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SEARCH EQUILIBRIUM AND THE CORE IN A DECENTRALIZED PURE EXCHANGE ECONOMY

by

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1. Introduction

In what has become General Competitive Analysis [1] the existence of two different relative prices for the exchange of the same two commodities is ruled out by assumption. This assumption can be interpreted in two ways. Either all trades take place in a single all inclusive market; e.g. Debreu [3], or price differentials vanish in the face of arbitrage; e.g. Walras [15]. Given the former interpretation, there is only one market while many markets act as one in the latter case. From the point of view of one interested in positive economic theory, the first interpretation is untenable. The required degree of centralization in the exchange activity is never observed and only rarely approximated even for subsets of commodities. The second interpretation is really a theorem in need of a proof.

The paper contains a study of a competitive-like pure exchange equilibrium in which more than one price vector can simultaneously prevail. This possibility arises only in the context of an exchange economy containing many markets and, then, only if arbitrage among them is costly or time consuming. We establish that these conditions are also sufficient given appropriate assumptions about the information available to the traders. Consequently, the paper provides a rationale for the differentials which are supposed to motivate search, a problem discussed by Rothschild [12]. Although the rationale is different from that recently suggested by Lucas and Prescott [6], in many other respects the two models are closely related. Finally, we present conditions which imply that differentials vanish as the number of traders becomes large.

The economy considered is composed on n traders and m markets in which ℓ non-storable commodities are exchanged. In general, we allow for the possibility that some traders may be unable to exchange in all of the markets and the possibility that not all commodities are exchanged in some markets. We imagine that exchange takes place in each of a sequence of periods and that each trader receives a fixed endowment at the beginning of every period. In addition, each trader can exchange in one and only one market per period, the allocation made to the traders in the market searched is competitive relative to the coalition which searches the market during the period and the corresponding competitive price vector is signalled to all traders at the end of the period. Generally then, it is impossible for any trader to know with certainty the price vector which will prevail in any market at the beginning of the period when he must decide which to search. However, he can construct an expectation from past observations.

The problem which concerns us is the choice of market by each trader. We allow mixed strategies so that each may search as if sampling from a probability distribution defined on the set of markets. Hence, any joint search strategy and the assumed price rule induce a probability distribution on the set of possible price vectors associated with each market. A set of these, one for each market, is a search equilibrium if and only if the joint strategy which generates it is such that each trader's component maximizes his expected utility given the equilibrium. In other words, optimal individualistic choice and "rational" price expectations characterize a search equilibrium. Existence is established by first showing that any joint search strategy generating an equilibrium is a Nash solution to the game of choosing membership in one of the m possible trading coalitions.

Because the core is a cooperative solution to this game, a comparative study of search allocations and core allocations suggests itself. But, by example, we establish that a random equilibrium search allocation is a possibility. Say, then, that a search allocation is in the core if and only if one of its certainty equivalents is in the core. In the case of search equilibria generated by joint strategies with the property that identical traders pursue the same search strategy, results in the spirit of the following statement hold: Search allocations are close to the core when the number of traders of each type is large if all have direct utility functions which are concave in commodities and indirect utility functions which are convex in prices. In the context of an economy composed of identical markets, the result implies that search equilibrium allocations and prices are approximately competitive. Consequently, limited search by many traders has the same effect as complete and instantaneous arbitrage by a few.

2. Preliminaries

The index set $L = \{1,\ldots,\ell\}$, $M = \{1,\ldots,m\}$ and $N = \{1,\ldots,n\}$ represent the set of commodity names, market names and trader names respectively. L_j, $j \in M$, is the subset of commodities which can be traded in market j and M_i is the subset of markets in which trader i is able to engage in exchange. We presume that

$$|L_j| \geq 2 \quad \forall j \in M \quad \text{and} \quad \bigcup_{j \in M} L_j = L \tag{1a}$$

$$|M_i| \geq 1 \quad \forall i \in N \quad \text{and} \quad \bigcup_{i \in N} M_i = M \tag{1b}$$

where $|\cdot|$ denotes the cardinality of a finite set.[1/] Markets differ either because the commodities exchanged in them differ or because the set of traders who can participate in exchange in them differ.

The process by which traders find one another we call search. Traders meet in markets. By assumption each trader can search one and only one market per period. In general random search is allowed in the sense that a trader may select a market to search as if sampling from a probability distribution defined on the set of markets. A particular probability distribution is called a search strategy.

A joint search strategy is a collection of n search strategies, one for each trader. It can be represented as an nxm vector in any of the following equivalent ways:

$$S = [s_{ij}] = (s_1, \ldots, s_i, \ldots, s_n) = (s^1, \ldots, s^j, \ldots, s^m).$$

The representative element s_{ij} is the probability that trade i searches market j during the period, $s_i = (s_{i1}, \ldots, s_{ij}, \ldots, s_{im})$ is the search strategy of trader i and $s^j = (s_{1j}, \ldots, s_{ij}, \ldots, s_{nj})$ is the vector of search probabilities associated with market j.

Let $\mathcal{S}_i$ denote the search strategy space of trader i and $\mathcal{S}$ denote the joint search strategy space. The former is the set of all probability distributions defined on M_i and the latter is the cartesian product of all the trader's strategy spaces.

$$\mathcal{S}_i = \{s_i \in R^m_+ \mid \sum_{j \in M} s_{ij} = 1 \quad \text{and} \quad s_{ij} = 0 \quad \forall j \notin M_i\},\ i \in N \tag{2a}$$

$$\mathcal{S} = \mathop{X}_{i \in N} \mathcal{S}_i \tag{2b}$$

Assumption (2a) allows for the possibility that barriers to exchange may exist; e.g. some markets may either be located far from trader i or may exclude trader i for non-economic reasons. But, the strategy choices open to trader i are not conditioned or restricted by the strategy chosen by any other trader by virtue of (2b). Clearly, a joint search strategy S is feasible if and only if $S \in \mathcal{S}$.

