period game is adopted.

4. A pbe characterizes beliefs at every information set after any history of the game. In the game studied, each uninformed buyer observes the complete play of the game after any period. (An information set h_t after any period t can be fully characterized by the sequence of pairs of price offers rejected by the seller. By assumption, both buyers observe this sequence. They must then form posterior beliefs over the distribution of the seller's type.) Given this similarity among buyers, henceforth, attention is restricted to pbe in which, after any history, the posterior belief of one buyer is the same as that of the other.

5. The original definition has been altered somewhat. i) Pse is not defined for games of more than two players. In this game in which all uninformed players observe the same history and have identical preferences, the extension will be made by requiring each to use the same updating rule.
ii) Pse as originally defined is more restrictive in that a belief can be rendered non-credible by a set k' by the existence of some best response σ' which justifies k's deviation. I require that the deviation be justified by the proposed equilibrium strategy.
ii) The updating rule g could be more general than the simple restriction of support characterized in the definition (and allowing different posterior distribution over q than the uniform distribution conditional on $q \in k$.) Lemma One in the appendix shows that this restriction is, in fact, implied by pse.

6. The terms `q is in $(q_t,1]$' and `buyers believe q_t' mean that the posterior beliefs are that the distribution of q is uniform over $(q_t,1]$.

7. It is clear that many other pbe can be constructed by appropriately specifying beliefs off the equilibrium path.

8. See Appendix Two for the formal argument. Notice that, unlike other games where sellers lose because of buyer collusion, there no such action occurring here. Buyers gain zero surplus. The strict preference of the seller for a monopsonistic buyer holds versus models with any number of buyers. Also note that this preference obtains by comparing the unique pse to the games.

9. For example, a rare wine merchant might wish to determine the best institution for auctioning bottles of wine. When the institution is created he may not know what wine he will sell but he is aware that when he actually acquires the wine to sell, he will have relevant private information. Christie's of Chicago hold wine auctions about four times a year. Sometimes, if a current bid is not high enough, they will withdraw the lot from the auction and put it up again some months later. If it were common knowledge that this is their policy, some additional information should be obtained from the observation that a wine had been withdrawn in the past. I am grateful to Beth Huntman of Christie's for this information.

10. Samuelson shows that the static mechanism which maximizes the seller's surplus is a take-it-or-leave-it game. By noting the identical roles played by discounting in dynamic trading mechanisms and played by the probability of trade functions in static mechanisms and recalling that buyers compete away all the surplus in the dynamic game described here, the same argument can be used to show that a single period game with buyers making simultaneous offers maximizes seller surplus among all finite-period games.

9 I am grateful to Ray Deneckere who suggested this more direct method of proof.

12. These facts are all proved in Vincent (1987).

Discussion Paper No. 770

DYNAMIC AUCTIONS[1]

by

Daniel R. Vincent*

March 1988

Abstract

A dynamic trading game is examined in which two uninformed buyers engage in Bertrand-like competition to attempt to purchase a single object of uncertain quality from an informed seller. It is shown that there exists a unique perfect sequential equilibrium. The game is compared to an analogous bargaining game in which a single uninformed buyer makes offers to a single seller. Despite the fact that in the equilibrium of the competitive game, buyers compete away their surplus, it is shown that sellers can often gain a higher expected surplus in the bargaining game.

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Buyers often compete to buy a good of uncertain quality from an informed seller. When the object is of high quality a seller will require to be paid more to part with the object than if it were of lower quality. Buyers, if they knew the good was of higher quality, would also be willing to pay more. However, if they do not know the quality of the object they must find ways to ensure that they do not pay too high a price for a bad good. This paper examines a specific extensive form trading game in which time is used as a screening device by competing buyers. While it is a commonplace that, in conditions of complete information, a seller benefits from truly competitive behaviour by buyers, it is shown here that, in other environments, a seller may actually prefer to limit the degree of competition among purchasers of his object.

