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EQUILIBRIUM OF REPEATED GAMES WITH COST OF IMPLEMENTATION

by

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Abstract

We study the structure of Nash Equilibrium set in repeated games assigning an increasing cost of implementation to increasing complexity. We present relations between the strategies supporting equilibria with and without cost of implementation. We drop the assumption about finite complexity of the equilibrium strategies done by Abreu-Rubinstein [1987]; and we obtain a 'full' Folk Theorem. A uniform finite approximation theorem is also presented.

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1.- Introduction

The theory of equilibrium of repeated games have become a topic of much research in recent years. Repeated play allows the players to react to each other's actions. The possibility of retaliations leads to some outcomes which are not supported in equilibrium in the one shot game.

Starting from the well know 'Folk Theorem' several lines of research were developed. A comprehensive survey of the main literature existing in this field prior to the 80's is given by Aumann [1981], see also Fundenberg-Maskin [1986], Kalai [1987] and Sorin [1988].

It was early observed that some strategies of the repeated game might be, at least intuitively, very complicated. Aumann [1981] proposed the use of finite automata for distinguishing between simple and complicated strategies . This line of research was followed by Neyman [1985], Ben Porath [1986], Ben Porath-Peleg [1987] and others. The interest of putting a bound on the complexity of the strategies that the players may choose, comes from the limited ability of the devises (secretaries or computers) they can use for implementing their strategies, see Simon [1972]. A measure of strategic complexity was considered by Kalai-Stanford [1986], (see also Stanford [1987]).

Besides this, it was studied how the cost of implementation affects the set of outcomes supported in equilibrium. This line of study was initiated by Rubinstein [1986] and Abreu-Rubinstein [1987]. A substantial reduction of the outcomes supported by Nash Equilibrium was obtained in a two-person repeated game when complexity costs are incorporated into the model. Also a severe discontinuity in the set of equilibrium outcomes was illustarted as the complexity costs go to zero (in page 27, Abreu-Rubinstein [1987] illustrate how the Folk Theorem looks for some of the most popular two-person games).

In this note, following Rubinstein [1986] and Abreu-Rubinstein [1987], we will focus on the structure of the set of Nash Equilibrium in repeated games with cost of implementation. We note that rules of behavior are costly to operate and it is the aim of the decision makers to minimize such costs. We will focus on this type of cost. We will not deal with the cost of selecting an optimal strategy. We will also deal with a measure of complexity for the strategies following Kalai-Stanford [1986].

The structure of this note is the following: In section 2, we present the model and some basic definitions. Section 3 is focussed on the study of relations between the strategies which are Nash Equilibrium without cost of implementation, with an infinitesimal and a non infinitesimal cost. These relations are strong for the case of two-person games (Theorem 1) and somewhat weaker for n-person games with $n \geq 3$ (Theorem 2 and Theorem 3).
Theorem 4 establish a relation between the complexity of the strategies of Nash Equilibrium with infinitesimal implementation cost. This is the version for n-person games of the result given by Abreu-Rubinstein [1987] for two-person games. Section 4 is centered on the study of the outcomes supported by Nash Equilibrium with an implementation cost. We do not restrict to strategies implemented by finite automata. Then we prove that a 'full' Folk Theorem (Theorem 5) holds for case of an infinitesimal implementation cost (i.e it coincides with the standard Folk Theorem without cost) . An 'almost full' Folk Theorem is presented if the cost of implementation is strictly positive (Corollary 1). We also present (Theorem 6) a technique for obtaining a uniform approximation of the strategies of equilibria by

strategies which are ε-Nash equilibrium with finite complexity. This result is similar to that presented by Kalai-Stanford [1986] (Theorem 4.1) for Subgame Perfect Equilibrium without cost of implementation.

2.- The Model

Let $G=(A,u)$ be a n-person game in normal form. $A = \prod_{i=1}^{n} A_i$ denotes the action combination of n-player. A_i, the set of action of player i, is a non-empty subset of a metric space. $u=(u_1,\dots,u_n)$ is a vector of utility functions. For each $i=1,\dots,n$; $u_i : A \to R$ is a real-valued function.

We describe a standard repeated game $G^{\infty}(A,u)$ associated with G as follows:

The set of histories of length 0 is a singleton set denoted by H^0. Its single element will be denoted by e. Let $H^m = \underbrace{A \times \dots \times A}_{m\text{-times}}$ be the set of histories of length m.

$H = \bigcup_{m=0}^{\infty} H^m$ is the set of all histories.

For each player $i=1,\dots,n$; a strategy f_i is a function $f_i : H \to A_i$.

Let F_i be the set of all strategies for player i. And $F = \prod_{i=1}^{n} F_i$

We define a path of length m as a sequence $P=(p(1),\dots,p(m))$ where $p(t) \in A$ for each $t=1,\dots,m$.

Given a strategy vector $f \in F$; Lets define a infinite path as follows:

$P(f) = (p(f)(1),\dots,p(f)(t),\dots\dots)$

where $p(f)(1) = f(e)$ and $p(f)(t) = f(p(f)(1),\dots,p(f)(t-1))$.