A <u>joint realization</u> of the search process is a set of m trader coalitions, one for each market. Let

$$T = [t_{ij}] = (t_1, \ldots, t_i, \ldots, t_n) = (t^1, \ldots, t^j, \ldots, t^m)$$

denote a realization where t_{ij} equals unity when trader i searches market j during the period and zero otherwise. Then, $t_i = (t_{i1}, \ldots, t_{ij}, \ldots, t_{im})$ is the vector of realizations associated with trader i and $t^j = (t_{1j}, \ldots, t_{ij}, \ldots, t_{nj})$ is the <u>trader coalition</u> which searches market j during the period. Since each trader searches one and only one market per period, T is feasible if and only if it is an element of $I(\mathcal{S})$ where $I(\cdot)$ denotes the integer points in any set.

As a set of probabilities, the joint search strategy is an <u>ex ante</u> concept. Obviously, a realization is the result of the random experiment called search. The two, S and T, are related by the fact that

$$\Pr\{t_{ij} = 1\} = s_{ij}, \quad (i,j) \in N \times M. \tag{3}$$

By assumption none of the commodities traded are storable and all markets are spot. In addition, preferences are intertemporally separable. These assumptions rule out the possibility of intertemporal speculation. Furthermore, we assume that each trader acts as a price taker within the market he searches during the period. Finally, trade takes place in each market in accord with a price vector which clears the market given the coalition which searches during the period. Formally, these assumptions require that a market clearing price vector exists corresponding to every possible coalition formable in market j and that a demand function exists for trader i given any possible price vector associated with market j.

Associated with market j is a <u>market clearing price function</u> f_j which maps the set of coalitions which can form in j, $\mathcal{T}^j$, to the price space associated with the market, $\mathcal{P}_j$. The coalitions are restricted in general as a consequence of the restrictions of the search of individual traders and the only meaningful prices are those for commodities which are traded in market j. In particular,

$$f_j : \mathcal{T}^j \to \mathcal{P}_j \tag{4}$$

such that

$$\sum_{i \in N} t_{ij} d_{ij}(f_j(t^j)) = \sum_{i \in N} t_{ij} \omega_i \tag{4a}$$

where $d_{ij}(p)$ denotes the demand vector of trader i given that he searches market j during the period and ω_i is the endowment of trader i. Note that (4) does not preclude the possibility that two or more market clearing price vectors exist and does not require that f_j be continuous. Furthermore,

$$\mathcal{P}_j = \{p \in R_+^{\ell} \mid \sum_{k \in L} p_k = 1 \text{ and } p_k = 0 \text{ if and only if } k \notin L_j\} \tag{4b}$$

and

$$\mathcal{T}^j = \{t^j \in R_+^n \mid 0 \leq t_{ij} \leq 1 \ \forall i \text{ such that } j \in M_i \text{ and } t_{ij} = 0 \text{ otherwise}\} \tag{4c}$$

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Finally, then a market is completely characterized by $(f_j, \mathcal{P}_j, \mathcal{T}^j)$.

A trader is characterized by the triplet $(u_i, \omega_i, \mathcal{S}_i)$ where u_i denotes his utility function. By assumption u_i is continuous, strictly quasi-concave and increasing. The endowment is non-zero and the strategy space satisfies (2). Consequently, for every $j \in M_i$ a demand function

$$d_{ij}: \mathcal{P}_j \rightarrow R^{\ell}_{+} \tag{5}$$

exists such that

$$u_i(d_{ij}(p)) = \max_x u_i(x) \text{ such that } px \leq p\omega_i \text{ and } x_k = \omega_{ik} \ \forall k \notin L_j \tag{5a}$$

and

$$pd_{ij}(p) = p\omega_i. \quad \underline{3}/ \tag{5b}$$

The indirect utility obtained by trader i during the period given that he searches market j is by definition

$$\varphi_{ij}(p) = u_i(d_{ij}(p)). \tag{6a}$$

Consequently, the utility obtained during the period can be expressed as

$$\emptyset_i(P, t_i) = \sum_{j \in M} t_{ij} \varphi_{ij}(p_j) \tag{6b}$$

where P is a vector of m price vectors, one for each market ; i.e.

$$P = [p_{kj}] = (p_1, \ldots, p_j, \ldots, p_m).$$

Of course, $\emptyset_i: \mathcal{P} \times \mathcal{S}_i \rightarrow R$ by virtue of (5) where $\mathcal{P}$ is the joint price space defined as follows:

$$\mathcal{P} = \underset{j \in M}{X} \mathcal{P}_j.$$

Given any particular feasible joint strategy $S \in \mathcal{S}$, each element of the feasible set of search realizations $I(\mathcal{S})$ will occur during the period with some probability by virtue of (2) and (3). Consequently, S and the price functions defined in (4) induce a probability distribution on the joint price space $\mathcal{P}$.[4/] By assumption, every trader must choose his search strategy, his component of S, at the beginning of the period prior to the realization of the search process. Therefore, he cannot know with certainty what the price vector will be in any one of the markets in general. However, given the probability distributions on the space of search realizations $I(\mathcal{S})$ and the price space generated by any S, the expected utility associated with each trader $E\phi_i(P,t_i)$ is well defined by virtue of (5) and (6). Indeed, since $s_{ij} = Et_{ij}$ from (3),

$$E\phi_i(P,t_i) = \sum_{j\in M} s_{ij}\, E\{\varphi_{ij}(p_j) \mid t_{ij} = 1\}$$

where $E\{\varphi_{ij}(p_j) \mid t_{ij} = 1\}$ is the expected utility that trader i would receive in market j given he were to search it during the period.

3. The Existence of Search Equilibrium

By a search equilibrium, we mean a probability distribution on the joint price space generated by a joint search strategy with the property that the search strategy of each trader maximizes his expected utility when calculated with respect to the equilibrium. Let E^* denote the expectation operator taken with respect to the joint distribution on $\mathcal{P}$ and $I(\mathcal{S})$ generated by some feasible joint strategy S^*.