Many examples of economic problems fit into the framework of the paper. Banks compete for loans to customers of uncertain reliability, university faculties compete for graduate students of uncertain abilities, insurance companies compete for clients of uncertain riskiness. The important common features of these problems include the fact that a potential seller has private information which is relevant for the value of the trade both to himself and to the buyers. Furthermore, uninformed agents have similar preferences and actively compete with each other for the object, there is a potential for gains from trade, and informed agents can use delay to agreement as a credible signalling device.[2]

The strategic structure in this paper consists of T periods. In each period, two uninformed buyers offer prices simultaneously to an informed seller. The seller can choose which of the prices to accept or to reject is welfare improving over the one-period game. On the other hand, analyses from the literature on mechanism design have suggested that one-period mechanisms are often welfare maximizing subject to individual rationality and incentive compatibility. Section VI shows that in the class of games examined in this paper, a seller prefers a one-period game over a multi-period game.

Section II: The Model

Two identical buyers compete to buy a single good from a single seller. The trading mechanism consists of T+1 periods (the first period is t = 0) -- in each period both buyers submit price offers simultaneously, the seller observes the offers and decides which if any to accept. If he accepts, the game ends. The buyer whose price is accepted gains the good and pays the seller the amount of the accepted offer. If both offers are rejected and the period is t, less than T, the game continues and the same play is repeated. If the period is the last one, T, and a rejection occurs the game ends with no trade taking place. After any period t, it is assumed that all players know the full history of the game to period t.

Let D_B be the real line and D_S, the space of probability distributions over the finite set {0,1,2} where {0} represents no trade, {1} represents trade with buyer one (b1) and {2} trade with buyer two (b2). Let H_{t-1} represent the space of histories of pairs of rejected price offers up to and including period t-1 and let σ_{it} be a function mapping H_{t-1} into D_B. Buyer i's strategy choice is a sequence of such functions $\sigma_i = \{\sigma_{it}\}$ for t from 0 to T. Similarly let σ_{sqt} be a function mapping H_t into D_S. The strategy choice of a seller of type q is the sequence of functions $\sigma_{sq} = \{\sigma_{sqt}\}$ for t from 0 to T.

support outcomes on the basis of what will be argued are implausible out of equilibrium beliefs. This paper uses a modified version of the more stringent refinement concept, perfect sequential equilibrium (pse) introduced in Grossman and Perry (1986). Any pse is a pbe but the converse does not hold in general.[4]

A meta-strategy is defined as a strategy choice of a player after a given history and a given belief. (For the informed seller, beliefs are trivial.) Let K be the space of subsets of [0,1] and now redefine σ_{it} to be a function from the cross-product space H_{t-1} and K into D_B. Perfect sequential equilibrium uses the concept of a meta-strategy to specify actions after any history and for any given candidate belief.

For a meta-strategy to form part of a pse it must be sequentially rational. Sequential rationality is defined in GP only for two-person games, however. What follows provides a natural adaptation for the specific three person game examined here (a similar technique is also used in Madrigal and Tan (1986)). Fix a period, t, history h_{t-1}, and a belief (held commonly between buyers) k_{t-1}. Fix some updating rule $g(h_{t-1},(p_{1t},p_{2t}),k_{t-1})$ which specifies the beliefs that will be held following any history h_{t-1} and rejected price offers (p_{1t},p_{2t}) when beliefs were k_{t-1}. A meta-strategy characterizes the play of the subform for every pair of price offers (p_1,p_2) via p's effect on the strategies directly and through its indirect effect on the updating rule. Given the strategies of the seller types, a belief over the types of the seller and the strategy of the other buyer as well as his own strategy following the play of a given period t, a buyer, i, can determine his expected payoff at period t by varying his price offer p_{it}. A meta-strategy $(\sigma_1,\sigma_2,\sigma_{sq})$ is said to be sequentially rational at time t with

Section III: An Example

This section provides an example of the game described in this paper which helps to illustrate the nature of a pse in the model. It shows, as well, why the equilibrium refinement is a persuasive one in this context.

Let $f(q) = q$, $v(q) = q+1/2$, $T = 2$ and $\delta \geq 3/4$. Here, the quality of the object from the buyers' points of view varies uniformly from 1/2 to 3/2 and it is always worth one-half more to the buyers than to the seller. Buyers submit bids in the first period. If the bids are rejected, they resubmit bids a second and final time. The following strategies form a perfect Bayesian equilibrium of the game but not a pse.[6]

Seller's strategies: (First period)

For $q \in [0,(1-\delta)/(1-\delta/2))$ accept $p \geq (1-\delta)q + \delta(1+q/2)$,

For $q \in [(1-\delta)/(1-\delta/2),\delta/(2-\delta))$ accept $p \geq 1$,

For $q \geq \delta/(2-\delta)$, accept $p \in [1,\delta(3/2)]$ and $p \geq (1-\delta)q + \delta(1+q/2)$.

