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A STRATEGIC FORM FOR A CONVEX GAME

by

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Abstract

A strategic interpretation of a convex game is given by formulating a sufficient condition for a convex game to be derived from a strategic form. Two well-known examples of convex games are examined to see how the sufficient condition is satisfied.

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1 Introduction.

The convex game, introduced by Shapley [6], is a cooperative game that exhibits an intuitively appealing property that incentives to join a coalition increase as the coalition becomes large. The property, often referred to as a snowballing or band-wagon effect is formally equivalent to the definition of the convex game. Therefore, the convex game itself does not explain why and how such an effect comes out when the game is played cooperatively.

In applications of cooperative games, we model the given situation by constructing the characteristic function, considering the relevant strategic opportunities open to coalitions. The game of "Jake" due to Shapley and Shubik [7] and the recent bankruptcy game due to O'Neill [5], Aumann and Maschler [1] are the examples of convex games in such applications.

The usual way to derive the characteristic function from strategies taken by the coalitions is the maximin principle of von Neumann and Morgenstern [8]. But, it is well known that the maximin principle only assures that two disjoint coalitions have incentives to join each other, i.e., that the derived characteristic function is super-additive.

This note considers the question that under what strategic environments, players will have incentives to join larger coalitions. We shall formulate a sufficient condition for a convex game to be derived from a strategic form through the maximin principle. The strategic form we set out is a usual one except for the cost function to each coalition. This is simply an expression of the fact that the net benefit to a coalition is given by the difference between the benefits and costs associated with a selection of strategies of all players.

Our condition for the convexity is concerned with the marginal contribution of players. It requires that the marginal contribution of players when they join a coalition must be greater than or equal to the amount they produce by full non-cooperation; namely, by playing their part of the strategies that induce the worst state to the coalition, and the marginal contribution must be smaller than or equal to the amount they produce by full cooperation; namely, by playing their part of the strategies that bring the best state to the coalition. We show this condition is sufficient to obtain a convex game, and illustrate how it is satisfied by two well-known examples: one is the game of a public good, and the other is the bankruptcy game.

2. The Strategic Form for a Convex Game.

Let $N=\{1,\dots,n\}$ be a finite set of players, and let S be a nonempty subset of N. We call S a coalition. A strategic form to be considered here is given by $G=(N,\{X_S\}_{S\subseteq N},\{u_i\}_{i\in N},\{c_S\}_{S\subseteq N})$, where

$X_S\equiv\Pi_{i\in S}X_i$: X_i is a strategy set of player i,

$u_i:X_N \to E^1$ is a utility function of i, and

$c_S:X_N \to E^1$ is a cost function of S.

We adopt the conventional assumption that the utilities are transferable, sidepayments are allowed and the costs are measured in terms of the same unit as the utility. Then, the classical von Neumann and Morgenstern characteristic function v can be defined as follows:

Definition 1. For all $S \subseteq N$.

$$v(S) = \max\min\ [\Sigma_{i \in S} u_i(x_S, x_{N-S}) - c_S(x_S, x_{N-S})],$$

where max is taken overall $x_S \in X_S$, and min is taken overall $x_{N-S} \in X_{N-S}$.

We assume the maximin is attained for each $S \subseteq N$ so that v is given by

$$v(S) = \Sigma_{i \in S} u_i(x^S_S, x^S_{N-S}) - c_S(x^S_S, x^S_{N-S}), \text{ for all } S \subseteq N.$$

Notice that if $S \subset T$, then $x^T_T = (x^T_S, x^T_{T-S}) \in X_T$ but not necessarily $x^T_S = x^S_S$.