Definition (D.1): The probability distribution on $\mathcal{P}$ generated by an $S^* \in \mathcal{S}$ is a search equilibrium if and only if

$$E^* \emptyset_i(t_i, P) = \max_{s_i \in \mathcal{S}_i} \sum_{j \in M} s_{ij} E^* \{\varphi_{ij} \mid t_{ij} = 1\} \quad \forall\ i \in N.$$

An equilibrium, then, is characterized by (i) optimal choice of a search strategy given each trader's expectation about prices in the various markets and by (ii) "rational" expectations in the sense of Muth [9].

The existence proof which follows consists primarily of a demonstration that the concept of a search equilibrium is equivalent to the concept of a Nash [10] solution to the n-person game of choosing trading partners. The demonstration is more than simply the "trick" by which existence is established. It also illustrates the structure of the problem which arises when each trader is assumed to use information only about prices in the various markets to make his search strategy decision.

The set of coalitions possible in market j consists of the integer points of $\mathcal{T}^j$, $I(\mathcal{T}^j)$. Denote the representative vector in this set, t^j, given that $t_{ij} = 1$ as $(1, \langle t^j_i \rangle)$ where $\langle t^j_i \rangle$ is t^j with its i^{th} component deleted. Because the joint strategy space $\mathcal{S}$ is the product of the individual trader's strategy spaces, the search of any market by two different traders are stochastically independent events. Consequently,

$$\Pr\{t^j = y \mid t_{ij} = 1\} = g(\langle y_i\rangle, \langle s^j_i\rangle) \tag{7}$$

$$= \begin{cases} \prod_{k\neq i} (s_{kj})^{y_k}(1-s_{kj})^{1-y_k} & \text{if } (1,\langle y_i\rangle) \in I(\mathcal{T}^j) \\ 0 & \text{otherwise.} \end{cases}$$

Note that the function g is continuous in its second argument.

Let $f_j^{-1}p$, $p \in f_j I(\mathcal{T}^j)$, denote the subset of coalitions associated with the price p. By virtue of (7),

$$E\,\phi_i(p,t_i) = \sum_{j\in M} s_{ij}\, E\{\varphi_{ij}(p) \mid t_{ij} = 1\}$$

$$= \sum_{j\in M} s_{ij} \sum_{p\in \mathcal{P}_j} \varphi_{ij}(p) \sum_{(1,\langle y_i\rangle)\in f_j^{-1}p} g(\langle y_i\rangle, \langle s^j_i\rangle)$$

$$= \theta_i(s_i, \langle S_i\rangle) \tag{8}$$

since $s_{ij} = E\,t_{ij}$ and

$$\langle S_i\rangle = (s_1,\ldots,s_{i-1}, s_{i+1},\ldots,s_n) = (\langle s^1_i\rangle,\ldots,\langle s^j_i\rangle,\ldots,\langle s^m_i\rangle). \tag{9}$$

Moreover, θ_i is linear in the trader's own strategy, s_i, and is continuous in all others, $\langle S_i\rangle$, by virtue of (8). In addition, definition D.1 implies that S^* generates a search equilibrium if and only if

$$\theta_i(s^*_i, \langle S^*_i\rangle) = \max_{s_i\in \mathcal{S}_i} \theta_i(s_i, \langle S^*_i\rangle) \quad \forall\, i \in N \tag{10}$$

But, (10) defines a Nash solution to the game of search.

<u>Theorem 1</u>: A joint strategy S^* generates a search equilibrium if and only if it is a Nash solution to the game $\{\theta_i,\ i=1,\ldots,n;\ \mathcal{S}\}$.

Corollary 1: A search equilibrium exists.

Proof. Because θ_i is continuous in S and linear in s_i $\forall$ i ϵ N and because the joint strategy space $\mathcal{S}$ is a compact convex set, Rosen's [12] existence theorem applies.

It is obvious that prices for the same commodity are likely to be different across markets in general simply because of the restrictions imposed on search and trade. However, one can show that *ex post* price differentials will exist in at least one search equilibrium when all traders are able to search every market and every market allows trade in all commodities. In this case, (4b) and (4c) imply that the price and coalition space are the same for all markets since $L_j = L$ $\forall$ j ϵ M and $M_i = M$ $\forall$ i ϵ N. Moreover, there is no reason for any trader's demand functions to differ across markets by virtue of (5a). Consequently, there is a price function f common to all markets by virtue of (4a).

Corollary 2: If all markets are identical, then random search with equal probability, $S = [\frac{1}{m}]$, generates a search equilibrium.

Proof. In this case, if $\langle S_i \rangle = [\frac{1}{m}] \in \underset{k \neq i}{X} \mathcal{S}_k$, the conditional distribution on p_j given $t_{ij} = 1$ is the same for all j. Hence, trader i can find no strategy which dominates random sampling.

In general, differences in the *ex post* price vectors in two identical markets can be attributed to the restriction that every trader can search but one market per period. If this restriction were relaxed, then each trader could spread his demands evenly among the markets. Price differences would not exist and the common price vector would be a competitive equilibrium. The indivisibility of the traders in their search activity, which in our model reflects the time required for search, is the crucial assumption.

Of course, a competitive search equilibrium does exist when all markets are identical.

<u>Corollary 3</u>: If all markets are identical, then the joint strategy characterized by $s_{ik} = 1$ for some $k \epsilon M$ and all $i \epsilon N$ generates a search equilibrium. Moreover, the price vector in market k is a competitive equilibrium with probability one.

Proof. Obvious.

Although the use of a single market by all traders is clearly a possibility, some cooperation in the form of a collective agreement regarding which one it will be is obviously necessary. No such agreement is required to achieve a random search equilibrium. Indeed, it is the likely outcome if each trader is initially totally ignorant about where the others plan to go.