(Second period): For all q, accept $p \geq q$.

Buyers' strategies: In period one, offer $p_{i1} = 1$, $i = 1,2$.

In period two, let $p^* = \max(p_{11},p_{21})$. Beliefs and strategies are:

a) $p^* \in [\delta,1)$, q is in $[(p^*-\delta)/(1-\delta/2),1]$ and offer $p_2 = .5(1+(p^*-\delta)/(1-\delta/2)) +.5$.

b) $p^* \in [1,\delta 3/2]$, q is in $[0,1]$, offer $p_2 = 1$.

c) $p^* \in (\delta 3/2, (1-\delta) + \delta 3/2]$, q is in $[q^*,1]$ where q^* satisfies $p^* = (1-\delta)q^* + \delta(1+q^*/2)$ and offer $p_2 = .5(q^*+1) + .5$.

The equilibrium path prescribes an offer of one in the first period by both buyers and acceptance by all sellers.

Note that if any price between one and $\delta 3/2$ is offered and rejected an `out of equilibrium message' has been sent by the seller. Perfect Bayesian following. Suppose a price $p = 1$ is offered and rejected. The only way a type $q < k$ could have been better off so rejecting is if the buyers, in fact, believe that these very types would not have rejected. In the intuitive criterion pbe described first, buyers believe that very low-quality sellers might have rejected the price offer of one as well as high quality sellers. However, buyers have this belief only if they believe that the low-quality sellers expect that the buyers will not have this belief, that the buyers will (over-optimistically) believe that some of the low-types did accept and, so, will offer a higher price in the continuation game than the pbe actually prescribes. An additional order of introspection should lead the buyers to reject this belief on the assumption that all sellers could mimic the same reasoning process. A rejection of a price of one in the first period should lead to a concentration of beliefs on higher sellers and a higher price offer as a result. This, in turn, destroys the first pbe profile.

The unique pse path to the game is calculated by exploiting this argument in a natural way.

Seller's strategies: (First period) Accept $p \geq (1-\delta)q + \delta(1+q/2)$.

(Second Period) Accept $p \geq q$.

Buyers' strategies: In period one offer $p(q^*)$ such that

$$\int_0^{q^*} x + 1/2 - p(q^*)dx = 0,$$

and $p(q^*) = (1-\delta)q^* + \delta(1+q^*/2)$. Note that this implies $q^* = 0$ and no trade occurs in the first period.

(Second period) For $p^* = (1-\delta)q^* + \delta(1+q^*/2)$, believe q is in $[q^*,1]$ and offer $p_2 = 1 + q^*/2$.

$$q_{i+1} - q_i = 2\epsilon(1-\delta)(1-(-\delta)^{i+1})/(1+\delta). \qquad (1)$$

Along the pse path, the first offer by the buyers is accepted by sellers of type less than q in $(q_{T^*-1}, q_{T^*-2}]$, the next offer is accepted by some q in the next interval and so on. The proof also shows that in each period, buyers make an offer exactly equal to the expected value of the object to them conditional on the seller types who accept, that is, in each period, buyers gain an expected utility of zero.

For games with periods less than T* there is also a unique pse path -- its form, though, is typically different from that described in the theorem. The example in Section 3 is an illustration of equilibria in such games. Notice in that example that the pse path consisted of a period (the first period) in which there was no possibility of trade. The theorem along with equation (1) indicates that this phenomenon does not remain true if the number of periods in the game is large enough since $q_{i+1} - q_i$ is greater than zero. However, the possibility of inefficiencies remains as the next section shows. Observe that for any fixed ϵ, as δ goes to one, (1) also indicates that seller types are essentially perfectly separated.

Section V: The Inefficiency of Competitive Markets

This section addresses the following question. Is the ex ante utility of the seller always higher when there is more than one buyer competing against each other? It is shown, here, that the seller may prefer to restrict the competitive behaviour of his buyers if there is uncertainty about the quality of the object he is selling. In particular, he may prefer to face a single buyer who is allowed to make all the offers than many competing buyers.

