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EFFICIENCY IN PARTNERSHIP WHEN THE JOINT OUTPUT IS UNCERTAIN[+]

by

Steven R. Williams[*]

and

Roy Radner[**]

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[*] Economics Department, Northwestern University, Evanston, IL 60208. This research was supported by NSF Grant SES-8705649.

[**] AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974.

Abstract

This paper concerns a model of partnership in which each partner privately chooses his input into a joint production process. The partners' inputs determine a probability distribution over a finite set of alternative output levels. Each partner's utility is the difference between his share of the output and the disutility of his input contribution; the partners are therefore risk neutral. Earlier work suggests that because of moral hazard, there cannot exist rules for fully sharing the joint output that sustain the Pareto optimal inputs as a Nash equilibrium. Our results are more positive. We show that in a generic problem, the corresponding first order conditions are solvable because uncertainty makes the budget constraint non-binding in the first order analysis. This allows us to construct examples in which moral hazard is overcome purely through the choice of the compensation scheme. The results are extended to a case in which the set of output levels is a continuum, and the case of risk aversion is also discussed.

1. Introduction.

A partnership is a group of agents who jointly produce some observable output. A production plan is efficient for the partnership if it is Pareto optimal, given the disutility to each partner of his input into the collective effort. Moral hazard may exist when each partner's input is not fully observable, for a partner may have an incentive to contribute less of his input than the amount that is needed for efficiency. Unless this problem is overcome, the partnership will be inefficient.

Monitoring is one possible solution. Either an outsider can be brought in to observe the partners, or the partners can monitor each other to insure that each partner contributes his proper input. This is a flawed solution, however, for the requisite monitoring may be costly, or physically difficult. The cost of monitoring may even exceed the gain in output that it permits.

This paper concerns a different approach. A sharing rule defines a partner's share as a function of the joint output. Rather than trying to learn about the partners' inputs, we consider the problem of choosing sharing rules that provide each partner with an incentive to contribute his proper input. To be more precise, we wish to design sharing rules that (i) sustain the efficient inputs as a Nash equilibrium, and (ii) balance the budget (i.e., the joint output must be fully distributed among the partners, no matter what level of output is observed). We call such a family of sharing rules a solution to the partnership problem. When such rules exist, moral hazard can be overcome without altering the decentralized nature of decision-making within the partnership.

Earlier work on this problem has produced mostly negative results. Holmstrom [3] showed that when the output is fully determined by the partners' inputs, the partnership problem is unsolvable. This was proven by showing that in order for each partner to have the proper incentive, his marginal payment must equal his marginal product at the efficient level of his input, given that the other partners have chosen to contribute the efficient levels of their inputs. It was then shown that paying each partner in this manner is incompatible with the budget constraint.

In contrast to this discouraging result, the conclusions of our paper are more positive. We consider a single-stage model of partnership. Each partner's utility is the difference between his share of the joint output and the disutility of his contribution. Utility is therefore transferable, and each partner is risk neutral. Each partner chooses an input from an interval on the real line. A finite number of different levels of output can result from the choice of the partners' inputs. The primary difference between our model and the model that underlies Holmstrom's [3] negative result is that the joint output in our model may not be completely determined by the partners' inputs. The <u>output function</u> in our model assigns a probability distribution over the range of output levels to each choice of the partners' inputs. The main conclusion of this paper is that when the joint output is uncertain, the partnership problem may be solvable. An intuitive explanation of the role of uncertainty follows an outline of our results.[1]

We begin by showing that when there are only two alternative output levels, even the presence of uncertainty does not permit the solution of the partnership problem; as in Holmstrom [3], balancing the budget is incompatible with the first order conditions for a Nash equilibrium at the efficient inputs. The case of two output levels, however, is exceptional. When there are three or more output levels, sharing rules that satisfy the budget constraint and the first order conditions for a Nash equilibrium at the efficient inputs exist for a generic choice of the output function and the partners' utility functions. (The precise meaning of "generic" will be specified later.) Under uncertainty (generically), each partner can be paid his marginal expected product at the efficient level of inputs without breaking the budget constraint.

It is easy to construct linear examples in which every family of sharing rules that satisfies both the budget constraint and the first order conditions for a Nash equilibrium also solves the partnership problem. More generally, we derive conditions on the output function under which at least one solution to the first order problem also solves the partnership problem. For simplicity, we derive these conditions only in the case where there are two partners and three levels of output. The conditions are awkward, and at present they have no economic interpretation. "Reasonable" examples exist, however, that satisfy them. Despite their awkwardness, they prove the existence of robust examples of partnerships in which moral hazard can be overcome through the proper choice of a compensation scheme.

This paper also contains a paradoxical result that concerns the nature of the solutions to the first order conditions when the output function satisfies stochastic dominance with respect to each partner's input. (For each partner, given the inputs of the other partners, stochastic dominance holds if the observation of a higher level of output allows one to infer, in a probabilistic sense, that the selected partner contributed a greater level of input; e.g., see Whitt [7].) Each of the three examples in this paper shows that stochastic dominance does not necessarily prevent the solution of the partnership problem. When stochastic dominance holds, one might expect that a partner's share should increase with the output; as Alchian and Demsetz [1, p. 778] suggested in their analysis of the internal structure of firms, a partner may have an incentive to "sabotage" the organization if his reward and the output are inversely related. We prove that the opposite is true: when stochastic dominance holds, for any sharing rules that satisfy the first order conditions, some (at least two) of the partners' shares must be nonincreasing over some subsets of the range of outputs levels. As our examples illustrate, moral hazard can be overcome in some problems in which stochastic dominance holds, but only if some partners do not always benefit when the joint output increases.