With the above construction we can extend the utility functions to be defined on F:

$$u_i(f) = \lim_{T \to \infty} 1/T \sum_{t=1}^{T} u_i(p(f)(t))$$

(As usual, in order to avoid the problem of nonexistence of this limit we might use a Banach limit; see Kalai [1987], page 11)

A strategy vector is a Nash Equilibrium for the repeated game $G^{\infty}(A,u)$ iff

$u_i(f) \geq u_i(f_{-i},g_i)$ for all $g_i \in F_i$ and for all $i=1,\ldots,n$.

We denoted $(f_{-i},g_i) = (f_1,\ldots,f_{i-1},g_i,f_{i+1},\ldots,f_n)$

Given a strategy $f_i \in F_i$ and a history $h \in H$ we denote $f_i|h$ the strategy defined by: $f_i|h\ (h') = f_i\ (h.h')$ for any $h' \in H$; where $h.h'$ is the concatenation of h and h'. This means to play the history h followed by the history h' (see Kalai-Stanford [1986]).

We will denote by comp f_i the complexity of a strategy f_i. It is the cardinality of the set $F_i(f_i)=\{ f_i|h\ ;\ h \in H\}$. We have to underline that comp f_i is the cardinality of the smallest automata implementing f_i (see Theorem 3.1, Kalai-Stanford [1986]).

Let $\succsim$ be a lexicographic preference relation on the set of strategy vectors F defined by:

$f \succsim_i g$ iff $u_i(f) > u_i(g)$ or: $u_i(f) = u_i(g)$ and comp $f_i <$ comp g_i.

(See Abreu-Rubinstein [1987]).

With this preference relation we say that $f \in F$ is a lexicographic Nash Equilibrium iff:

for any $i=1,\ldots,n$ and for all $g_i \in F_i$: $f \succsim_i (f_{-i},g_i)$.

(We will refer to this as Nash Equilibrium with u_{lex}).

We are going to assign a cost to increasing complexity by defining the following nondecreasing cost function:

$k : N \to R^+$

(N stands for the set of positive integers and R^+ for the positive real numbers) such that :

$$\lim_{n \to \infty} k(n) = K$$

(K might eventually be infinite).

The utility functions with cost k are defined by:

$u_i^k(f) = u_i(f) - k(f_i)$

where we denoted $k(f_i) = k(\text{comp } f_i)$.

Note: We assume that the cost function is the same for all the players. By dropping this assumption we will get essentially the same type of results with added notational complexity.

The preference induced by u^k belongs to the class of preferences considered for two person games by Abreu-Rubinstein [1987].

A strategy f is a Nash Equilibrium with cost k iff f is a Nash Equilibrium with u^k, i.e.:

$u_i^k(f) \geq u_i^k(f_{-i}, g_i)$ for all $g_i \in F_i$ and for all $i=1,\ldots,n$.

A strategy $f \in F$ is an ε-Nash Equilibrium with u^k iff:

for any $g \in F$ $u_i^k(f) \geq u_i^k(f_{-i}, g_i) - \varepsilon$.

3.- Connections between the Nash Equilibrium strategies with u, u_{lex} and u^k.

In this section we will assume that for each $i=1,\ldots,n$; A_i is a finite set of actions (pure strategies in the one shot game).

We will establish some relations between the strategies which are Nash Equilibrium with the lexicographic order and those which are Nash Equilibrium with u^k.

This connection is very close for two-person games. We will use the result given by Abreu-Rubinstein [1987] in order to obtain a necessary condition for Nash Equilibrium with u^k. This is shown in Theorem 1. That is no longer the case for $n \geq 3$. This will be illustrated through an example. For

$n \geq 3$, we obtain (Theorem 2) a weaker version of Theorem 1.

Definition 1: Given two cost functions k and k'. k increases faster than k' (we denote $k \succ k'$) if and only if $k(n+1)-k(n) \geq k'(n+1)-k'(n)$ $\forall n \in N$.

In the next Theorem, we will consider n=2.

Theorem 1:

a) If $f=(f_1,f_2)$ is a Nash Equilibrium with u_i^k then f is a Nash Equilibrium with u_i.

b) If $f=(f_1,f_2)$ is a Nash Equilibrium with u_i^k then f is a Nash Equilibrium with u_i with the lexicographic order.

c) If $f=(f_1,f_2)$ is a Nash Equilibrium with u_i^k then f is a Nash Equilibrium for every $u_i^{k'}$ with $k \succ k'$.

Proof:

a) Let g_2 be a strategy such that:

$$u_2(f) < u_2(f_1,g_2)$$

If comp $(g_2) \leq$ comp (f_2), then $k(g_2) \leq k(f_2)$, which implies:

$$u_2^k(f) < u_2^k(f_1,g_2).$$

If comp $(g_2) >$ comp (f_2) then by Lemma 1 (Abreu-Rubinstein [1987]) there exists an strategy $\hat{g}_2$ such that comp $(\hat{g}_2)$ = comp (f_1) and $u_2(f_1,g_2) \leq u_2(f_1,\hat{g}_2)$. Then we have: $u_2(f) < u_2(f_1,\hat{g}_2)$ and comp $(\hat{g}_2)$ = comp (f_1) .