It is obvious from the preceding comments that much must be added to the model before a meaningful stability theory can be derived. The remainder of the paper is an indirect approach to that problem. In particular, our purpose is to find conditions under which the efficiency losses associated with equilibria of the type generated by random search are small. As a preview, note that equation (3) and the equations of (4) imply that the price vector in every market would be distributed about a competitive equilibrium with a small variance given random search if all markets were identical and the number of traders of each of n types were large.

4. Search Equilibria and the Core

We have shown that a search equilibrium is a non-cooperative solution to the game of choosing ones trading partners. The core can be regarded as the set of cooperative solutions to this same game. For this reason a comparison of the allocations associated with search equilibria with those in the core suggests itself as a first step in the study of the efficiency properties of the search process. As a means of simplifying the study, we restrict our attention to the case in which all markets are identical. [5/]

An allocation $x = (x_1, \ldots, x_i, \ldots, x_n) \in R^{\ell \times n}$ is an assignment of commodity bundles to traders where $x_i \in R^{\ell}$, is the bundle assigned by x to trader i. A feasible allocation satisfies

$$\Sigma\, x_i \leqq \Sigma\, \omega_i$$

where ω_i, $i \in N$, is the endowment of trader i. A coalition is a vector $y = (y_1, \ldots, y_i, \ldots, y_n)$ where $y_i = 1$ indicates membership of trader i in y and $y_i = 0$ indicates non-membership. A feasible allocation is in the core if and only if it is not blocked. An allocation is blocked if a pair (y,x') exists such that

$$u_i(x_i') > u_i(x_i) \quad \forall\, i \quad \text{such that} \quad y_i \neq 0 \tag{11a}$$

and

$$\sum_{i \in N} y_i x_i' \leqq \sum_{i \in N} y_i \omega_i \tag{11b}$$

where u_i is the direct utility function of trader i. When x is blocked by (y,x') it is also said to be dominated by x' via y. These definitions are the usual ones used in the theory of the core. [6/]

The commodity bundle actually received in a search equilibrium by trader i is given by

$$x_i = \sum_{j \in M} t_{ij} d_i(p_j), \quad t_i \in I(\mathscr{S}_i) \tag{12}$$

where t_i is the search realization associated with trader i and p_j is the price vector in market j. Because competitive allocations are in the core, (12) and Corollary 3 imply that at least one search equilibrium allocation is in the core. However, most of the realizable allocations associated with a search equilibrium like that generated by random search will not be in the core.

In the sequel we are not interested in whether a particular allocation realized during the exchange period is in the core. Instead, we ask the following question: Can a coalition and a feasible allocation be found such that the commodity bundle assigned to each member is preferred to the gamble he faces given some search equilibrium? In other words, would each prefer not to gamble in search as of the beginning of the period?

To answer this question we make use of the concept of a certainty equivalent. Let $\tilde{x}$ denote a random allocation. A certain allocation x equivalent to $\tilde{x}$ is any which yields to every trader a utility equal to the expected utility of $\tilde{x}$: i.e.,

$$u_i(x_i) = E\, u_i(\tilde{x}_i) \quad \forall\, i \in N.$$

A certainty equivalent to an equilibrium search allocation is, then, any which every trader is just willing to accept in lieu of the equilibrium. Hence, for our purpose the following definition is natural.

Definition (D.2): An equilibrium search allocation is in the core if and only if one of its certainty equivalents is in the core.

Let $\tilde{X}^*$ denote a random allocation generated by a particular equilibrium joint strategy S^*. Because every t_{ij} in (12) is either unity or zero, a certain bundle equivalent to $\tilde{x}^*$ is any x^* satisfying

$$u_i(x_i^*) = \theta_i(s_i^*, \langle S_i^* \rangle) \quad \forall\ i \in N \tag{13}$$

Because s_i^* maximizes θ_i given $\langle S_i^* \rangle$ and because θ_i is linear in s_i, the following result holds as an implication of (8).

<u>Proposition 1</u>: For every $i \in N$, the certainty equivalent to the search equilibrium bundle generated by S^* is any $x_i^* \in R_+^\ell$ such that

$$u_i(x_i^*) \geqq E^*\{\varphi_i(p_j) \mid t_{ij} = 1\} \quad \forall\ j \in M$$

and

$$u_i(x_i^*) = E^*\{\varphi_i(p_j) \mid t_{ij} = 1\} \quad \forall\ j \text{ such that } s_{ij}^* \neq 0.$$ 7/

Since search allocations are random in general, the attitudes of the traders toward risk are relevant. In our context, we need to distinguish between attitudes toward stochastic variations in prices on the one hand and preferences with regard to risk in commodities on the other.

Definition (D.3): Trader i is said (a) not to prefer risk in commodities if and only if

$$E\ u_i(\tilde{x}) \leqq u_i(E\ \tilde{x})$$

and is said (b) not to be averse to risk in prices if and only if

$$E\,\varphi_i(\tilde{p}) \geqq \varphi_i(E\,\tilde{p}).$$ 8/

When the direct utility function u_i is quasi-concave, as we assume, a trader does not prefer risk in commodities if and only if he is not a risk lover in the Arrow-Pratt sense; i.e. iff u_i is concave in commodities. Similarly, the trader is not averse to risk in prices if the indirect utility function is convex in prices.

Although the indirect utility function $\varphi_i(p)$ is quasi-convex when the direct utility function $u_i(x)$ is quasi-concave by virtue of the principal theorem of duality theory, φ_i is not necessarily convex if u_i is concave. 9/ Nevertheless, no trader is averse to risk in prices everywhere independent of his attitudes toward risk in commodities. For example, consider a probability distribution on prices with an expectation equal to that vector at which the trader demands his own endowment. Since he prefers to trade at any other price vector, he prefers the gamble of trading at any randomly drawn vector to trading at the mean with certainty. This example implies that a trader may not prefer risk in commodities and simultaneously not be averse to risk in prices. The results which follow suggest the importance of this case.