This paradox may explain why our results seem surprising, and why they have been overlooked in the literature on partnership. We note that Radner [5, p.46] did analyze the problem that we consider, and he correctly argued that the Nash equilibria of partnerships are typically inefficient when the budget is balanced. His argument does not contradict the conclusions of our paper, for it assumes: (i) each partner's share is an increasing function of the output; (ii) for each state of a random environment, the output is an increasing function of each partner's input. This second assumption implies that the output function satisfies stochastic dominance.

We also explain how each of the results that we prove when the range of output levels is finite can be extended to the case where this set is a subinterval of the real line. The purpose of this discussion is to show that our results are not just a special feature of a finite model. It is possible to solve the partnership problem in our model, not because of any special assumption about the range of output levels, but because the joint output is uncertain.

Our paper is focused on the case in which the partners are risk neutral. In the final section of this paper, we emphasize the importance of risk neutrality to our analysis by briefly discussing the more general case in which the partners may be risk averse. In a generic problem with risk aversion, efficiency cannot be sustained as a Nash equilibrium with budget-balancing sharing rules. By completing our analysis of the uncertainty case, this discussion allows us to conclude the paper by summarizing the relationship between the partnership model and the principal-agent model.

The explanation of the role of uncertainty in our results rests upon the following point: while the decision problem that a partner faces is the same in both the certainty and the uncertainty cases, the mechanism designer's problem is very different when there is uncertainty, for he would typically have available more methods for influencing the strategic choices of the partners. For fixed sharing rules, a partner verifies that an input profile is a Nash equilibrium by solving the same maximization problem in both cases; when there is uncertainty, he simply solves this problem in terms of expected (rather than certain) shares. The set of sharing rules that is available to the mechanism designer is also the same in both cases. Now regard the certainty case as a special instance of the uncertainty case. For given sharing rules, consider the functions of the inputs that assign to each partner his expected share under the given sharing rules; define these functions as the (induced) _expected sharing rules_. Except for degenerate instances of our model, the mechanism designer can choose from a wider class of expected sharing rules in the uncertainty case than in the certainty case. This is clearly significant, because the expected sharing rules completely determine the incentives of a given set of sharing rules.

The extra freedom that uncertainty provides is illustrated by the following simple observation: in the certainty case, the expected sharing rules can be written as functions of the (expected) joint output, but when the joint output is uncertain, the expected shares may not be expressible as functions of the expected joint output. For instance, two input profiles may define probability distributions that determine the same expected output; this does not mean that they would determine the same expected shares for the partners. This discussion does not explain how this extra freedom may permit the implementation of efficiency with budget-balancing sharing rules. To explain this central issue and to get a better sense of how uncertainty alters the design problem, we return to the first order viewpoint.

Our explanation of the role of uncertainty in the first order analysis begins with Holmstrom's simplest model of the case in which the output is fully determined by the inputs. Holmstrom showed that the efficient inputs cannot be implemented with budget-balancing sharing rules. In fact, the following argument shows that a generic input profile cannot be implemented in this fashion. At an equilibrium input profile, the marginal disutility of each partner's contribution must equal his marginal share of the output. The budget constraint, however, restricts the values of these marginal shares; at a generic input profile, the marginal disutilities would not satisfy these restrictions. A generic input profile therefore cannot be a Nash equilibrium. Holmstrom's proof that the efficient profile cannot be a Nash equilibrium is really a verification that it is always in this sense generic. It is also clear that the efficient profile can be sustained as a Nash equilibrium with budget-breaking sharing rules (e.g., see Holmstrom [3, Thm. 2]).

A similar analysis holds when the partners' inputs determine a probability distribution over only two different output levels. Suppose one wishes to choose budget-balancing sharing rules to sustain some particular profile as a Nash equilibrium. The marginal expected shares of the partners at the given profile are again determined by the first order incentive constraints. The budget constraint imposes an additional condition upon these marginal expected shares; hence, a generic input profile cannot be a Nash equilibrium. As in the certainty case, it can then be shown that the efficient inputs are always in this sense generic.

The analysis changes completely when the partners' inputs determine a probability distribution over three or more output levels, for the budget constraint no longer imposes a restriction upon the marginal expected shares at a generic input profile. As the number of output levels increases, there are more variables (the shares of each of the different output levels) that one can adjust to determine the partners' marginal expected shares. These extra variables are "wasted" in the case of two output levels, for the elementary nature of the marginal probabilities makes the formulas for the players' marginal expected shares into a degenerate linear system. When there are at least three output levels, however, this extra freedom permits one to choose budget-balancing sharing rules that make the marginal expected shares at a generic input profile assume whatever values one wants. (This is proven in Theorem 1 of this paper.) At a generic input profile in this case, budget-balancing sharing rules can be chosen that solve the first order conditions for a Nash equilibrium; hence, this can usually be done at the efficient profile. Uncertainty is therefore significant in our model because it makes the budget constraint non-binding in the first order analysis when there are three or more possible output levels.