By Abreu-Rubinstein [1987]: comp (f_2) = comp (f_1) then:

$$u_2^k(f) < u_2^k(f_1,\hat{g}_2).$$

b) Suppose that $f=(f_1,f_2)$ is a Nash Equilibrium with u^k but it is not a Nash Equilibrium with the lexicographic order. Then there exists g_2 such that:

I) $u_2(f) < u_2(f_1,g_2)$

or

II) $u_2(f) = u_2(f_1,g_2)$ and comp g_2 < comp f_1.

Case I is not possible because of part a). Then f is Nash Equilibrium.

If II holds then $u_2^k(f) < u_2^k(f_1,g_2)$.

c) $f=(f_1,f_2)$ is a Nash Equilibrium with u_i^k . Then

$$u_2(f)-k(f_2) \geq u_2(f_1,g_2)-k(g_2) \text{ for any strategy } g_2.$$

$f=(f_1,f_2)$ is also a Nash Equilibrium with u_i (by part a)). Then

$$u_2(f) \geq u_2(f_1,g_2).$$

Thus, if comp $g_2 \leq$ comp f_2 then

$$u_2(f)-k'(f_2) \geq u_2(f_1,g_2)-k'(g_2) \text{ for any cost function } k'.$$

Assume that comp g_2 < comp f_2. Then for any $k \succ k'$, we have :

$$k(f_2)-k(g_2) \geq k'(f_2)-k'(g_2). \text{ Thus } u_2^{k'}(f) < u_2^{k'}(f_1,g_2). \qquad \text{(q.e.d.).}$$

<u>Remark 1:</u> It is immediate that Nash Equilibrium lexicographic implies Nash Equilibrium.

The results stated in Theorem 1 do not hold (as it is) for n≥3. It is illustrated through the following example:

<u>Example 1:</u>

Lets consider the following 3-person one-shot game G:

I

	C	D
C	3 , 3 ,5/2	1/2, 4 , 0
D	4 ,1/2, 0	1 , 1 ,3/2

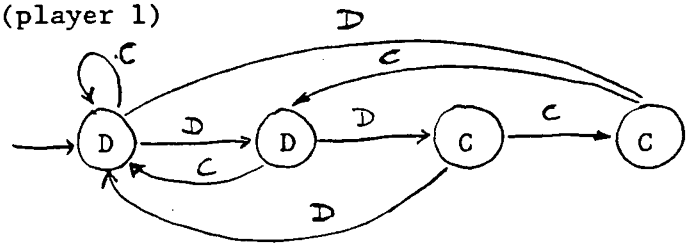
II

	C	D
C	2 , 2 , 3	0 , 3 , 0
D	3 , 0 , 0	1 , 1 ,1/2

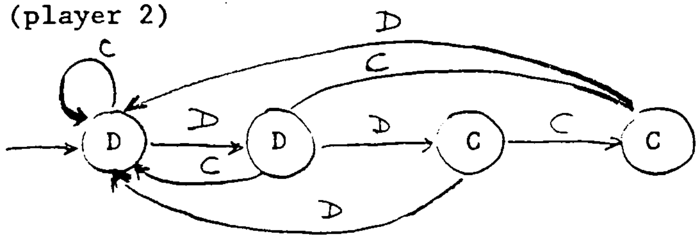
Player 1 chooses the row (C or D), player 2 the column (C or D) and player 3 the matrix (I or II).

Lets f_1,f_2,f_3 be the strategies of the repeated game G^∞ implemented by the following automata:

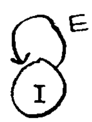
f_1 (player 1)



f_2 (player 2)



f_3 (player 3)

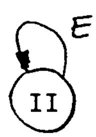


C and D on the arcs in player 1 (player 2) automata, mean that the transition function depends only on player 2 (player 1) actions. E means that the transition function does not depend on the previous action.

We have that:

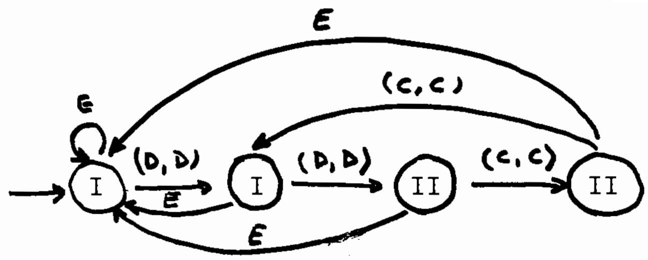
$u_1(f_1,f_2,f_3)=7/3$; $u_2(f_1,f_2,f_3)=7/3$ and $u_3(f_1,f_2,f_3)=13/6$

If f_3^* is the strategy described by the following automata



Then we obtain again $u_3(f_1,f_2,f_3^*)=13/6$

$f=(f_1,f_2,f_3)$ is not a Nash Equilibrium with u because if we define f_3' by the following automata:



then $u_3(f_1,f_2,f_3')=15/6 > u_3(f_1,f_2,f_3)=13/6$ (indeed (f_1,f_2,f_3') is a Nash Equilibrium with u).

However, if we define a cost function k such that:
$k(1)=0$ and $k(2)\geq 1/3$, then $f=(f_1,f_2,f_3)$ is a Nash Equilibrium with u^k, because for all $g_3 \neq f_3$ we have:

$$u_3(f_1,f_2,g_3) - u_3(f_1,f_2,f_3) \leq u_3(f_1,f_2,f_3^*) - u_3(f_1,f_2,f_3) =$$
$$= 1/3 \leq k(g)-k(f_3).$$

Remark 2:

There does not exist a strategy f_2 such that: for any f_1 with comp $f_1 < \infty$, $u_2^k(f_1,f_2) \geq u_2^k(f_1,g_2)$ for any g_2.
(The same statement holds for u_i with the lexicographic order).