Let (N,v) denote the characteristic function game. It is well known that if $c_S \equiv 0$ for all $S \subseteq N$, the game (N,v) thus defined is super additive, i.e., $v(S)+v(T) \leq v(S \cup T)$ for all disjoint $S, T \subseteq N$. Following Shapley [6], the convex game is defined as follows:

Definition 2. (N,v) is a convex game iff for all $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

The convexity implies the super additivity. Our main assumption is the following:

Assumption I. If $S \subseteq T \subseteq N$, then

$$\text{(i)} \quad v(T) \geq \Sigma_{i \in T} u_i(x^S_S, x^S_{N-S}) - c_S(x^S_S, x^S_{N-S}),$$

$$\text{(ii)} \quad v(S) \geq \Sigma_{i \in S} u_i(x^T_T, x^T_{N-T}) - c_T(x^T_T, x^T_{N-T}).$$

To interprete this assumption, let us rewrite (i) and (ii) as follows:

$$\text{(iii)} \quad \Sigma_{i \in T-S} u_i(x^S_S, x^S_{N-S}) \leq v(T) - v(S) \leq \Sigma_{i \in T-S} u_i(x^T_T, x^T_{N-T}).$$

The amount $v(T)-v(S)$ measures the marginal contribution of the members in T-S. Thus, inequalities (iii) gives the lower and upper bounds to the marginal contribution. The first inequality states that it must be at least equal to the amount they would produce by playing the part of the strategies x^S_{N-S}: that is, by playing most antagonistically against S. This would give them the incentive to join S and take cooperative strategies. On the other hand, the second inequality states that the contribution of T-S cannot exceed the amount they would produce by fully cooperating as members in T.

We can now state our theorem.

Theorem. Under Assumption I, the game (N,v) defined by Definition 1 is convex.

Proof. Let $S, T \subseteq N$. Then,

$$v(S) + v(T) = \Sigma_{i\in S} u_i(x^S_S, x^S_{N-S}) - c_S(x^S_S, x^S_{N-S}) + \Sigma_{i\in T} u_i(x^T_T, x^T_{N-T}) - c_T(x^T_T, x^T_{N-T}).$$

We may assume without loss of generality that

$$\Sigma_{i\in T-S} u_i(x^S_S, x^S_{N-S}) \geq \Sigma_{i\in T-S} u_i(x^T_T, x^T_{N-T}).$$

Then,

$$\begin{aligned} v(S) + v(T) &= \Sigma_{i\in S} u_i(x^S_S, x^S_{N-S}) + \Sigma_{i\in T-S} u_i(x^T_T, x^T_{N-T}) - c_S(x^S_S, x^S_{N-S}) \\ &\quad + \Sigma_{i\in T\cap S} u_i(x^T_T, x^T_{N-T}) - c_T(x^T_T, x^T_{N-T}) \\ &\leq \Sigma_{i\in S\cup T} u_i(x^S_S, x^S_{N-S}) - c_S(x^S_S, x^S_{N-S}) \\ &\quad + \Sigma_{i\in T\cap S} u_i(x^T_T, x^T_{N-T}) - c_T(x^T_T, x^T_{N-T}) \\ &= v(S) + \Sigma_{i\in (S\cup T)-S} u_i(x^S_S, x^S_{N-S}) \\ &\quad + v(T) - \Sigma_{i\in T-(T\cap S)} u_i(x^T_T, x^T_{N-T}). \end{aligned}$$

Hence, by Assumption I or (iii), we have

$$v(S) + v(T) \leq v(S\cup T) + v(S\cap T). \qquad \text{QED}$$

We now consider a variant G' of the strategic form G by introducing an outcome function h as follows:

$$c_S(x) = c(h(x)) \quad \text{for all } S \subseteq N \text{ and } x \in X_N,$$

$$u_i(x) = b_i(h(x)) \quad \text{for all } i \in N \text{ and } x \in X_N.$$

where

$h: X_N \to E^m$ is an outcome function,

$c: E^m \to E^1$ is a cost function, and

$b_i: E^m \to E^1$ is a utility function.

Assumption II. Assume that $S \subseteq T \subseteq N$. Then,

(i) for each $z_S \in X_S$, there is a $y_T \in X_T$ such that $h(y_T, x^T_{N-T}) = h(z_S, x^S_{N-S})$.

(ii) for each $y_T \in X_T$, there is a $z_S \in X_S$ such that $h(z_S, x^S_{N-S}) = h(y_T, x^T_{N-T})$.

This assumption states that if $S \subseteq T$ then as far as the worst state is concerned S and T are in the same strategic environment, i.e., they can realize the same outcome.