A similar argument resolves the paradox of why some of the partners' sharing rules must be somewhere nonincreasing when stochastic dominance holds. As shown in Theorem 2, the budget constraint and the assumption that each sharing rule is an increasing function together imply that each partner's marginal expected share must exceed the marginal disutility of his contribution at the efficient profile of inputs. Efficiency therefore cannot be sustained as a Nash equilibrium in the prescribed manner. The paradox illustrates how any constraint upon the sharing rules may cause inefficiency if it restricts the values of the partners' marginal expected shares at the efficient profile. At a Nash equilibrium, a partner's expected share must be an increasing function of his input, for his marginal expected share must equal the (positive) marginal disutility of his contribution; the paradox rests upon the trivial observation that a partner's expected share can be an increasing function of his input even if his share does not always increase with the output.

The results of this paper are significant because of the questions that they pose for theories of the firm. The presence of moral hazard in partnerships has been used to explain why firms are typically organized in a hierarchical form (e.g., see [1], [3]). Because the partnership is (allegedly) inefficient, the partners have an incentive to change their organization by bringing in a principal, either to monitor inputs, and/or to administer budget-breaking compensation schemes. The partnership therefore evolves into a hierarchical form. Our paper questions whether or not the partners have any incentive to alter their organization, for they may be able to achieve optimal production without having to pay a principal. It may in fact be true that partnership is an inherently inefficient form of economic organization, but our results show that the existing theories are incomplete.

We conclude this introduction by discussing the limitations of our approach. Aside from the assumption of risk neutrality, the most obvious flaw is that we derive specific sharing rules for each particular choice of the output function and the partners' utility functions. The solution process requires information that may be dispersed among the partners (the utility functions), or not even known with certainty (e.g., the output function). Our analysis thus rests upon strong informational assumptions. We have also not proven that the sharing rules we derive for a particular partnership are in any sense optimal for an open class of partnerships. From a practical viewpoint, it may not be possible to redesign a compensation scheme as either technology or the composition of the partnership changes.

Our solutions seem more complex than the simple sharing rules that are often seen in partnerships. It is possible that these simple rules are just inefficient, or it may be that they have certain advantages because of their simplicity. Complexity is not considered in this paper.

Finally, partners in many production processes contribute their inputs over time. A partner may receive information about the output during production. Our model clearly does not include these cases (e.g., if production is ending, and a partner believes that his share would now decrease with an increase in output, then he may have an incentive to stop providing his input). The role of time in production may justify Alchian and Demsetz's [1] claim that compensation should increase with the output.

These points criticize both the informational hypotheses of our model, and the questions that we try to answer. The main conclusion of this paper is that when the output is not completely determined by the partners' inputs, it is not per se impossible to implement efficiency through the proper choice of a compensation scheme; it may, however, be impractical to actually do so. Richer models that perhaps reflect the above points may provide a better understanding of the performance of partnerships.

2. The First Order Approach.

We begin by describing our model. There are $m > 1$ partners. The ith partner chooses his input a_i from some closed and bounded subinterval A_i of the real line. His choice is his own private information. Let $a \equiv (a_1,\ldots,a_m)$ denote an input profile, and let a_{-i} denote the $(m-1)$-tuple $(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_m)$.

Once the partners have chosen their inputs, one of several levels of output results. This output is publicly observable. Let Ω denote the range of output levels of the partnership. Except where otherwise noted, the reader should assume that Ω is some finite subset of $\mathbb{R}$ with $n \geq 2$ elements,

$$\Omega = \{y_1 < y_2 < \ldots < y_n\}.$$

The partners' inputs determine a probability distribution over Ω. For the input profile a, let $F(\cdot,a)$ denote the cumulative distribution that is determined by a, and let $f(\cdot,a)$ denote the corresponding density function. These functions are common knowledge, and each is a C^1 function of the inputs. For simplicity, let $F_i(y,a) \equiv \frac{\partial F}{\partial a_i}(y,a)$ and $f_i(y,a) \equiv \frac{\partial f}{\partial a_i}(y,a)$.

The ith partner's utility $u_i(s_i,a_i)$ consists of whatever share $s_i(y)$ he receives of the observed output y, minus the disutility $Q_i(a_i)$ of his contribution of the input a_i:

$$u_i(s_i,a_i) \equiv s_i(y) - Q_i(a_i).$$

By assumption, $Q_i(\cdot)$ is a C^1 function of the ith partner's contribution. Let $q_i(\cdot) \equiv \partial Q_i/\partial a_i(\cdot)$. We assume that $q_i(\cdot)$ is strictly positive.

Since utility is transferable, an input profile $\hat{a}$ is Pareto optimal if and only if it maximizes the difference of the expected total output minus the total disutility of the input contributions:

$$\hat{a} \in \underset{a}{\operatorname{argmax}}\ E(y|a) - \sum_{i=1}^{m} Q_i(a_i). \tag{2.0}$$

We assume that there exists a solution to this maximization problem in the interior of $\Pi_{i=1}^{m} A_i$. Efficiency therefore requires each partner to make a positive input contribution.