This is a consequence of Lemma 1 Abreu-Rubinstein [1987], which states that: For any f_1 with comp $f_1 < \infty$ and for any f_2, $\exists$ g_2 with comp $g_2 \leq$ comp f_2 and $u_2(f_1,f_2) \leq u_2(f_1,g_2)$.

Thus the result presented by Gilboa and Samet [1987] in Theorem A, is not longer true with the utility functions u_i^k or even with the lexicographic order.

For n≥3 we obtain the following weaker version of Theorem 1.

Theorem 2:

If f is a Nash Equilibrium with u_i and u_i^k for some cost function k, then f is a Nash Equilibrium for u_i with the lexicographic order.

The proof follows from Lemma 1 and Lemma 2.

Lemma 1:

Let f be a Nash Equilibrium with u_i and u_i^k for some cost function k. Then there exists a sequence of cost functions k^s with $\lim_{s \to \infty} K^s = 0$ such that f is a Nash Equilibrium with $u_i^{k^s}$ for all s.

Proof:

Let $k^s: N \to R^+$ be a sequence of cost functions with the following properties:

i) $\lim_{s \to \infty} K^s = 0$ and ii) $k^s(n+1) - k^s(n) \leq k(n+1) - k(n)$

We will prove that f is a Nash Equilibrium with $u_i^{k^s}$ for any k^s.

By hypothesis we have that for any strategy g_i:

(3.1) $\quad u_i(f) \geq u_i(f_{-i}, g_i)$

and

(3.2) $\quad u_i(f_{-i}, g_i) - u_i(f) \leq k(g_i) - k(f_i)$

We consider two cases:

a) comp $g_i \geq$ comp f_i

If this holds, then $k^s(g_i) \geq k^s(f_i)$ for every s. Then by 1) we have :

$$u_i^{k^s}(f_{-i}, g_i) \leq u_i^{k^s}(f) \text{ for all s.}$$

b) comp $g_i <$ comp f_i.

This implies $k^s(g_i) < k^s(f_i)$ and then:

$$k^s(f_i) - k^s(g_i) \leq k(f_i) - k(g_i) \text{ for all s.}$$

Then, by 2) we have:

$$u_i(f_{-i}, g_i) - u_i(f) \leq k(g_i) - k(f_i) \text{ for all s.} \qquad \text{(q.e.d).}$$

Lemma 2:

Let $k^s: N \to R^+$ be a sequence of cost functions such that $\lim_{s \to \infty} K^s = 0$

(Where $K^s = \lim_{n \to \infty} k^s(n)$).

If $f=(f_1,\ldots,f_n)$ is a Nash Equilibrium with u^{k^s} for all s, then f is a Nash Equilibrium for u with the lexicographic order.

Proof:

Suppose that f is not a Nash Equilibrium for u with the lexicographic order. Then for some i there exists g_i such that

a) $u_i(f_{-i},g_i) > u_i(f)$

or

b) $u_i(f_{-i},g_i) = u_i(f)$ and comp $g_i <$ comp f_i.

If a) is true then we have:

$B = u_i(f_{-i},g_i)-u_i(f) > 0$

Let s' be such that $K^{s'} \leq B$. Then

$u_i(f_{-i},g_i)-u_i(f) > k^{s'}(g_i)-k^{s'}(f_i)$ then $u_i^{k^{s'}}(f_{-i},g_i) > u_i^{k^{s'}}(f)$

which is a contradiction because f is Nash Equilibrium with $u_i^{k^{s'}}$.

If b) holds then for all k^s we have $u_i^{k^s}(f_{-i},g_i) > u_i^{k^s}(f)$ leading again to a contradiction. (q.e.d.).

The following result complement Theorem 2, by giving a sufficient condition for the existence of Nash Equilibrium with cost.

Theorem 3:

If f is a Nash Equilibrium with the lexicographic order, then there exists a cost function k such that $f=(f_1,\ldots,f_n)$ is a Nash Equilibrium with u_i^k.

Proof:

We will first analyze the case when comp $f_i = \infty$.

f is a Nash Equilibrium with the lexicographic order, then for any strategy

g_i we have:

a) $u_i(f_{-i},g_i) = u_i(f)$ and comp $g_i=\infty$

or

b) $u_i(f_{-i},g_i) < u_i(f)$.

If a) holds, then f is a Nash Equilibrium with u_i^k for any cost function k.

If b) holds. Lets define

$$B^n=\{b \setminus b=u_i(f_{-i},g_i) \text{ and comp } g_i=n\}$$

As we assumed A_i to be a finite set, then B^n is also finite for any $n\in N$.

Let $b_n = \max_{b\in B^n} b$

We define inductively the sequence t_n.

$t_1 = u_i(f) - b_1$ and we choose t_n such that:

$$0 < t_n < \min (t_{n-1},u_i(f)-b_n).$$

t_n is a decreasing sequence and $\lim_{n\to\infty} t_n = 0$

Now, we define the cost function k.