The similarity of the game form and payoffs in this model with that of

Let t2 be the period in which sellers of type q = 1/2 accept. Therefore, for q = 0 to accept the equilibrium price of one-half, it must be the case that

$$1/2 > \delta^{t2}(1) \text{ or } 1/2 > \delta^{t2}.$$

Since sellers gain all the surplus in this game, this equation shows that with probability one-half, seller types incur a waiting cost of δ^{t2}. The total surplus that sellers can hope to gain in this game is less than 1/4 + (1/2)(1/2)(1/2) = 3/8.

Now consider the total surplus available when, instead of a competitive game the seller is in a bargaining game in which a single buyer makes offers in each period. This game is examined in Vincent (1988) and has a unique pbe (and pse) path. As δ goes to one, an argument based on the Gul-Sonnenschein-Wilson proof of the Coase conjecture shows that the first price offer is arbitrarily close to the last price offer of one. That is, the game ends arbitrarily quickly and the total surplus gained by the sellers is close to one-half. They gain almost all the surplus in the game.[8]

The bargaining game dominates the dynamic auction game from the seller's point of view because of the informational asymmetry. In the bargaining game, the single buyer is willing to take advantage of the fact that he alone is purchasing the object to take the risks represented by offering a price close to one early on even though it will be accepted by sellers of objects which are worth strictly less than one to him. He is willing to do this because a rejection tells him that he can gain a positive surplus from higher quality types in the continuation game. This gain compensates him (only just) for the risk taken earlier on. The absence of another buyer allows him to appropriate to himself the information contained in a rejection to go after the high-quality seller types. A consequence is a model of the form $v(q) = q + \epsilon$, $f(q) = q$. With a low ϵ, it can be shown that in a one-period game, trade only occurs when q is less than 2ϵ, thus generating welfare losses of $\epsilon(1-2\epsilon)$. Nevertheless, an adaptation of a result by Samuelson shows that the seller will always prefer a one-period game.[10] The costs of satisfying self-selection constraints in a dynamic game will always outweigh the additional potential gains from trade that a multi-period game may make possible.

Section VI: Conclusion

This paper analyzes equilibrium behaviour in a dynamic auction game. In conventional models of static auctions, information transmission only occurs -- if at all -- through the process of bidding. When it is the seller who enjoys private information, then, this information cannot be transmitted in the play of the game. In the dynamic game examined in this paper, not only can the seller's information be transmitted but sometimes it must be transmitted, potentially at costs to efficiency. The example in Section III illustrates this point where the opportunity for buyers to gain information forces trade in equilibrium to occur only in the last period.

Dynamic auctions are also compared to dynamic bargaining games in this paper. In finite period models of complete information, a seller always prefers to face many buyers who are making offers (where he wins all the surplus) than to face just one (where he loses all the surplus). When the seller is privately informed about the quality of the good, however, this preference may be reversed. As Section V illustrates, the screening mechanism which allows buyers to gain necessary information about the object they are purchasing can be more costly in a competitive game than in a bilateral monopoly game. This cost is borne, in part, by the seller, and

Appendix One

Theorem One: Let $f(q) = q$, $v(q) = q + \epsilon$, $\epsilon > 0$, $0 < \delta < 1$. There is a T^* such that for all T - period games, $T > T^*$, there exists a unique and identical pse path.

Proof; The proof of the theorem is preceded by four Lemmas. Define $p_t = (p_{1t}, p_{2t})$ $p^*_t = \max(p_{1t}, p_{2t})$, $p^-_t = \min(p_{1t}, p_{2t})$.

Lemma 1.1: Fix a pbe to the game. Suppose at any period t (with history h_t) and beliefs by both buyers that q is in $[q_t, 1]$, price p_t is offered. If `reject p^*_t' is a weak best response for $q' < 1$, then it is a strictly best response for all $q'' > q'$.