Our concern is the existence of sharing rules $s_1(\cdot),\ldots,s_m(\cdot)$ that satisfy the budget constraint

$$\sum_{i=1}^{m} s_i(y) = y, \text{ all } y \in \Omega, \tag{2.1}$$

and that also make the efficient profile $\hat{a}$ into a Nash equilibrium,

$$\hat{a}_i \in \underset{a_i}{\operatorname{argmax}}\ E\big(s_i(y)\,|\,(a_i,\hat{a}_{-i})\big) - Q_i(a_i) \quad \text{for} \quad 1 \leq i \leq m.$$

The problem of devising sharing rules with these properties is the partnership problem. The following first order conditions are necessary: if $\hat{a}$ is efficient, then each partner's marginal expected product must equal the marginal disutility of his contribution at $\hat{a}$,

$$\sum_{j=1}^{n} y_j f_i(y_j,\hat{a}) = q_i(\hat{a}_i) \quad \text{for all} \quad 1 \leq i \leq m, \tag{2.2}$$

and if $\hat{a}$ is a Nash equilibrium, then each partner's marginal expected share must equal the marginal disutility of his contribution at $\hat{a}$,

$$\sum_{j=1}^{n} s_i(y_j)f_i(y_j,\hat{a}) = q_i(\hat{a}_i) \quad \text{for all} \quad 1 \leqslant i \leqslant m. \tag{2.3}$$

The problem of devising sharing rules that satisfy the first order conditions (2.2), (2.3) and the budget constraint (2.1) is the <u>first order problem</u>.

Our approach is to solve the first order problem and then to determine whether or not these solutions also solve the partnership problem. In analyzing the first order problem, we focus upon the solvability of the first order Nash equilibrium conditions, together with the budget constraint; the role of efficiency is merely to determine a particular instance of this system. We illustrate this viewpoint by reconsidering a simple model in Holmstrom [3] in which the partners' inputs fully determine the joint output. Let Ω be the real line, and let $F(a)$ denote the output determined by a. Assume that $F(\cdot)$ is a differentiable function of the inputs, and that the sharing rules $s_1(\cdot),\ldots,s_m(\cdot)$ are differentiable functions of the output. An input profile a^* is a Nash equilibrium only if each partner's marginal share equals his marginal disutility at a^*, i.e.,

$$s_i'(y^*) \cdot dF/da_i(a^*) = q_i(a^*) \quad \text{for} \quad 1 \leqslant i \leqslant m, \tag{2.4}$$

where $y^* \equiv F(a^*)$. The budget constraint implies that

$$\textstyle\sum_{i=1}^{m} s_i'(y) = 1 \tag{2.5}$$

at any $y \in \Omega$. Equations (2.4) and (2.5) define a system of $m + 1$ equations in the m unknowns $s_1'(y^*),\ldots,s_m'(y^*)$. In a generic problem, this system is unsolvable for all a^* in some open, dense subset of $\Pi_{i=1}^{m} A_i$.[2] A generic input profile in the certainty model hence cannot be sustained as a Nash equilibrium with budget-balancing sharing rules. At the efficient profile $\hat{a}$, the marginal disutility of any partner's input equals its marginal product,

$$dF/da_i(\hat{a}) = q_i(\hat{a}), \quad \text{for} \quad 1 \leq i \leq m. \qquad (2.6)$$

When (2.6) is substituted into (2.4) we obtain $s_i'\big(F(\hat{a})\big) = 1$ for $1 \leq i \leq m$, which contradicts (2.5). This shows that the system defined by (2.4) and (2.5) is unsolvable at the efficient profile in every problem; hence, efficiency can never be implemented in the desired fashion.

The above analysis can be illustrated with a simple example. Let $A_1 = A_2 = [0,2]$, $y = F(a) = a_1 + a_2$, and $u_i\big(s_i(y),a_i\big) = s_i(y) - a_i^2/2$ for $i = 1,2$. The first order conditions for a Nash equilibrium at a^* are $s_i'(a_1^*+a_2^*) = a_i^*$, $i = 1,2$. By (2.5), the budget constraint implies that $a_1^* + a_2^* = 1$; hence, an input profile must be on the line $a_1 + a_2 = 1$ if it can be a Nash equilibrium under some set of budget-balancing sharing rules. It is easy to show that the efficient inputs are $\hat{a}_1 = \hat{a}_2 = 1$, which therefore cannot be sustained in the prescribed manner.

The main conclusion of Theorem 1 is that the first order problem is solvable for a generic choice of $F(\cdot)$ and $Q_1(\cdot),\ldots,Q_m(\cdot)$ when $F(\cdot,a)$ defines a probability distribution over at least three output levels. As in the above analysis of the certainty case, efficiency plays a relatively minor role in our proof. We actually prove the stronger result that in a generic problem when there are at least three output levels, a generic input profile can be a Nash equilibrium under budget-balancing sharing rules. Our emphasis in this paper, however, is upon implementing efficiency.

We specify our set of problems with the following notation. For a given range of output levels Ω and input intervals $A_1,\ldots,A_m$, let Ψ denote the set of (m+1)-tuples $\left(F(y,a),\ Q_1(a_1),\ldots,Q_m(a_m)\right)$ such that: (i) $F(\cdot,a)$ is a cumulative distribution on Ω that is C^1 in a; (ii) each $Q_i(\cdot)$ is a C^1 function of a_i, and $q_i(a_i) > 0$; (iii) there is at least one efficient point (i.e., a solution to (2.0)) in the interior of $\Pi_{i=1}^{m} A_i$. The set Ψ is given the Whitney C^1 topology.