Let K be any real number such that $K > t_1$ and let be $k(n) = K - t_n$.

It defines a (increasing) cost function k with $\lim_{n\to\infty} k(n) = K$.

We will proof that f is a Nash Equilibrium with u_i^k.

For any strategy g_i we have:

$$u_i^k(f) - u_i^k(f_{-i},g_i) = u_i(f) - u_i(f_{-i},g_i) - k(f_i) + k(g_i) =$$

$$u_i(f) - u_i(f_{-i},g_i) - t_{n'}$$

where $n'=$ comp g_i. However, by construction:

$$u_i(f) - u_i(f_{-i},g_i) > t_{n'}$$

Then $u_i^k(f) - u_i^k(f_{-i},g_i) > 0$ and f is a Nash Equilibrium with u_i^k.

Now, if comp $f = n$. Then f is a Nash Equilibrium with u_i^k for any cost function k such that :

$$0 < k(n) - k(n+1) < b$$

where $b = \min_{b' \in B'(n)} b'$ and

$$B'(n) = \{ b' = u_i(f) - u_i(f_{-i}, g_i) \text{ and comp } g_i < n \}$$

$B'(n)$ is finite and $\forall\, b' \in B'(n)$: $b'>0$. Then k is well defined. (q.e.d.)

We conclude this section with a result establishing a relation between the complexities of the strategies of a lexicographic Nash Equilibrium. This is the version for n-players of that presented for two players by Abreu-Rubinstein [1987].

This result goes much in the direction of the Corollary 5.1 given by Kalai-Stanford [1986].

<u>Theorem 4:</u>

Let f be a Nash Equilibrium for u with the lexicographic order and comp $f_i < \infty$ for all i. Then:

$$\text{comp } f_i \le \prod_{j \ne i} \text{comp } f_j$$

<u>Proof:</u>

Without lost of generality (w.l.o.g.) we assume that

$$\text{comp } f_1 \le \text{comp } f_2 \le \ldots\ldots \le \text{comp } f_n.$$

Consider the following two players game:

$$G = ((A_{-n}, A_n), u_{-n}, u_n) \text{ where } A_{-n} = \mathop{\mathrm{X}}_{i=1}^{n-1} A_i$$

$u_{-n} : A \to R$ defined by

$$u_{-n}(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} u_i(a_1, \ldots, a_n)$$

Let G^* be the infinite repeated game of the one shot game G.

and let $f_{-n} : H \to A_{-n}$ be the following strategy:

$$f_{-n}(h) = (f_1(h), f_2(h), \ldots, f_{n-1}(h)) \text{ and comp } f_{-n} \le \mathop{\mathrm{X}}_{i \ne n} \text{comp } f_i.$$

Then by (Abreu-Rubinstein [1987]) there exists f'_n such that

$$u_n(f_{-n},f'_n) \leq u_n(f_{-n},f_n) \text{ and comp } f'_n = \text{comp } f_{-n}.$$

If comp $f_n >$ comp f_{-n}, then there would exist f'_n such that:

comp $f'_n \leq$ comp f_{-n} and $u_n(f_{-n},f'_n) = u_n(f_{-n},f_n)$, which is impossible because $f=(f_{-n},f_n)$ is a Nash Equilibrium for u_k with the lexicographic order.

(q.e.d.)

Remark 3: The results of this section also holds if, instead of considering the lim of the mean criteria, the utility functions are defined for a given discount parameter.

4.- Folk Theorems and Finite Complexity Approximation

In this section we drop the assumption of finiteness of A_i. We consider a n-person (one-shot) game $G=(A,u)$, where A_i is a compact set and u_i are bounded functions.

Abreu-Rubinstein [1987] showed, for two-person repeated games, that the set of outcomes supported by Nash Equilibrium is drastically reduced when restricted to strategies implemented by finite automata under an infinitesimal implementation cost.

We show that by dropping the restriction on the finite complexity of the strategies, we obtain a 'Full' Folk Theorem for the case of infinitesimal cost. A slightly different version of this theorem is obtained when the cost of implementation is not infinitesimal. We also present a Finite Approximation Theorem. It shows in a constructive way how to uniformly approximate any outcome supported by a Nash Equilibrium under a non infinitesimal cost, by considering ε-equilibrium strategies of finite complexity.

We associate with the game (A,u), n n-tuples of actions $s_1,\dots,s_n$ such that:

$$s_i = (s_i^1,\dots,s_i^n) = (s_i^{-i},s_i^i)$$

$$s_i^{-i} \in \arg\min_{a_{-i}} \max_{a_i} u_i(a_{-i},a_i) \quad \text{and}$$

$$u_i(s_i) = u_i(s_i^{-i},s_i^i) = \max_{a_i} u_i(s_i^{-i},a_i)$$

$$v_i = u_i(s_i)\ ;\ U = \{u(a) \text{ with } a\in A\}\ ;\ U^* = \text{Convex Hull of } U \quad \text{and}$$

$$V = \{x\in U^* \text{ with } x_i > v_i\}$$

W.l.o.g. we assume that $v_i=0$ for all i.

<u>Theorem 5:</u>

For all $x\in V$ there exists an strategy f which is a Nash Equilibrium for u with the lexicographic order and $u(f)=x$.