Proposition. Assumption II implies Assumption I.

Proof. Let $S \subseteq T$. Then, Assumption II implies

$$\Sigma_{i \in T} b_i(h(x^S_S, x^S_{N-S})) - c(h(x^S_S, x^S_{N-S}))$$
$$= \Sigma_{i \in T} b_i(h(y_T, x^T_{N-T})) - c(h(y_T, x^T_{N-T})) \le v(T),$$

and

$$\Sigma_{i \in S} b_i(h(x^T_T, x^T_{N-T})) - c(h(x^T_T, x^T_{N-T}))$$
$$= \Sigma_{i \in S} b_i(h(z_S, x^S_{N-S})) - c(h(z_S, x^S_{N-S})) \le v(S). \quad \text{QED}$$

Although Assumption II is strong, there exists a typical economic example, called a public good game, which satisfies it in a natural way. We show this in the next section.

3. Examples.

3.1 Provision of a Public Good. Our theorem has a natural economic application. Consider a public good economy where each coalition S can supply $y \geq 0$ amount of a public good at cost $c(y)$ in terms of money. The cost $c(y)$ is born by the members of S, and possibly by players in N-S if they are willing to do so. Each player i obtains the monetary benefit $b_i(y)$ when consuming y amount of the public good.

Assume that each b_i is continuous, increasing, bounded from above and $b_i(0)=0$, and that c is continuous, increasing and $c(0)=0$. The inverse of c is denoted by q. The strategy x_i of each player i is taken to be the amount of willingness to pay for the public good. Then, the strategic form G^p for the public good provision may be given by

$$X_i = [0, \infty) \quad \text{for each } i \in N,$$

$$u_i(x_S, x_{N-S}) = b_i(q(\Sigma_{i \in N})), \quad \text{for all } i \in S \text{ and all } S \subseteq N,$$

$$c_S(x_S, x_{N-S}) = \Sigma_{i \in N} x_i - \Sigma_{i \in N-S} x_i, \quad \text{for all } S \subseteq N.$$

The characteristic function v is then given by

$$v(S) = \max\{\Sigma_{i \in S} b_i(q(\Sigma_{i \in S} x_i)) - \Sigma_{i \in S} x_i \mid x_S \in X_S\}, \text{ for all } S \subseteq N.$$

We call the game (N,v) the public good game. Notice that for each $S \subseteq N$, $x^S_{N-S}=0 \in X_{N-S}$. That is, no payment is the maximal damage that coalition N-S can inflict upon S.

Corollary 1. The strategic form G^p for the public good game satisfies Assumption II.

Proof. By the assumptions on b_i and c. v(S) is well-defined. By Proposition, it is sufficient to verify (i) and (ii) of Assumption II. But, these follow from the fact that for all $R, W \subseteq N$, and all $x_R \in X_R$, there is a $y_W \in X_W$ such that

$$\Sigma_{i \in R} x_i = \Sigma_{i \in W} y_i .$$

and the fact that $x^T_{N-T} = 0 \in X_{N-T}$ for all $T \subseteq N$. QED

The public good game (N,v) can be also given a priori by

$$v(S) = \max\{\Sigma_{i \in S} b_i(y) - c(y) \mid y \geq 0\}, \quad \text{for all } S \subseteq N.$$

The convexity of this game was proved in Kaneko and Nakayama [4]. Champsaur [2] has also given a set function with the convexity in a model of public goods.

3.2 The Bankruptcy Game. The bankruptcy game (O'Neill [5], Aumann and Maschler [1]) is also an example of our theorem. Let E be the estate of a bankrupt, and let $d_i > 0$ be the debt to creditor i. N is the set of all creditors. Assume that

$$0 \leq E \leq D \equiv \Sigma_{i \in N} d_i .$$

The bankruptcy game is then defined to be the characteristic function game (N,v) such that

$$v(S) = \max\ (0,\ E - \Sigma_{j \in N-S} d_j) \quad \text{for all } S \subseteq N.$$

Curiel, Maschler and Tijs [3] have shown that the bankruptcy game is convex.