Is there a feasible certain allocation which all traders are willing to accept in lieu of an equilibrium search allocation? An affirmative answer is not always possible. However, intuition suggests that the certainty equivalent is not feasible only if some traders prefer risk in the Arrow-Pratt sense. Below, we verify this conjecture.

Theorem 2: If no trader prefers risk in commodities, then a feasible certain allocation exists which is equivalent to any search equilibrium.

Proof. D.3a, equation (12) and the fact that $E^* t_{ij} = s^*_{ij}$ imply

$$u_i(x^*_i) \leqq u_i(E^* \tilde{x}^*_i) = u_i(\sum_{j \in M} s^*_{ij} E^*\{d_i(p_j) \mid t_{ij} = 1\} \quad \forall i \in M.$$

The price vector in every market is market clearing; i.e., p_j satisfies

$$\sum_{i \in N} t_{ij} d_i(p_j) = \sum_{i \in N} t_{ij} \omega_i, \quad \forall j \in M.$$

By taking the expectation of both sides, we obtain

$$\sum_{i \in N} s^*_{ij} E^*\{d_i(p_j) \mid t_{ij} = 1\} = \sum_{i \in N} s^*_{ij} \omega_i, \quad \forall j \in M.$$

By monotonicity and continuity of preferences, the inequality above implies that an x^*_i can be found such that

$$x^*_i \leqq \sum_{j \in M} s^*_{ij} E^*\{d_i(p_j) \mid t_{ij} = 1\} \quad \forall i \in N.$$

Therefore,

$$\sum_{i \in N} x^*_i \leqq \sum_{i \in N} \sum_{j \in M} s^*_{ij} \omega_i = \sum_{i \in N} \omega_i$$

since

$$\sum_{j \in M} s^*_{ij} = 1 \quad \forall i \in N.$$

The search process is not competitive in spirit to the extent that each trader takes account of his presence in the market when choosing one to search. He does so in the sense that the distribution on price used to calculate his expected indirect utility derived from trade in a market is conditional on the fact that he will have to search it to trade. However, when this impact on price is negligible for all traders, a search allocation is not blocked provided that none is averse to risk in prices. The proof is an adaptation of that used by Debreu and Scarf [4] to establish membership of competitive allocations in the core. Let $\bar{p}_j = E\, p_j$ and $\bar{p}_{ij} = E\{p_j \mid t_{ij} = 1\}$ denote the unconditional and conditional expected prices respectively.

<u>Theorem 3:</u> If no trader is risk averse in prices, then an equilibrium search allocation is not blocked when $\bar{p}_{ij} = \bar{p}_j$ for all $i \in N$ and every $j \in M$ searched by some trader with positive probability.

Proof. Suppose the theorem is false. Then a pair (y,x') satisfying (11) exists when x^* replaces x and x^* is any certainty equivalent allocation associated with the search equilibrium generated by an $S^* \in \mathcal{S}$. Let $k \in M$ be any market for which $s^{k*} \neq 0$. There must be at least one since every trader searches at least one market. By virtue of (11b), then,

$$\sum_{i \in N} y_i \bar{p}_k (x_i' - \omega_i) \leqq 0.$$

The hypothesis, Proposition 1 and (11a) imply

$$u_i(x_i') > u_i(x_i^*) \geqq E\{\varphi_i(p_k) \mid t_{ij} = 1\} \geqq \varphi_i(\bar{p}_{ik}) = u_i(d_i(\bar{p}_k))$$

for all i such that $y_i \neq 0$. This fact and the assumptions of (5) imply $\bar{p}_k(x_i' - \omega_i) > 0$ for all such $i \in N$. Hence, we obtain the contradiction

$$\sum_{i \in N} y_i \bar{p}_k (x_i' - \omega_i) > 0.$$

To sum up, then, a certainty equivalent to a stochastic search equilibrium allocation exists which is in the core when no trader has an effect on the price vector associated with any market searched if none prefers risk in commodities and none is averse to risk in prices. When prices and the allocation are not random, the restrictions on risk preferences are not needed of course. These results suggest, then, that the welfare loss associated with a particular search equilibrium which is not in the core will be small when each trader is one among many of the same type if either the equilibrium is non-stochastic or the stated restrictions on attitudes toward risk are satisfied. For the purpose of making the meaning of the word "small" precise we apply the concept of the ϵ-core.

An allocation is close to the core, if it is an element in the ϵ-core corresponding to a small positive number ϵ. An allocation x is in the ϵ-core if and only if it is feasible and not blocked. It is blocked if and only if a coalition-allocation pair (y,x') exists which satisfies both of the following:

$$u_i(x_i') > u_i(x_i) \quad \forall \ i \text{ such that } y_i \neq 0. \tag{14a}$$

$$\sum_{i\in N} y_i x_i' \leqq \sum_{i\in N} y_i(\omega_i \ \theta \ \epsilon) \tag{14b}$$

where $\omega_i \ \theta \ \epsilon$ is the vector whose k^{th} coordinate is max $\{\omega_{ik} - \epsilon, 0\}$. Because the intersection of all the ϵ-cores is the core, our definition of closeness is meaningful. 10/

It is a simple matter to find conditions under which every certain equivalent to a random search equilibrium is blocked even when risk preferences satisfy the hypothesis of both Theorem 1 and Theorem 2. Suppose, for example, that all traders are strictly risk averse in commodities; i.e. $Eu_i(\tilde{x}) < u_i(E\tilde{x}) \; \forall \; i \in N$ and any random bundle $\tilde{x}$. Because the expected search allocation is always feasible (every realizable allocation is feasible), the coalition of the whole blocks via the expected equilibrium search allocation in this case.