We state the following Lemma which will be used during the proof of Theorem 5.

<u>Lemma 3:</u>

Let $(r_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ be (weakly) increasing sequences of natural numbers and $b_i>0$. If $(r_i/b_i)_{i=1}^{\infty}$ defines a sequence of rational numbers Q, which is also (weakly) increasing. Then:

a) $$\lim_{i\to\infty} r_i/b_i = \lim_{n\to\infty} \sum_{i=1}^{n} r_i \Big/ \sum_{i=1}^{n} b_i .$$

b) $$\lim_{i\to\infty} r_i/b_i = \lim_{n\to\infty} \sum_{i=1}^{n} r_i \Big/ \Big(n + \sum_{i=1}^{n} b_i \Big).$$

<u>Proof of Lemma 3:</u>

Part a)

$(r_i/b_i)_{i=1}^{\infty}$ is monotone. Thus, the limit exists (it might be ∞).

(4.1) $\sum_{i=1}^{n} r_i \,/\, \sum_{i=1}^{n} b_i \;\leq\; r_n/b_n.$

(This follows immediately because $(r_i/b_i)_{i=1}^{\infty}$ is increasing).

(4.2) $\forall$ k $\exists$ n such that:

$$r_k/b_k \;\leq\; \sum_{i=1}^{n} r_i \,/\, \sum_{i=1}^{n} b_i$$

The above statement is equivalent to:

$$\sum_{i=1}^{n} (r_i\, b_k - r_k\, b_i) \geq 0$$

Again as $(r_i/b_i)_{i=1}^{\infty}$ is increasing:

$$r_i\, b_k - r_k\, b_i \leq 0 \quad \text{for } i \leq k \quad \text{and} \quad r_i\, b_k - r_k\, b_i \geq 0 \quad \text{for } i \geq k.$$

Then $\sum_{i=1}^{k} r_i\, b_k - r_k\, b_i \leq 0$ and $\sum_{i=k+1}^{n} r_i\, b_k - r_k\, b_i \geq 0$

Moreover

$$\forall\, j \geq i \geq k \qquad b_j\, (r_i\, b_k - r_k\, b_i) \leq b_i (r_j\, b_k - r_k\, b_j)$$

Using that $(b_i)_{i=1}^{\infty}$ is also increasing, we obtain:

$$\forall\, j \geq i \geq k \qquad r_i\, b_k - r_k\, b_i \leq r_j\, b_k - r_k\, b_j.$$

Then, choosing n big enough

$$\sum_{i=1}^{n} (r_i\, b_k - r_k\, b_i) \geq 0$$

(4.1) and (4.2) imply the assertion done in a).

Part b)

It follows from part a) and the fact that $\lim_{n \to \infty} n \,/\, \sum_{i=1}^{n} b_i = 0.$ (q.e.d)

<u>Proof of Theorem 5:</u>

Let $x = \sum_{h=1}^{k} \alpha_h x_h$ with $x_h \in A$; $\alpha_h \geq 0$ and $\sum_{h=1}^{k} \alpha_h = 1$

Let $(x^m)_{m=1}^{\infty}$ be a sequence of vectors in V, satisfying:

i) x^m goes to x when m tends to ∞.

ii) There exists $r_h^m \in Q$ such that:

$$x^m = \sum_{h=1}^{k} r_h^m x_h \quad ; \quad \sum_{h=1}^{k} r_h^m = 1 \quad \text{and} \quad r_h^m \geq 0$$

iii) $r_h^m \leq r_h^{m+1}$ for h=1,...,k-1

We note that i) means that r_h^m goes to α_h when m tends to ∞ for any h=1,...,k.

We write :

$$\tilde{x}^m = \sum_{h=1}^{k} \tilde{r}_h^m x_h$$

where

$$\tilde{r}_h^m \in N \ ; \quad b_m = \sum_{h=1}^{k} \tilde{r}_h^m \quad \text{and} \quad x^m = \tilde{x}^m / b_m .$$

We will construct the vectors: $x_h(i)$, with i=1,...,n.

They will consist of a reordering of the vectors x_h for h=1,...,k such that there exists a pure strategy $a^i \in A$ with :

(4.3) $\quad u(a^i) = x_1(i)$ and $a_i^i \neq s_i^i$

<u>Note:</u> If there does not exist a strategy $a^i \in A$ satisfying (4.3) then, in the construction of the vector P, replace s_i^i by any other strategy $\hat{s}_i^i$ for, which (4.3) holds.

We will denote by $\tilde{r}_h^m(i)$ the coefficient corresponding to $\tilde{r}_h^m$ after the above stated reordering.