Theorem 1. When the range of output levels Ω has only $n = 2$ elements, the first order problem is unsolvable. When Ω has $n > 2$ elements, the first order problem is solvable for each $(F,Q_1,\ldots,Q_m)$ in some open, dense subset of the set Ψ of problems that we consider, and the solution set for $(F,Q_1,\ldots,Q_m)$ forms an (mn-m-n) - dimensional affine space.

Proof. We regard (2.3) and (2.1) as a system of m + n linear equations in the mn variables $\left(s_i(y_j)\right)_{1 \leq i \leq m,\ 1 \leq j \leq n}$ whose coefficients are determined by the efficient profile $\hat{a}$. The left-hand side of the first m equations (from (2.3)) are the marginal expected shares of the partners, and the last n equations (from (2.1)) form the budget constraint. Efficiency is used in the analysis of the $n \geq 3$ case in this proof only to determine the values of these coefficients; the argument could be carried out in terms of a generic input profile.

Let S_i denote the n-vector of the ith partner's shares, $S_i \equiv \left(s_i(y_1),\ldots,s_i(y_n)\right)$, and let S denote the mn-vector $(S_1,S_2,\ldots,S_m)$. The system of linear equations can be represented in matrix form as

$$DS^T = \left(q_1(\hat{a}_1),\ldots,q_m(\hat{a}_m),\ y_1,\ldots,y_n\right)^T, \qquad (2.7)$$

where D is an (m+n) × mn matrix. The matrix D consists of m blocks in a row,

$D = D_1 D_2 \cdots D_m$, where D_i is the $(m+n) \times n$ matrix

$$D_i = \begin{matrix} Z_{i-1,n} \\ \left(f_i(y_j,\hat{a})\right)_{1 \leq j \leq n} \\ Z_{m-1-i,n} \\ I_{n,n} \end{matrix} \quad .$$

Here, $Z_{k,n}$ denotes the $k \times n$ zero matrix, and $I_{n,n}$ is the $n \times n$ identity matrix.

To prove the existence of a $[mn - (m+n)]$-dimensional affine space of solutions to the first order problem, it is sufficient to show that the rank of D is $m + n$. This would imply that the budget constraint (the last n equations) does not affect the solvability of the first order incentive constraints (the first m equations). We shall show that this holds for a generic choice of $(F,Q_1,\ldots,Q_m)$ when $n > 2$. Note that since $\sum_{j=1}^{n} f_i(y_j,\hat{a}) = 0$ for each i, through column operations the last column of each D_i could be replaced by a vector whose top m entries are zero and whose bottom n entries are one. The rank of D is therefore bounded above by $m(n-1) + 1$. When $n = 2$, rank $D \leq m + 1 < m + 2 = m + n$; the budget constraint therefore restricts the values that the partners' marginal expected shares can assume, and our argument breaks down. We return to the $n = 2$ case below.

The following argument holds for an open, dense set of problems $(F,Q_1,\ldots,Q_m)$ when $n > 2$. We outline a procedure for choosing $m + n$ linearly independent columns of D:

(i) for $1 \leq i \leq n$, define C_i as the ith column of D_1;

(ii) for $2 \leq k \leq m$, let C_{n+k} be a column of D_k whose kth entry is nonzero;

(iii) finally, let C_{n+1} be any remaining column in D_2.

It is clear that $C_1,\ldots,C_n$ are linearly independent. For $2 \leq k \leq m$, C_{n+k} is the only column in the set $\{C_1, C_{n+2},\ldots,C_{n+m}\}$ that has a nonzero entry in row k; hence, this set is also linearly independent.

It remains to be shown that C_{n+1} is not a linear combination of the other columns. Let C_{n+1} and C_{n+2} be the pth and qth columns (respectively) of D_2. We assume that $C_{n+1} = \sum_{t \neq n+1} \lambda_t C_t$, and we shall derive a contradiction. Since C_{n+1} has a zero in row k for $3 \leq k \leq m$, λ_t must be zero for $t > n + 2$. By examining the last n rows in the equation $C_{n+1} = \sum_{t=1}^{n} \lambda_t C_t + \lambda_{n+2} C_{n+2}$, it follows that $\lambda_t = 0$ for $t \neq p,q$, and $\lambda_p = 1$, $\lambda_q = -\lambda_{n+2}$, i.e., $C_{n+1} = C_p + \lambda_q C_q - \lambda_q C_{n+2}$. This equation implies that

$$-f_1(y_p,\hat{a})/f_1(y_q,\hat{a}) = \lambda_q = -f_2(y_p,\hat{a})/f_2(y_q,\hat{a}).$$

When n = 2, this equality is satisfied; when n > 2, it does not hold for all problems in some open and dense subset of Ψ.[3] The set $\{C_t | 1 \leq t \leq n + m\}$ is therefore linearly independent.

