Discussion Paper No. 764

# UNIQUENESS OF NASH EQUILIBRIUM POINTS IN BIMATRIX GAMES

by

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February 1988

Abstract

This note is concerned with techniques for constructing bimatrix games with predetermined unique Nash equilibrium points. The results obtained here extend the constructions presented by Heuer (1979) and Quintas (1988).

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The author gratefully acknowledges the support from CONICET: Consejo de Investigaciones Cientificas y Tecnicas, Republica Argentina. Thanks are due to E. Marchi and M. Nakayama for helpful comments.

The concept of equilibrium introduced by Nash (1951) *[1] is considered a landmark in noncooperative game theory. The set of Nash equilibrium points is non-empty for any finite game if mixed strategies are allowed ( Nash (1951) ). However in general there is a multiplicity of such equilibrium. This origins the problem to decide which equilibria is taken as a solution *[2]. If the players cannot communicate it is not clear how they can coordinate their actions to play in a specific equilibria. Even is they can communicate it remains the problem of agreeing which equilibrium they will play because the utilities can be quite different from one equilibrium point to another. This problem does not arise if the equilibrium is unique.

Many studies have been done on uniqueness of Nash equilibrium points. On one hand it was studied some sufficient conditions to guarantee uniqueness ( Rosen (1965), Gale and Nikaido (1965) ) *[3]. On the other hand it has been investigated under what conditions it is possible to construct games with predetermined unique equilibrium points. Most of these constructions have been done on bimatrix games: Millham (1972), Kreps (1974), Raghavan (1970), Heuer (1975) (1979), Jansen (1981) and Quintas (1988). (Kreps (1981) studied the problem on n-person games).

In the present note we will construct a broad family of bimatrtix games

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1 The Cournot (1838) equilibrium in oligopoly theory and the saddle point equilibrium of von Neumann (1928) in two-person zero-sum games are precursor of the Nash equilibrium concept. Both concepts are a special instance of the notion introduced by Nash (1951).

2 This problem does not arise in two-person zero-sum games because the strategies of equilibrium are interchangeable and the utility remains constant over any equilibrium point.

3 There is also a vast bibliography on refinements of the Nash equilibrium concept. Some of them drastically reduce the multiplicity of equilibrium. For a comprehensive discussion of the characteristics of the more important refinements introduced see Van Damme (1983).

with an arbitrary prefixed unique equilibrium points. It will generalize some of the above mentioned constructions.

A bimatrix games is defined by a pair $(A,B)$ of $m$ by $n$ matrices over an ordered field $F$. We denote with $S_1$ and $S_2$ the sets of (pure) strategies available to player 1 and player 2 respectively. The cardinality of $S_1$ and $S_2$ is $m$ and $n$ respectively. We will use the following notation for pure strategies: $i \in S_1$, $j \in S_2$ and the respective payoffs $A(i,j) = a_{ij}$ and $B(i,j) = b_{ij}$. A mixed strategy for player 1 (player 2) will be a probability vector $x = (x_1,\dots,x_m)$ with $x_i \geq 0$ and $\sum_{i=1}^{m} x_i = 1$ ( $y = (y_1,\dots,y_n)$; $y_j \geq 0$ and $\sum_{j=1}^{n} y_j = 1$). We denote the set of mixed strategies for player 1 (player 2) by $S^m$ ($S^n$). We will represent by $A_{i.}$ and $B_{.j}$ the i-th row of A and the j-th column of B respectively. Let $y^t$ be the transpose of the row matrix y. $xAy^t$, $xB_{.j}$, $A_{.i}y^t$, etc will denote the usual matrix products.

A pair of mixed strategies $(x,y)$ is an equilibrium point for the game $(A,B)$ if and only if $xAy^t \geq x'Ay^t$ for any $x' \in S^m$ and $xBy^t \geq xBy'^t$ for any $y' \in S^n$.

Let $M_1(x) = \{i\colon x_i > 0\}$, $N_1(y) = \{j\colon y_j > 0\}$, $M_2(A,y) = \{i\colon A_{i.}y^t = \max_k A_{k.}y^t\}$ and $N_2(x,B) = \{j\colon xB_{.j} = \max_k xB_{.k}\}$.