Whenever a search allocation is blocked, the extent to which it is outside the core can be measured by the smallest number ε_o specifying a ε-core containing at least one certainty equivalent. Moreover, the value of ε_o is a measure of the benefit, in terms of commodities, which each member of the blocking coalition would obtain if the coalition were formed. However, if it were costly to form the coalition because identifying its members absorbs time and resources, for example, then the benefit would not compensate for the costs when ε_o is sufficiently small. Since the non-cooperative solution to the game of choosing trading partners, a search equilibrium, does not require that anyone know the identities of any of the traders, the stochastic equilibrium search allocation may dominate that associated with the blocking coalition when the costs of forming coalitions in a deterministic manner are taken into account. In the next section we establish this possibility for large economies by proving that ε_o vanishes as the number of traders of each type becomes large.

5. Symmetric Equilibria and the ε-Core

In this section we establish that allocations associated with equilibria generated by joint search strategies with the property that like traders pursue the same strategy are in the ε-core when the number of traders of each type is large if certain restrictions on risk preference hold. For this purpose it is convenient to reinterpret N as the set of trader types. Let n_i denote the number of traders of type $i \in N$. Fix these and distribute the aggregate endowment for each type, ω_i, equally among the traders of type $i \in N$. Then, each trader of type $i \in N$ can be characterized by the utility function u_i and endowment ω_i/n_i when the markets are all identical. Let the representative element of the sequence $\{\mathcal{E}_r\}$ denote an economy composed of $n_i(r) = rn_i$ traders of each of the n types. Therefore, the sequence of economies unambiguously increases in size and the set of competitive equilibrium allocations is invariant with respect to r.

An allocation which is feasible for $\mathcal{E}_r$ is said to be symmetric if and only if it assigns the same bundle to all the traders of a given type. It is well known that core allocations are symmetric.[11/] In an earlier version of this paper [8], we show that any symmetric allocation not in the ε-core is dominated by a symmetric allocation. Therefore, if we interpret $x = (x_1,\ldots,x_i,\ldots,x_n)$ as a symmetric allocation which assigns the common bundle x_i to all traders of type i and $y = (y_1,\ldots,y_i,\ldots,y_n)$ as a coalition which contains the proportion y_i of the total number of traders of type i, the definitions of the core and ε-core above apply to this section as well.

A certainty equivalent to an equilibrium search allocation is not symmetric in general. However, if every trader with a given characteristic pursues the same search strategy, then a symmetric certainty equivalent exists because within a market traders of the same type receive equal treatment. A <u>symmetric joint search strategy</u> can be represented as an n×m vector $S = (s_1,\ldots,s_i,\ldots,s_n) \in \mathcal{S}$ if we reinterpret s_i as the search strategy common to all traders of type i. Since random search with equal probability is symmetric, at least one symmetric equilibrium allocation exists for each $\mathcal{E}_r$.

When the joint search strategy is symmetric, the effect of the presence of any trader on the price vector in any market is the same for all traders of the same type because they have identical demands and endowments. Let $\bar{p}_{ij}$ denote the conditional expected vector in market j given that any trader of type i searches. A sequence of symmetric strategies $\{S(r)\}$, then, generates a sequence of differences $\{\bar{p}_{ij}(r) - \bar{p}_j(r)\}$, where, of course, $\bar{p}_j(r)$ is the unconditional expected price vector in the j^{th} market of $\mathcal{E}_r$. The following result is an implication of the law of large numbers.

Proposition 2: The sequence $\{\bar{p}_{ij}(r) - \bar{p}_j(r)\}$ converges to zero for all $(i,j) \in N\times M$ as $r \to \infty$.

<u>Proof</u>. The representative element of $S(r), s_{ij}(r)$, is the probability that any trader of type i searches market j of $\mathcal{E}_r$. Because the search of a market by any two traders is stochastically independent, the probability that t_i of the rn_i traders search j is distributed according to the binomial with parameters $s_{ij}(r)$ and rn_i for all $i \in N$. Its value, then, is given by

$$b(t_i, s_{ij}(r), rn_i) = \frac{rn_i!}{t_i!(rn_i - t_i)!}(s_{ij}(r))^{t_i}(1-s_{ij}(r))^{rn_i - t_i} .$$

For the same reasons the conditional probability that t_i-1 of the remaining rn_i-1 traders of type i will search j given that one searches is $b(t_i-1,\ s_{ij}(r),\ rn_i - 1)$. The market clearing price function common to all markets, f, is homogeneous of degree zero in the vector $t = (t_1,\ldots,t_i,\ldots,t_n)$ since d_i and ω_i are common to all traders of type i. Moreover, without loss of generality we can define $f(0) = 0$. Therefore,

$$\bar{p}_j(r) = \sum_{i=0}^{n} \sum_{t_i=1}^{rn_i} f(t/r) \prod_{i=0}^{n} b(t_i, s_{ij}(r), rn_i) \tag{15a}$$

and

$$\bar{p}_{ij}(r) = \sum_{\langle t_i \rangle = 0} \sum_{t_i=1} f(t/r)\, b(t_i-1, s_{ij}(r), rn_i) \prod_{k\neq i} b(t_k, s_{kj}(r), rn_k) \tag{15b}$$

Since

$$b(t_i-1, s_{ij}(r), rn_i-1) = \frac{t_i}{rn_i s_{ij}(r)} \cdot b(t_i, s_{ij}(r),\ rn_i),$$

it follows that

$$\bar{p}_{ij}(r) = \sum_{t=0} \frac{t_i}{rn_i s_{ij}(r)}\ f(t/r) \cdot \prod_{k=1}^{n} b(t_k, s_{kj}(r),\ rn_k)$$

by virtue of (15b). By subtracting the corresponding sides of (15a), we obtain

$$\bar{p}_{ij}(r) - \bar{p}_j(r) = E\left\{ \frac{t_i/rn_i - s_{ij}(r)}{s_{ij}(r)}\ f\ (t/r)\right\}$$

where the expectation is taken with respect to the joint distribution on t.

Because f is bounded, it suffices to show that the random variable $(t_i/rn_i - s_{ij}(r)) / s_{ij}(r)$ converges to zero in probability as $r \to \infty$.