Proof: Reject is a weak best response for q' implies

$$p_t - f(q') \leq \max(0, E(\delta^{t'}(p_{t'} - f(q')))) \qquad (1)$$

where E(.) is the expected value of the possibly random path $\{p_{t'}\}_{t'>t}$ determined by the pbe when history (h_{t-1}, p_t) has occurred. $q'' > q'$ and (1) imply

$$p_t - f(q'') < \max(0, E(\delta^{t'}(p_{t'} - f(q'')))). \qquad ||$$

Corollary: In any pse, in any period t with beliefs $[q_t, 1]$, if $p^*_t \leq (1 - \delta)f(1) + \delta v(1)$ is rejected, then the credible updating rule must set q in $[q_{t+1}, 1]$ (or $(q_{t+1}, 1]$), $q_{t+1} \geq q_t$. If $p^*_t > (1 - \delta)f(1) + \delta v(1)$, no restriction can be placed on beliefs. ||

A consequence of the corollary is that for any history (with $p_t \leq (1 - \delta)f(1) + \delta v(1)$) the updating rule $g_t(h_t)$ can be characterized by q_{t+1}, the bottom end of the support. This convention is used throughout the proof.

Lemma 1.2: Let T > 1. Define $q_1 = 1-2\epsilon(1-\delta)$. For all t < T-1, if q is in $(q_1, 1]$, q follows a strategy

Accept $p_t \geq P_t(q) = (1-\delta)q + \delta((1+q)/2 + \epsilon)$,

$$P_t(q_i)-q_i = \delta(P_t(q_{i-1})-q_i) \qquad (b).$$

Note that a) and b) imply

$$(P_t(q_{i-1})-q_i) = 2(\epsilon-\delta(P_t(q_{i-2})-q_{i-1}), \text{ so}$$

$$q_i-q_{i+1} = 2\epsilon-\delta(2\epsilon-\delta(P_t(q_{i-2})-q_{i-1})))$$
$$= 2\epsilon(1-\delta)(1-(-\delta)^{i+1})/(1+\delta)$$
$$\geq 2\epsilon(1-\delta)^2.$$

Define $T^* = \min\{i \mid q_i \leq 0\}$.

Let $P:(q_i,1]$ to R be as in Lemma 2. Define $q^*(q):(q_2,q_1]$ to $(q_1,1]$ implicitly by

$$P(q^*(q)) = .5(q^*(q)+q)+\epsilon \qquad (ZP)$$

Observe that $q^*(q_1) = 1$ and $q^*(q_1) = q_1$ by (a).

Given $P':(q_i,1]$ to R and $q^{*\prime}:(q_i,q_{i-1}]$ to $(q_{i-1},q_{i-2}]$ we can extend the domain of P' and $q^{*\prime}$ iteratively by the rules:

i) $q^*:(q_{i+1},q_i]$ to $(q_i,q_{i-1}]$ satisfies $q^*(q) \in (q_i,q_{i-1}]$ such that $.5(q^*(q)+q)+\epsilon = P'(q^*(q))$ and then

ii) $P : (q_{i+1},1]$ to R satisfies $P(q) = P'(q)$ if $q > q_i$ and

$P(q) = (1-\delta)q + \delta P'(q^*(q))$ for $q \in (q_{i+1},q_i]$.

P and q^* are thus defined over [0,1]. Note that $P(.)$ and $q^*(.)$ are continuous and strictly increasing. Note also that since the derivative of $P(q)$ inside the interval $(q_1,1]$ is $1-\delta/2$, the derivative of $P(q)$ (left or right derivative) is not less than $1-\delta/2$.

The next two lemmas rely on the following two induction hypotheses:

A1) Suppose that in any pse, σ, for any $q' \in (q_j,q_{j-1}]$, $j \leq i$, and for any history $h_{t'-1}$, $t' \leq T-j$, the pse strategy of q' is given by: If $p^*_{t'} \geq P(q')$, accept p^*, else reject, and that all $q < q'$ accept $p \geq P(q)$.