We now show that solutions to the first order problem cannot exist when there are only two outputs. In this case, $f_i(y_1,a) = -f_i(y_2,a)$ for all i. Equations (2.2) and (2.3) reduce to

$$(y_1-y_2)f_i(y_1,\hat{a}) = q_i(\hat{a}_i) \qquad (2.8)$$

and

$$\big(s_i(y_1) - s_i(y_2)\big)f_i(y_1,\hat{a}) = q_i(\hat{a}_i), \qquad (2.9)$$

respectively, for all i. Since $q_i(\hat{a})$ is positive, $f_i(y_1,\hat{a})$ is nonzero. We can therefore solve (2.8) and (2.9) to obtain $y_1 - y_2 = s_i(y_1) - s_i(y_2)$ for all i. Using the budget constraint, summing this expression over $1 \leqslant i \leqslant m$ gives

$$m(y_1 - y_2) = \sum_{i=1}^{m} s_i(y_1) - \sum_{i=1}^{m} s_i(y_2) = y_1 - y_2,$$

which is a contradiction. Q.E.D.

Example 1. We now illustrate that the first order problem can be solved, and that its solutions may also solve the partnership problem. We consider a simple example where the output rule is linear in each partner's input, and the disutility of each partner's input is quadratic. This insures that each solution to the first order problem is also a solution to the partnership problem.

There are two partners (m=2) and three output levels ($y_1 = 0$, $y_2 = 1$, $y_3 = 2$). Each partner's strategy space is [0,1]. The density $f(\cdot,a)$ is defined by

$$f(y_1,a) = 1 - 5a_1/12 - a_1/3$$

$$f(y_2,a) = a_1/6 + a_2/12$$

$$f(y_3,a) = a_1/4 + a_2/4.$$

Partner i's utility function is $u_i(s_i,a_i) = s_i - a_i^2$.

The pair $\hat{a} = (\hat{a}_1,\hat{a}_2)$ is efficient if and only if

$$\hat{a} \in \underset{a}{\text{argmax}}\ E(y|a) - (a_1^2+a_2^2) = \underset{a}{\text{argmax}}\ 2a_1/3 + 7a_2/12 - (a_1^2+a_2^2).$$

The efficient inputs are therefore $\hat{a}_1 = 1/3$ and $\hat{a}_2 = 7/24$.

Given these values of $\hat{a}_1$ and $\hat{a}_2$, solving the first order problem is equivalent to finding sharing rules $s_1(\cdot)$, $s_2(\cdot)$ such that

$$s_1(y_1)(-5/12) + s_1(y_2)(1/6) + s_1(y_3)(1/4) = 2/3$$

$$s_2(y_1)(-1/3) + s_2(y_2)(1/12) + s_3(y_3)(1/4) = 7/12$$

$$s_1(y_j) + s_2(y_j) = y_j, \qquad 1 \leqslant j \leqslant 3.$$

This is a system of five linear equations in the six unknowns $s_i(y_j)$, $i = 1,2$, $1 \leqslant j \leqslant 3$. The set of all solutions is the one dimensional space

$$s_1(y_1) = t \qquad s_2(y_1) = -t$$

$$s_1(y_2) = 8 + t \qquad s_2(y_2) = -7 - t$$

$$s_1(y_3) = -8/3 + t \qquad s_2(y_3) = 14/3 - t$$

where the choice of $t \in \mathbb{R}$ determines a particular solution. As noted above, each of these solutions also solves the partnership problem because of the nature of the functions in this example.

Several remarks should be made about Theorem 1. The theorem states that when there are at least three output levels, there exists a multiplicity of solutions to the first order problem in the generic case. We note that if $s_1(y),\ldots,s_m(y)$ form a solution to the partnership problem and if $k_1,\ldots,k_m$ are constants such that $\sum_{i=1}^{m} k_i = 0$, then $k_1 + s_1(y),\ldots,k_m + s_m(y)$ also solve the partnership problem. This is a very simple procedure for adjusting the relative expected shares of the partners, once one solution has been found.

A linear sharing rule $s_i(y)$ has the form $s_i(y) = \lambda_i y + \delta_i$. The proof of Theorem 1 can be modified to show that regardless of the number of output levels, the Nash equilibria under linear sharing rules are necessarily inefficient. The analysis of the case of linear sharing rules is similar to the two output case in Theorem 1; the key issue in both cases is that there are an insufficient number (i.e., two) of variables per partner that the mechanism designer can adjust. Looking ahead, it is interesting to note that in Example 3 (Section 4) of this paper, a set of linear sharing rules plus appropriate bonuses and penalties are efficient. The bonuses and penalties provide the mechanism designer with the extra freedom he needs to induce the proper choices by the partners.

Finally, we emphasize that Theorem 1 states that the two output case is degenerate. For simplicity, research on partnership has sometimes focused on this case. Though it may be of interest, such research may not reflect what is achievable in a partnership when there are more than two output levels.[4]

We conclude this section by examining the nature of the solutions to the first order problem when the output function satisfies stochastic dominance with respect to each partner's input. Mathematically, this is given by $F_i(y,a) \leq 0$ for all $1 \leq i \leq m$, $y \in \Omega$, and $a \in \Pi_{i=1}^{m} A_i$.

Theorem 2. Assume that the output function $F(\cdot)$ satisfies stochastic dominance. If $s_1(\cdot),\ldots,s_m(\cdot)$ is some solution to the first order problem, then at least two of the functions $s_i(y)$ are nonincreasing over some subsets of the range of output levels.

Proof. For $1 \leq i \leq m$, let $s_{-i}(y) \equiv \sum_{k\neq i} s_k(y)$. The proof is mostly a matter of rewriting the first order conditions for a Nash equilibrium in terms of $s_{-1}(\cdot),\ldots,s_{-m}(\cdot)$.