It is well known the following characterization of equilibrium points: $(x,y)$ is an equilibrium point of $(A,B)$ if and only if $M_1(x) \subseteq M_2(A,y)$ and $N_1(y) \subseteq N_2(x,B)$. (1)

An equilibrium point $(x,y)$ is said to be completely mixed if $x_i > 0$ for any $i \in S_1$ (i.e $M_1(x) = S^m$) and $y_j > 0$ for any $j \in S_2$ (i.e $N_1(y) = S^n$). The reader is referred to Raghavan (1970) for his work on completely mixed strategies on bimatrix games, and Heuer (1975) who extended some results of that paper. * [4]

Millham (1972) [Theorem 2] showed that a necessary and sufficient condition for the existence of a game having (x,y) as its unique completely mixed equilibrium point is that m=n * [5]. Kreps (1974) extended this condition to the case when the prefixed unique equilibrium is not necessarily completely mixed. The main result presented by Kreps (1974) asserts that: A necessary and sufficient condition for the existence of a bbimatrix game having (x,y) as its unique equilibrium point is that $|M_1(x)|=|N_1(y)|$ (the bars $|\ |$ stand for the cardinality of the corresponding set). However it have not been described the sets of all the bimatrix games having (x,y) as its unique equilibrium point. Some constructions quite generals have been obtained (see Heuer (1979) and Quintas (1988)). A broader family of games with this property will be given here and we will estabiish how the new construction involves those mentioned above.

Let (x,y) be given with $v_1=A(x,y)\neq 0$, $v_2=B(x,y)\neq 0$, $M_1(x) = S^m$ and $N_1(y)=S^n$. We will focus our attention to the case when (x,y) results the unique completely mixed equilibrium point (i.e m=n). A general construction for the case when $m \neq n$ can be easily obtained along the lines of section 3, Quintas (1988).

Let $f_{21} : S_2 \to S_1$ and $f_{12} : S_1 \to S_2$ be two bijective functions such

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4 Among the results proved by Raghavan (1970) I mention the following: (a) Let (A,B) be a game having only completely mixed equilibrium points then m=n and the equilibrium point is unique. (b) If (x,y) is an equilibrium point with m>n then there is a non-completely mixed equilibrium point (x',y). This last result was improved by Heuer (1975) who proved that x' may be found with n or fewer positive components. See also Theorem 3.12 ( Jansen(1981))

5 Millham (1972) also characterized the set of all the bimatrix games having a given equilibrium point (x,y).

that:

For each $(j',j") \in S_2xS_2$ if $f_{12}[f_{21}(j')]=j"$ then $f_{12}[f_{21}(j")]\neq j'$ (or equivalently For each $(i',i") \in S_1xS_1$ if $f_{21}[f_{12}(i')]=i"$ then $f_{21}[f_{12}(i")]\neq i'$). (2)

We remark that there is a large set of functions fulfilling the above condition *[6]. For example:

$$f_{21}(j)=j \text{ and } f_{12}(i)=i-1 \bmod m. \quad (3)$$

Let A be defined by:

$$a_{ij}=\begin{cases} v_1+\dfrac{1}{y_j}\displaystyle\sum_{f_{21}(t)\neq i}\varepsilon_t\,(1-y_t) & \text{for } f_{21}(j)=i\\ v_1-\dfrac{(1-y_j)\,\varepsilon_j}{y_j} & \text{for } f_{21}(j)\neq i\end{cases}$$

and B defined by: (4)

$$b_{ij}=\begin{cases} v_2+\dfrac{1}{x_i}\displaystyle\sum_{f_{12}(t)\neq j}\varepsilon'_t\,(1-x_t) & \text{for } f_{12}(i)=j\\ v_2-\dfrac{(1-x_i)\,\varepsilon'_i}{x_i} & \text{for } f_{12}(i)\neq j\end{cases}$$

We choose $\varepsilon_j$ and $\varepsilon'_i$ fulfilling:

$$\sum_{j=1}^{m}(1-y_j)\,\varepsilon_j>0 \quad\text{and}\quad \sum_{i=1}^{m}(1-x_i)\,\varepsilon'_i>0 \quad (5)$$

<u>Theorem:</u> For any (x,y) with $|M_1(x)| = |N_1(y)|=m$ and nonzero values v1 and v2. The game (A,B) , defined by (4) with $\varepsilon_j$ and $\varepsilon'_i$ fulfilling (5) and $f_{12}$ , $f_{21}$ satisfying (2), has (x,y) as its unique equilibrium point and $v_1=A(x,y)$, $v_2=B(x,y)$.

<u>Proof:</u>

It is immediate to verify that (x,y) is a completely mixed equilibrium point for the game (A,B) ( it satisfies (1) with $M_2(A,y)=M_1(x)$ and $N_2(x,B)=N_1(y)$ ) and $v_1=A(x,y)$ , $v_2=B(x,y)$.

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6 A detailed interpretation of this condition is given by Quintas(1988).