Let P_i^m be a sequence consisting of b_m n-tuples of actions.

$$P_i^m = (\underbrace{a_1^i, \ldots, a_1^i}_{\tilde{r}_1^m(i)\text{-times}}, \ldots, \underbrace{a_k^i, \ldots, a_k^i}_{\tilde{r}_k^m(i)\text{-times}})$$

where $u(a_j^i) = x_j(i)$. We note that the average payoff of P_i^m is x^m. Lets define:

$P_i(z) = (P_i^1, P_i^2, \ldots, P_i^z)$ It is a finite sequence of length $\sum_{m=1}^{z} b_m$

and let P be an infinite sequence of n-tuples of actions

$$P = (P_1(1), s_2, P_2(2), s_3, P_3(3), \ldots, s_n, P_n(n), s_1, P_1(n+1), s_2, \ldots,$$
$$s_j, P_j(kn+j), \ldots\ldots)$$

It will be the equilibrium path for a strategy $f: H \to A$ which will be defined as follows:

Let $h = (h(1), \ldots, h(t)) \in H$ then $f_i(h) = s_i^j$ iff for some t' with $1 \le t' \le t$ and only one j:

$[h(t')]_j \neq [P(t')]_j$ and otherwise $f(h) = P(t+1)$

The above constructed strategy f is a Nash Equilibrium of the repeated game and comp $f = \infty$.

We claim that:

a) $u(f) = x$

b) It does not exist a strategy g_i with comp $g_i < \infty$ and $u_i(f) = u_i(f_{-i}, g_i)$.

a) holds because :

$$u_i(f) = \lim_{T \to \infty} 1/T \sum_{t=1}^{T} u_i(P(t)) =$$

$$= \lim_{n \to \infty} \sum_{h=1}^{k} x_h \left[\sum_{j=1}^{n} (n+1-j)\, r_h^j \right] / \left(n + \sum_{j=1}^{n} (n+1-j) b_j\right)$$

and by Lemma 3 we have that: $u_i(f) = x_i$.

Now in order to prove b), Lets assume that there would exist a strategy g_i such that comp $g_i < \infty$ and $u_i(f) = u_i(f_{-i}, g_i)$.

By the definition of f we have that:

(4.4) $(f_{-i}, g_i)\,(P(1), \ldots, P(t)) = P(t+1)$ for all t

(because otherwise $u_i(f_{-i}, g_i) = 0$).

For $c = 0,1,2,\ldots$ and $i = 1,\ldots,n$; lets define :

$$t(c,i) = nc + i - 1 + \sum_{j=1}^{nc+i-1} (nc + i - j)\, b_j$$

Comp $g_i < \infty$, then there exists $c' < c''$ such that:

(4.5) $\quad g_i|((P(1),\ldots,P(k_1)) = g_i|((P(1),\ldots,P(k_2))$ where we denote

$k_1 = t(c',i-1)$; $k_2 = t(c'',i-1)$ and $k_3 = t(c',i)$. Note that $P(t(c',i)) = s_i$

By the definition of P(t) we have :

$$(P(k_1+1),\ldots,P(k_3-1)) = (P(k_2+1),\ldots,P(k_2+k_3-k_1-1))$$

and by (4.3) we have:

(4.6) $\quad s_i^i = [P(k_3)]_i \neq [P(k_2+k_3-k_1)]_i = a_i^i$

But, by (4.4) and (4.5) we have:

$$[P(k_3)]_i = g_i|((P(1),\ldots,P(k_1))\ (P(k_1+1),\ldots,P(k_3-1)) =$$
$$= g_i|((P(1),\ldots,P(k_2))\ (P(k_2+1),\ldots,P(k_2+k_3-k_1-1)) =$$
$$= g_i|(P(1),\ldots,P(k_2+k_3-k_1-1)) = [P(k_2+k_3-k_1)]_i$$

and this contradicts (4.6) (q.e.d.).

Lets denote $V^K = \{\, x\in V$ such that $x+(K,\ldots,K)\in V\}$

<u>Corollary 1:</u>

Consider a cost function $k:N\to R^+$ with $\lim_{n\to\infty} k(n) = K$.

Let $x \in V^K$ then there exists a strategy f of the repeated game such:

$u^k(f) = x$ and f is a Nash Equilibrium with u^k.

<u>Proof:</u>

Lets consider $x'=x+(K,\ldots,K)$ then x' belongs to V.

Let f be the strategy constructed in the proof of Theorem 5. Then f is a lexicographic Nash Equilibrium and $u(f)=x'$, we have:

$$u_i^k(f) = u_i(f) - k(f_i) = x_i' - K = x_i$$

Then we only need to prove that f is a Nash Equilibrium with u^k.

By construction, comp $f = \infty$. Then for all g_i with comp $g_i < \infty$ we have:

$$u_i(f_{-i},g_i) = v_i$$

Then

$$u_i(f_{-i},g_i) - k(g_i) \leq u_i(f) - K$$

With this and having into account that f is a lexicographic Nash Equilibrium, we obtain that f is a Nash Equilibrium with u^k. (q.e.d.).

<u>Remark 4:</u> Corollary 1 is not a complete Folk Theorem. It is immediate to observe that it does not exist f with comp $f_i = \infty$ for all i and $u^k(f) \in V-V^K$. However it might exist f being a Nash Equilibrium for u^k with comp $f_i < \infty$ for some i and $u^k(f) \in V-V^K$.

<u>Remark 5:</u> For all x in the interior of V there exists a cost function k and a strategy f such that $u^k(f) = x$ and f is a Nash Equilibrium with u^k.