A2) Suppose that for any pse σ, for any $t \leq T-j$, $j \leq i$, if buyers' beliefs straightforward to show. To see that it is the only pse strategy consider h_{t+1} after some $p_t \in (P(q_{i+1}, P(q_i))$ has been rejected. Let buyers' beliefs be $(k,1]$. A1) ensures $k < q_i$. If $k \in (q_{i+1}, q_i]$, then Lemma 3 shows that the unique next period pse price offer is $P(q^*(k))$. If $k > P^{-1}(p_t) = q_t$, by definition of $P(.)$, $p_t - k > \delta(P(q^*(k))-k)$ so there are types in the interval (q_t, k) who would have done better to accept p_t as well. Similarly for $k \in (q_{i+1}, q_t]$. For $k \leq q_{i+1}$, an argument similar to the last part of Lemma 2 shows that the belief $(q_t, 1]$ renders this belief 'incredible'. ||

Proof of the theorem: Using Lemma 2 to validate A1 and A2 and iteratively applying Lemmas 3 and 4 implies that for any game of length $T \geq T^*$, the unique pse price offer in the first period is a price which (in the event of a rejection) would lead to next period beliefs $(q_{t+1}, 1]$, $q_{t+1} \in (q_{T^*-1}, q_{T^*-2}]$, the next in $(q_{T^*-2}, q_{T^*-3}]$, and so on. In each subsequent period, the assumptions of the hypotheses A1 and A2 hold and determine equilibrium behaviour in the continuation game. ||

the first period price offer of the single buyer approaches one. The argument is essentially the same as in Gul/Sonnenschein, Wilson, Theorem Three, the proof of the Coase Conjecture. Note that the bargaining game has a unique pbe (which is also the unique pse path) and that sellers' strategies can be denoted by an acceptance function P(q) which is continuous, strictly increasing and independent of the history of the game.[12]

Lemma 2.1: For every $\lambda > 0$ there is a δ^* such that for all $\delta > \delta^*$, the first period price in the bargaining game is above $1-\lambda$.

Proof: The proof follows the lines of Gul/Sonnenschein/Wilson and can be provided by the author upon request. ||

If as δ goes to one, the first period price goes to one, then bargaining ends almost instantaneously and the buyer gains an expected surplus of almost zero. Thus the sellers gain virtually all the surplus in the game which, since it is arbitrarily efficient, is arbitrarily close to 1/2. As δ goes to one, $B(\delta)$ goes to 1/2 and exceeds $A(\delta) < 3/8$. ||

1. This paper is a revised version of the third chapter of my thesis, defended in Princeton University, November, 1987.

2. Just after finishing the revised version of this paper, I received a connected paper, developed independently by Georg Noldeke and Eric van Damme, "Signalling in a Dynamic Labour Market". Their model has a similar extensive form game but different specifications of payoffs and uncertainty and is motivated by the desire to examine the screening properties of a slightly different equilibrium concept than that used here.

3. In typical bargaining models attention is focussed on infinite horizon games. With three-person games, however, it is difficult to ensure uniqueness of equilibrium outcomes in infinite horizon extensive forms even with complete information. Many outcomes other than the competitive outcome can be supported as equilibria (see Vincent(1987)). The purpose of this paper is to observe the effects of competitive behaviour and, so, a finite period game is adopted.

4. A pbe characterizes beliefs at every information set after any history of the game. In the game studied, each uninformed buyer observes the complete play of the game after any period. (An information set h_t after any period t can be fully characterized by the sequence of pairs of price offers rejected by the seller. By assumption, both buyers observe this sequence. They must then form posterior beliefs over the distribution of the seller's type.) Given this similarity among buyers, henceforth, attention is restricted to pbe in which, after any history, the posterior belief of one buyer is the same as that of the other.

5. The original definition has been altered somewhat. i) Pse is not defined for games of more than two players. In this game in which all uninformed players observe the same history and have identical preferences, the extension will be made by requiring each to use the same updating rule.
ii) Pse as originally defined is more restrictive in that a belief can be rendered non-credible by a set k' by the existence of some best response σ' which justifies k's deviation. I require that the deviation be justified by the proposed equilibrium strategy.
ii) The updating rule g could be more general than the simple restriction of support characterized in the definition (and allowing different posterior distribution over q than the uniform distribution conditional on $q \in k$.) Lemma One in the appendix shows that this restriction is, in fact, implied by pse.

6. The terms `q is in $(q_t,1]$' and `buyers believe q_t' mean that the posterior beliefs are that the distribution of q is uniform over $(q_t,1]$.

7. It is clear that many other pbe can be constructed by appropriately specifying beliefs off the equilibrium path.