The first order conditions for efficiency can be rewritten as

$$q_i(\hat{a}) = \sum_{j=1}^{n} f_i(y_j,\hat{a})y_j$$

$$= F_i(y_1,\hat{a})y_1 + \sum_{j=2}^{n} [F_i(y_j,\hat{a}) - F_i(y_{j-1},\hat{a})]y_j$$

$$= \sum_{j=1}^{n-1} F_i(y_j,\hat{a})(y_j - y_{j+1}).$$

Since $q_i(\hat{a}) > 0$, $F_i(y_j,\hat{a})$ must be nonzero for at least one value of j between 1 and n-1.

In a similar fashion, each of the first order conditions for a Nash equilibrium can be rewritten as

$$0 = \sum_{j=1}^{n} f_i(y_j,\hat{a})s_i(y_j) - q_i(\hat{a}) \tag{2.10}$$

$$= [\sum_{j=1}^{n} f_i(y_j,\hat{a})y_j - q_i(\hat{a})] - \sum_{j=1}^{n} f_i(y_j,\hat{a})s_{-i}(y_j)$$

$$= \sum_{j=1}^{n} f_i(y_j,\hat{a})s_{-i}(y_j)$$

$$= \sum_{j=1}^{n-1} F_i(y_j,\hat{a})[s_{-i}(y_j) - s_{-i}(y_{j+1})]. \qquad (2.11)$$

Since $F_i(y_j,\hat{a}) \leq 0$ for all $1 \leq j \leq n$, and $F_i(y_j,\hat{a}) < 0$ for at least one value of j, one of the terms $s_{-i}(y_j) - s_{-i}(y_{j+1})$ must be nonnegative; for some $k \neq i$, $s_k(\cdot)$ is nonincreasing over some subset of Ω. Such a value k exists for every $1 \leq i \leq m$; hence, there must be at least two partners whose shares are nonincreasing over some subsets of Ω. Q.E.D.

The reader should note that a stronger conclusion holds if $F_i(y_j,a)$ is strictly less than zero for all $1 \leq i \leq m$ and $1 \leq j < n$; in this case, we can conclude that the shares of at least two partners must somewhere decrease.

3. The Sufficiency of the First Order Approach.

In this section we describe conditions on $F(\cdot)$ under which some solution to the first order problem also solves the partnership problem. We restrict our attention to the simple case in which there are two partners (m=2), and three levels of output (n=3). The conditions that we shall derive can be extended to larger partnerships with many possible levels of output; this extension is not very illuminating, however, and it is therefore omitted. Throughout this section, we also assume that $F(\cdot)$ satisfies stochastic dominance.

We begin by rewriting the partnership problem. Using the budget constraint, the ith partner's expected utility given his input a_i and the input $\hat{a}_{-i}$ of the other partner is

$$s_i(y_1)f(y_1,a_i,\hat{a}_{-i}) + s_i(y_2)f(y_2,a_i,\hat{a}_{-i}) - Q_i(a_i)$$

$$= \left[y_1 f(y_1,a_i,\hat{a}_{-i}) + y_2 f(y_2,a_i,\hat{a}_{-i}) - Q_i(a_i)\right] \quad (3.0)$$

$$- \left[s_{-i}(y_1)f(y_1,a_i,\hat{a}_{-i}) + s_{-i}(y_2)f(y_2,a_i,\hat{a}_{-i})\right] \quad (3.1)$$

Since $\hat{a}$ is efficient, (3.0) is maximized at $a_i = \hat{a}_i$. The remaining term (3.1) is the expected share of the $-i$th partner when he contributes $\hat{a}_{-i}$ and the ith partner contribute a_i. It is clear from (3.0) and (3.1) that $s_1(\cdot)$, $s_2(\cdot)$ form a solution to the partnership problem if each partner minimizes the expected share of the other partner by choosing $\hat{a}_i$, given that the other partner contributes $\hat{a}_{-i}$.

It is helpful if we now take a slightly different viewpoint. A partner's share of the joint output can be regarded as a rule that provides him with some fixed fraction of the lowest possible output level (i.e., the "certain" output) plus fractions of each additional increment of output. The partner chooses his level of input to maximize the expected value of the additional fractions that he receives as the observed joint output increases. This decision does not depend upon the size of his share of the smallest level of output. (Note the importance of risk neutrality in this argument.) Similar remarks hold for his evaluation of the other partner's shares.

We now rewrite the above problem to reflect this viewpoint. For i = 1,2, let dy denote the two-vector whose jth component is $y_j - y_{j+1}$, let v_i denote the two-vector whose jth component is $s_i(y_j) - s_i(y_{j+1})$, and let $\mathcal{F}_i(a_i)$ denote the vector-valued function $\mathcal{F}_i(a_i) \equiv \left(F_i(y_1,a_i,\hat{a}_{-i}),\ F_i(y_2,a_i,\hat{a}_{-i})\right)$. Using the method between (2.10) and (2.11) in the proof of Theorem 2, it is easy to show that the first order condition for the ith partner to minimize the expected share of the -ith partner is

$$\mathcal{F}_i(\hat{a}_i)\cdot v_{-i} = 0. \tag{3.2}$$

The budget constraint can also be rewritten as

$$v_1 + v_2 = dy. \tag{3.3}$$

The problem of devising sharing rules that satisfy the first order problem is equivalent to the problem of finding two-vectors v_1, v_2 that satisfy (3.2) and (3.3). For the remainder of this section, we shall work with this form of the first order problem.