<u>Theorem 6:</u> Given the game $G^\infty(A,u)$, the cost function k and $\varepsilon > 0$. There exists a positive integer $W=W(\varepsilon,k)$ such that for each $f \in F$ with f a Nash Equilibrium with u^k and comp $f_i = \infty$ for all i, $\exists$ $g \in F$ fulfilling :

i) For $i=1,\dots,n$ comp $g_i \leq W$

ii) $| u_i^k(f) - u_i^k(g) | < \varepsilon$

iii) g is an ε-Nash Equilibrium with u^k.

<u>Proof:</u>

Let $N(r^j, \varepsilon/2)$ be a neighborhood of center $r^j \in V$ and radio $\varepsilon/2$ such that:

$$\bigcup_{j=1}^{y} N(r^j, \varepsilon/2) \supset V^K \quad \text{and} \quad r_k^j = r^j + (K,\dots,K) \in V.$$

<u>Claim:</u> For each $j=1,\dots,y$, there exists a strategy $g^j \in F$ such that

a) For $i=1,\dots,n$ comp $g_i^j < \infty$

b) $u^k(g^j) \in N(r^j, \varepsilon/2)$

c) g is an ε-Nash Equilibrium with u^k.

The claim completes the proof of the theorem. In order to check this, lets define $W = \max_{j,i} \text{comp } g_i^j$.

For a given $f \in F$ such that $u^k(f) \in V^k$ there exists j such that:

$u^k(f) \in N(r^j, \varepsilon/2)$

Then g^j fulfills the conditions stated in the theorem. (q.e.d)

<u>Proof of the Claim:</u>

Let z be a vector in V such that:

(4.7)
(i) $| z - r_k^j | < \varepsilon/4$

(ii) $z = \sum_{\ell=1}^{m} r_\ell x^\ell$ where $x^\ell = u(a^\ell)$ with $a^\ell \in A$ and $r_\ell \in Q^+$

(iii) $z - (K,\dots,K) > 0$

W.l.o.g. we assume that there exists a pure strategy $c=(c_1,\dots,c_n) \in A$ such that:

(4.8) $c_i \neq [a^1]_i$ for all i.

We have that:

$$z = \sum_{\ell=1}^{m} r'_\ell x^\ell / b \quad \text{with } r'_\ell \text{ being a positive integer and } r_\ell = r'_\ell / b.$$

Let P' be a sequence consisting of b n-tuples of actions:

$$P' = (\underbrace{a^1,\dots,a^1}_{r'_1\text{-times}},\dots,\underbrace{a^m,\dots,a^m}_{r'_m\text{-times}})$$

We note that the average payoff of P' is z.

Lets define:

$$P^q = (P',c,\underbrace{P',P'}_{2\text{-times}},c,\dots,c,\underbrace{P',\dots,P'}_{q\text{-times}},c)$$

For this path we have:

$$\lim_{q\to\infty} u(P^q) = z$$

We will define the equilibrium strategy g^q as follows:

Let $h = (h(1),\dots,h(t))\in H$ then $g_i^q = s_i^i$ iff for some t' with $1\le t'\le t$ and only one j:

$[h(t')]_j \neq [P^q(t')]_j$ and otherwise $g^q(h) = P^q(t+1)$.

This strategy g^q fulfills that $u(g^q) = u(p^q)$ and for all strategy g' such that comp $g_i' < q$: $u_i(g^q_{-i},g_i') \le 0$

Now we can choose a q' big enough such that:

$$(4.9)\qquad \begin{aligned} &\text{(i)}\quad |(K,\dots,K) - (k(q'),\dots,k(q')| \le \varepsilon/4 \\ &\text{(ii)}\quad |r_K^j - u(g^{q'})| \le \varepsilon/4 \\ &\text{(iii)}\ u(g^{q'}) > (K,\dots,K) \end{aligned}$$

We will check that the strategy $g^{q'}$ satisfies the Claim's conditions.

a) holds because by construction comp $g_i^{q'} < \infty$ for all i.

In order to check b) we have:

$$|u^k(g^{q'}) - r^j\ | = |u(g^{q'}) - (k(g^{q'}_1),\dots,k(g^{q'}_n)) - r_k^j + (K,\dots,K)| \le$$
$$|u(g^{q'}) - r_k^j\ | + |(K,\dots,K) \ - (k(g^{q'}_1),\dots,k(g^{q'}_n))| \le \varepsilon/2$$

Now we prove c):

If comp $g'_i < q'$ then we have:

$$u_i(g^{q'}_{-i}, g'_i) \leq 0$$

and as $g^{q'}$ is a Nash Equilibrium then :

$$u_i(g^{q'}_{-i}, g'_i) - k(g'_i) - u_i(g^{q'}) + k(g^{q'}_i) < 0 < \varepsilon$$

by condition (4.9) part (iii)

If comp $g'_i > q'$ then we have:

$$u_i(g^{q'}_{-i}, g'_i) - k(g'_i) - u_i(g^{q'}) + k(g^{q'}_i) \leq \varepsilon/4$$

because $g^{q'}$ is a Nash Equilibrium and by (4.9) (i):

$$k(g^{q'}_i) - k(g'_i) \leq K - k(g'_i) \leq K - k(q') < \varepsilon/4$$

This completes the proof of the Claim. (q.e.d.)

<u>Remark 6:</u> An approximation result for lexicographic Nash Equilibrium obviously holds because every strategy which is a Nash Equilibrium is an ε-lexicographic Nash Equilibrium for any $\varepsilon>0$.

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