Chebychev's inequality

$$P_r\{ t_i/rn_i - s_{ij}(r) \geq \varepsilon\ s_{ij}(r)\} \leq \frac{1-s_{ij}(r)}{rn_i \varepsilon},\quad \forall\ \varepsilon > 0$$

and the fact that $0 \leq s_{ij}(r) \leq 1$ imply the result.

Consider a sequence of equilibrium symmetric search strategies $\{S^*(r)\}$ which generates a sequence of symmetric certainty equivalent allocations $\{x^*(r)\}$ the representative element of which is symmetric allocation in the economy $\mathcal{E}_r$. The economies $\{\mathcal{E}_r\}$ increase in size with r in the sense defined earlier and each contains m identical markets and n trader types. The following result is a consequence of Proposition 2.

Theorem 4: If each trader type has a strictly positive endowment and if no trader type either prefers risk in commodities or is averse to risk in prices, then search allocations generated by symmetric equilibrium joint strategies are in the ϵ-core when the number of traders of each type is sufficiently large.

Proof. Feasibility is implied by Theorem 2. Suppose that the conclusion is false none the less. If false, the sequence of symmetric strategies contains a subsequence $\{S^*(r_t)\}$, $r_t \geqq t$, generating a subsequence of certainty equivalents $\{x^*(r_t)\}$ which is blocked from the ϵ-core. Because $\{S(r_t)\} \subset \mathcal{S}$ and $\mathcal{S}$ is a compact set, we may assume that $\{S^*(r_t)\}$ converges to an $S^* = (s^{1*},\ldots,s^{j*},\ldots,s^{m*})$ without loss of generality. Moreover, $s^{k*} \neq 0$ for some $k \in M$. Because the subsequence converges, θ_i is continuous in S and u_i is continuous, equation (13) implies that $\{x^*(r_t)\}$ converges to some x^* as $t \to \infty$.

Kannai [5] has established that the ϵ-core is lower semi-continuous in the sense that $x_r \to x$ implies x_r is in the ϵ-core of $\mathcal{E}_r$ if x is in the $\epsilon/2$-core of the limiting economy given continuous and monotonic preferences. Therefore, the supposition, the equations of (14), continuity of preferences, the lack of aversion to risk in prices and Proposition 1 imply that a coalition $y \neq 0$ and an allocation x' exist such that

$$\sum_{i\epsilon N} y_i x_i' \leq \sum_{i\epsilon N} y_i(\omega_i \ominus \frac{\epsilon}{2})$$

and

$$u_i(x_i') \geq u_i(x_i^*(r_t)) \geq u_i(d_i(\bar{p}_{ik}^*(r_t))) \quad \forall \quad i \quad \text{such that } y_i \neq 0$$

for all large t.

As implications of these inequalities we obtain

$$\sum_{i\epsilon N} y_i \bar{p}_k^* (r_t) (x_i' - (\omega_i \ominus \frac{\epsilon}{2})) \leq 0$$

and

$$\sum_{i\epsilon N} y_i \bar{p}_{ik}^* (r_t) (x_i' - \omega_i) \geq 0.$$

Because ω_i is strictly positive by assumption, $(\omega_i \ominus \frac{\epsilon}{2}) - \omega_i = \frac{1}{2}(\epsilon,\ldots,\epsilon)$ for all $\epsilon > 0$ sufficiently small. Because $f(z)$ is in the simplex of R_+^ℓ when $z \neq 0$ and $f(0) = 0$ by convention, equation (15a) implies

$$\bar{p}_k(r)\cdot(\epsilon,\ldots,\epsilon) = \epsilon(1-\Pr\{z^k(r) = 0\})$$

where $z^k(r)$ is the random coalition to be found in market k of $\mathcal{E}_r$. Finally, then, we have

$$\sum_{i\epsilon N} y_i(\bar{p}_{ik}(r_t) - \bar{p}_k^*(r_t)) (x_i' - \omega_i) \geq \frac{\epsilon}{2} (1-\Pr\{z^k(r_t) = 0\}) \sum_{i\epsilon N} y_i.$$

But, $z^k(r_t)$ converges to s^{*k} in probability by the proof to Proposition 2. Therefore, $s^{*k} \neq 0$, $y_i \neq 0$ for some i and Proposition 2 imply the contradiction:

$$0 \geq \frac{\varepsilon}{2} \sum_{i \in N} y_i > 0.$$

That every sequence of symmetric search equilibria converges to the intersection of the sequence of cores is an implication of Theorem 4.[12/] But, this set is the set of competitive equilibria in our model because the latter is invariant with respect to size given our replication method. Consequently, a search allocation generated by a symmetric joint strategy is approximately competitive by virtue of the Debreu-Scarf [4] Theorem when the number of traders is large in the sense of r. This fact suggests that the realized price vectors in markets in which trade takes place are all approximately equal to the same competitive equilibrium price vector when the number of traders of each type is large.

Theorem 5: Given the hypothesis to Theorem 4, the price vectors associated with those markets searched with positive probability converge in probability to a common competitive equilibrium price vector as the number of traders of each type becomes large if the common price rule is continuous except at the null coalition.[13/]

Proof. In the proof to Proposition 2 we established that the variance in the coalition which forms in any market vanishes in the limit as r becomes large. Since the price vector in each market is a function f of the coalition which forms and the function is continuous except at the origin by assumption, the variance in the price vector vanishes if the market is searched with positive probability in the limit.