Suppose that there exists another completely mixed equilibrium point $(x',y')$ with $A(x',y')=v_1'$ and $B(x',y')=v_2'$. Then (1) implies $M_2(A,y)=M_2(A,y')$ and $N_2(x,B)=N_2(x',B)$. It means that:

$$\sum_{j=1}^{m} a_{ij}\, y_j = v_1 \quad \text{for } i=1,\dots,m \; ; \quad \sum_{i=1}^{m} b_{ij}\, x_i = v_2 \quad \text{for } j=1,\dots,m \quad \text{and}$$

$$\sum_{j=1}^{m} a_{ij}\, y_j' = v_1' \quad \text{for } i=1,\dots,m \; ; \quad \sum_{i=1}^{m} b_{ij}\, x_i' = v_2' \quad \text{for } j=1,\dots,m$$

A and B are non singular matrices ( $\det A = v_1 \left( \sum_{j=1}^{m} (1-y_j)\, \varepsilon_j \right)^m \prod_{j=1}^{m} y_j \neq 0$ and $\det B = v_2 \left( \sum_{i=1}^{m} (1-x_i)\, \varepsilon_i' \right)^m \prod_{i=1}^{m} x_i \neq 0$ ) and $x,x',y,y'$ are probability vectors, then $x=x'$ and $y=y'$.

Now suppose $(x',y')$ would be an equilibrium point with $M_1(x)=M_1(x')$ and $N_1(y')\subseteq N_1(y)$ (or $M_1(x')\subseteq M_1(x)$ and $N_1(y)=N_1(y')$). By (1) and $N_1(y')\subseteq N_1(y)$, we have: $\sum_{i=1}^{m} a_{ij}\, y_j' = v_1'$ for $i=1,\dots,m$ and again, it implies $y'=y$.

Finally suppose that $(x',y')$ is an equilibrium point with $M_1(x')\subseteq M_1(x)$ and the strict inclusion : $N_1(y')\subset N_1(y)$ (or $M_1(x')\subset N_1(x)$ and $N_1(y')\subseteq N_1(y)$).

For each $j \in N_1(y)-N_1(y')$ we have:

$$A_{f_{21}(j)} \,.\, y'^{\,t} = v_1 \sum_{s\in N_1(y')} y_s' - \sum_{s\in N_1(y')} \frac{y_s'\,(1-y_s)\,\varepsilon_s}{y_s}$$

and for each $r \in N_1(y')$ we have:

$$A_{f_{21}(j)} \,.\, y^{t} = v_1 \sum_{s\in N_1(y')} y_s' - \sum_{\substack{s\in N_1(y')\\ s\neq r}} \frac{y_s'\,(1-y_s)\,\varepsilon_s}{y_s} + \frac{y_r'}{y_r} \sum_{f_{21}(t)\neq f_{21}(r)} (1-y_t)\,\varepsilon_t$$

Then $A_{f_{21}(r)} \,.\, y'^{\,t} > A_{f_{21}(j)} \,.\, y'^{\,t}$ (the strict inequality holds because $y_r'>0$)

This implies that $f_{21}(j) \not\in M_2(A,y')$ and $(M_1(x') \subseteq M_2(A,y'))$ then $x'_{f_{21}(j)} = 0$.

Now we choose $j \in N_1(y) - N_1(y')$ such that $L(j)=f_{12}[f_{21}(j)] \in N_1(y')$. ( the existence of such j is guarantee by (2)). Then we have:

$$x'\, B_{.L(j)} = v_2' \sum_{s\in M_1(x')} x_s' - \sum_{s\in M_1(x')} \frac{x_s'\,(1-x_s)\,\varepsilon_s'}{x_s}$$

and for each r such that :

$$f_{21}(r) \in M_1(x') \tag{6}$$

we have:

$$x'\,B_{.L(r)} = v_2 \sum_{s\in M_1(x')} x'_s - \sum_{\substack{s\in M_1(x') \\ s\neq f_{21}(r)}} \frac{x'_s\,(1-x_s)\,\varepsilon'_s}{x_s} + \frac{x'_{f_{21}(r)}}{x_{f_{21}(r)}} \sum_{L(t)\neq L(r)} (1-x_t)\,\varepsilon'_t$$

and then:

$$x'\,B_{.\,L(r)} \geq x'\,B_{.\,L(j)} \tag{7}$$

However, $L(r) \in N_1(y')$ and $(x',y')$ is an equilibrium point, then:

$$x'\,B_{.\,L(j)} \geq x'\,B_{.\,L(r)} \tag{8}$$

(7) and (8) are incompatible because the equalities hold only if $x'_{f_{21}(r)}=0$ (which is impossible because of (6)). And this completes the proof of uniqueness.

We note that the present construction is quite general. In particular if we choose $\varepsilon_j = v_1 \dfrac{y_j}{1-y_j}$, $\varepsilon'_i = v_2 \dfrac{x_i}{1-x_i}$ and $f_{21}$, $f_{12}$ fulfilling (3) we obtain the construction given by Heuer (1979) in the proof of his Theorem (5.3). Choosing $\varepsilon_j > 0$ for any $j=1,..,m$ and $\varepsilon'_i > 0$ for $i=1,...,m$ we obtain the construction presented in section 2 , by Quintas (1988).

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