The bankruptcy game can be derived from the following strategic form

G^b. The strategy of each player i consists of making a concession x_i of his demand; namely, he can obtain the amount d_i only if he pays the cost x_i. Then G^b may be given as follows:

$$X_i = [0, d_i] \quad \text{for all } i \in N,$$

$$u_i(x_S, x_{N-S}) = d_i \quad \text{if } \Sigma_{i \in N} x_i \geq D-E,$$
$$= 0, \quad \text{otherwise}.$$

$$c_S(x_S, x_{N-S}) = \Sigma_{i \in S} x_i \quad \text{for all } S \subseteq N.$$

Note that d_i is not the net amount that player i finally obtains: he must bear its cost, and therefore, will end up with only a reduced amount.

That the bankruptcy game is derived from G^b can be seen below:

If $\Sigma_{i \in S} d_i \geq D-E$, then for any $x_{N-S} \in X_{N-S}$ there is a joint strategy $\hat{x}_S \in X_S$ such that

$$\Sigma_{i \in S} \hat{x}_i + \Sigma_{i \in N-S} x_i \geq D-E.$$

Hence,

$$v(S) = \max\{\Sigma_{i \in S} u_i(x_S, x^S_{N-S}) - \Sigma_{i \in S} x_i \mid \Sigma_{i \in S} x_i \geq D-E\}$$
$$= \max\{\Sigma_{i \in S} d_i - \Sigma_{i \in S} x_i \mid \Sigma_{i \in S} x_i \geq D-E\}$$
$$= \Sigma_{i \in S} d_i - (D-E).$$

If $\Sigma_{i \in S} d_i < D-E$, then for any $x_S \in X_S$ there is a joint strategy $\hat{x}_{N-S} \in X_{N-S}$ of N-S such that

$$\Sigma_{i \in S} x_i - \Sigma_{i \in N-S} \hat{x}_i < D-E.$$

Hence,

$$v(S) = \max\{0 - \Sigma_{i \in S} x_i \mid \Sigma_{i \in S} x_i < D-E\} = 0.$$

Hence (N,v) is the bankruptcy game.

We now show that the condition (iii) is satisfied. Let $S \subseteq T$, and assume that $v(T)>0$. If $\Sigma_{i\in S}d_i \geq D-E$, then

$$v(T)-v(S) = \Sigma_{i\in T-S}d_i$$

$$= \Sigma_{i\in T-S}u_i(x^T_T, x^T_{N-T}) = \Sigma_{i\in T-S}u_i(x^S_S, x^S_{N-S}).$$

If $\Sigma_{i\in S}d_i<D-E$, then $v(S)=0$ and

$$0 = \Sigma_{i\in T-S}u_i(x^S_S, x^S_{N-S}) < v(T)-v(S) = \Sigma_{i\in T}d_i-(D-E)$$

$$< \Sigma_{i\in T-S}d_i = \Sigma_{i\in T-S}u_i(x^T_T, x^T_{N-T}).$$

Finally, if $v(T)=0$, then $\Sigma_{i\in S}d_i<D-E$ and

$$0 = \Sigma_{i\in T-S}u_i(x^S_S, x^S_{N-S}) = v(T)-v(S) \leq \Sigma_{i\in T-S}u_i(x^T_T, x^T_{N-T}).$$

This completes the proof of the following Corollary.

Corollary 2. The strategic form G^b for the bankruptcy game (N,v) satisfies Assumption I.

4. Concluding Remarks.

We have given a sufficient condition for a convex game to be obtained from a strategic form, and two examples illustrating this. Since the class of convex games is narrow, any sufficient condition as considered in this note would tend to be strong. Therefore, there will be convex games which are not covered by our assumption. In particular, the strategic form G^b for the bankruptcy game does not satisfy Assumption II, though the strategic form G^p for the public good game does. Notice that, in characteristic function form, the bankruptcy game is a degenerate public good game. Hence, this would imply that the public good game is, in a sense, generic to our sufficient condition.

We have confined ourselves to games with sidepayments. A more challenging question would be to ask if it is possible to give a strategic interpretation to convex games without sidepayments. This would merit further study.

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