With these facts in mind suppose that the theorem is false. Then a sequence $\{S^*(r)\}$ exists which contains a subsequence $\{S^*(r_t)\}$, $r_t \geqq t$, converging to some S^* as $t \to \infty$ and a pair of markets (j_1, j_2) exists such that $s^{*j1} \neq 0$ and $s^{*j2} \neq 0$ and

$$\lim_{t\to\infty} \bar{p}_{j1}(r_t) = \bar{p}_{j1} = f(s^{*j_1}) \neq f(s^{*j_2}) = \bar{p}_{j2} = \lim_{t\to\infty} \bar{p}_{j2}(r_t).$$ [14/]

Moreover, because all uncertainty vanishes in the limit

$$u_i(x_i^*) = u_i(d_i(\bar{p}_j)) \quad \forall\ (i,j) \in N\text{x}M \ \text{ such that } s_{ij}^* \neq 0$$

by virtue of Propositions 1 and 2, where x_i^* denotes the limit of $\{x_i^*(r_t)\}$. Hence, we are free to choose the sequence of certainty equivalents such that

$$x_{i_1}^* = d_{i_1}(\bar{p}_{j1}) \quad \text{and} \quad x_{i_2}^* = d_{i_2}(\bar{p}_{j2})$$

for any pair of types (i_1, i_2) such that $s_{i_1 j_1}^* \neq 0$ and $s_{i_2 j_2}^* \neq 0$. But, this fact implies that some trader of type i_1 and another of type i_2 are in effect trading at different certain prices in the limit. Hence, the limiting certainty equivalent x^* is not Pareto optimal which contradicts Theorem 4. That the price vector common to all markets searched in the limit is a competitive equilibrium is obvious.

6. Summary and Future Research

The following picture emerges from our analysis: In a pure exchange economy with many markets, the process of matching traders (forming trading coalitions) is part of the process by which initial endowments are reallocated. The extent to which gains from trade are exploited depends on the ways in which traders with complementary preferences and endowments are matched. In an economy operated as a single market this function is performed by the mythical auctioneer. In particular, a matching of trading partners which maximizes gains from trade in the sense of Pareto is implicit in any competitive equilibrium in the standard general competitive market model.

In our formulation the process by which traders are matched, the search process, is not centrally directed. It is the joint outcome of the limited, uncoordinated and self interested search behavior of all the individual traders. As we have shown, this joint decision problem can be formulated as a non-cooperative n-person non-sum zero game in which trading coalitions are the outcomes. Under appropriate informational assumptions, optimal individual search decisions correspond to a Nash equilibrium solution to the game. This fact enables us to establish the existence of a search equilibrium in general. In at least some equilibria, realized prices for the same commodity differ across markets. Moreover, because traders may search in accord with mixed strategies, equilibrium allocations and prices are random in general. Price differentials are a possibility because each trader is limited to search one market per period and no trader knows with certainty what prices will be when he takes his search decision.

Although equilibrium price differentials may persist, they are small when the number of traders of each type is large relative to the number of markets if all traders do not prefer risk in the commodity bundle received and are not averse to risk with regard to the prices they face. In particular the certainty equivalent to the generally random equilibrium search allocation is in every ε-core of all economies containing a sufficient number of traders of each type when all markets are identical and all traders of the same type pursue the same strategy. This result implies that the price vector in every market approximates some competitive equilibrium price vector when the number of traders is large. In other words, a large multi-market economy acts almost like an economy with one market under the assumed restrictions on risk preferences.

A number of restrictive assumptions could be relaxed without altering the basic logic used in the proofs. In the paper we assume that each trading coalition once formed in some market allocates to its members bundles which represent a competitive equilibrium allocation for that coalition. Possibly it would be more realistic to assume that they obtain an allocation in the core of the sub-economy represented by the coalition. Although the "message space" would have to be enlarged, the existence proof would go through. Moreover, because the number of traders in the coalition formed in each market searched with positive probability will increase in size in proportion as the size of the economy increases, any such core allocation approaches some competitive allocation in the limit. Hence, one would expect that Theorems 4 and 5 would continue to hold under this alternative specification. Of course, they will not hold if any one of the trader's sets prices as either a monopsonist or monopolist as Mortensen and Phelps and Winter respectively assume in two somewhat related models studied in [11].

It would be of interest to include some form of intertemporal speculation in the model. This could be done either by allowing the traders to hold inventories or to exchange futures contracts of some type. In either case it is optimal for each trader to set reservation prices, prices above or below which the trader will not join the coalition as either a buyer or seller respectively. The micro economics of this problem is the central issue in the literature on price search. The model developed in [7] represents an initial effort in this direction. The results obtained suggest that the principal conclusions presented here continue to hold.

Questions concerning the dynamic stability of search equilibria are probably the principal ones left unanswered here. To formulate such questions, of course, an adjustment process must be specified. The process by which traders learn about the price distribution in each market would presumably be an important component in a dynamic formulation of the search process.

FOOTNOTES

1/ It goes without saying that there are at least two traders and two markets,

2/ Later we reinterpret N as the set of trader types. In that case t_{ij} is reinterpreted as the proportion of the traders of type i who belong to coalition t^j. It is for this reason that we want $\mathcal{T}^j$ to be a dense set but restrict the set of possible coalitions to $I(\mathcal{T}^j)$ when there is only one trader of each type.

3/ The term px represents the inner product of the two vectors p and x.

4/ Note that each individual market can be thought of as an economy with "random agents." The probability distribution defined on the set of possible agents is determined by S, the joint search strategy.

5/ The general case is considered in an earlier draft [8] of this paper.

6/ See Arrow and Hahn [1] p. 184.

7/ Because all markets are now identical, demand and indirect utility functions are independent of the market searched. For that reason j is dropped as a subscript on d_i and ω_i here and in the sequel.

8/ Note that both properties are local in the sense that they depend on the "location" of the distribution in question.

9/ See Hanoch [16] for a discussion of this point as well as a summary of known results concerning preference for stochastic variation in prices.

10/ The concept of the ε-core was introduced by Shapley and Shubik [14]. Our definition corresponds to that of the "weak ε-core" in Kannai [5].

11/ See Debreu and Scarf [4].

12/ See Kannai [5].

13/ The market clearing price function f can never be continuous at the origin because it is homogeneous of degree zero.

14/ If only one market is searched, the theorem is trivial. Indeed, this case is covered by Corollary 3 to Theorem 1.

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