Our sufficient conditions on F(•) are motivated by the following geometric analysis. By stochastic dominance, $\mathcal{F}_1(\hat{a}_1)$ and $\mathcal{F}_2(\hat{a}_2)$ lie in the lower left-hand quadrant of $\mathbb{R}^2$ (see Figure 1). When these vectors are colinear (e.g., in the symmetric case), solutions to (3.2) and (3.3) do not exist. In a generic case, these vectors are independent; we assume that $\mathcal{F}_1(\hat{a}_1)$ lies on a line with smaller slope than $\mathcal{F}_2(\hat{a}_2)$. The solution to (3.2) and (3.3) is unique; the one parameter family of solutions to the first order problem is generated by the different divisions of the smallest output y_1. The solution vector v_1 lies in the upper left-hand quadrant, while v_2 lies in the lower right-hand quadrant (as illustrated). Our goal is to find conditions upon F(•) such that each $\mathcal{F}_i(a_i)\cdot v_{-i}$ changes from negative to positive at $a_i = \hat{a}_i$. The most obvious way to insure this is to assume that $\mathcal{F}_1(a_1)$ moves clockwise as a_1 increases, while $\mathcal{F}_2(a_2)$ moves counterclockwise as a_2 increases: as $\mathcal{F}_i(a_i)$ changes from determining an obtuse angle with v_{-i} to an acute angle, $\mathcal{F}_i(a_i)\cdot v_{-i}$ changes from negative to positive. This motivates the following theorem.

Theorem 3. Consider the model of partnership in which there are two partners ($m=2$), and three levels of output ($n=3$). Assume that $F(\bullet)$ satisfies stochastic dominance. A one parameter family of solutions to the partnership problem exists if the following two additional hypotheses are satisfied at the efficient profile $\hat{a}$:

i) $F_1(y_1,\hat{a})F_2(y_2,\hat{a}) - F_1(y_2,\hat{a})F_2(y_1,\hat{a}) > 0;$ (3.4)

ii) $F_1(y_2,a_1,\hat{a}_2)/F_1(y_1,a_1,\hat{a}_2)$ is an increasing function of a_1, and $F_2(y_2,\hat{a}_1,a_2)/F_2(y_1,\hat{a}_1,a_2)$ is a decreasing function of a_2. (3.5)

Proof. We first note that (3.4) states that $\mathscr{F}_1(\hat{a}_1)$ lies on a line with smaller slope than the line determined by $\mathscr{F}_2(\hat{a}_2)$. Hypothesis (3.5) states that $\mathscr{F}_1(a_1)$ moves clockwise as a_1 increases, while $\mathscr{F}_2(a_2)$ moves counterclockwise as a_2 increases.

Given the preceding discussion, the proof is straightforward. The solution to (3.2) and (3.3) is

$$v_1 = \frac{q_2(\hat{a}_2)}{K}\left(F_1(y_2,\hat{a}),\ -F_1(y_1,\hat{a})\right)$$

$$v_2 = \frac{q_1(\hat{a}_1)}{K}\left(-F_2(y_2,\hat{a}),\ F_2(y_1,\hat{a})\right)$$

where K is the left-hand expression in (3.4). A one parameter family of solutions to the first order problem is obtained by varying the division of the "certain" output level y_1. One can easily verify that each of these solutions also solves the partnership problem by using the argument that immediately precedes the statement of the theorem. Q.E.D.

Example 2. The three levels of output are $y_1 = 1$, $y_2 = 2$, $y_3 = 3$. The ith partner chooses an input $a_i \in [0,1]$, and the disutility of his contribution is $Q_i(a_i) = a_i^2/2$. The output function $F(\cdot)$ is given by

$$F(y_1,a) = [(2 + a_1^3 - 3a_1) + (1 - a_2^3)]/6$$

$$F(y_2,a) = [(3 - a_1^3) + (3 + a_2^3 - 3a_2)]/6.$$

Note that $F(\cdot)$ satisfies stochastic dominance. By a direct computation, it can be shown that the efficient inputs are $\hat{a}_1 = \hat{a}_2 = 1/2$. It is then easy to verify that the remaining hypotheses of Theorem 3 hold.

The solutions to the first order problem are

$$s_1(y_1) = t \qquad s_2(y_1) = (1-t)$$

$$s_1(y_2) = 3/2 + t \qquad s_2(y_2) = -1/2 + (1-t)$$

$$s_1(y_3) = 1 + t \qquad s_2(y_3) = 1 + (1-t)$$

or alternatively, in the notation of (3.2) and (3.3), $v_1 = (3/2,1/2)$, $v_2 = (1/2,-3/2)$. With these sharing rules, the marginal expected return to the ith partner when he chooses a_i and his partner chooses $\hat{a}_{-i} = 1/2$ is

$$\left(\mathcal{F}_i(a_i)\cdot v_i\right) - a_i = -(a_i+3/2)(a_i-1/2).$$

This changes from positive to negative at $a_i = 1/2$. The efficient inputs are therefore a Nash equilibrium under these sharing rules.

4. The Continuum Case.

In this section, we discuss how each of the three theorems in this paper can be extended to the case in which the range of outputs Ω is an interval $[\underline{y},\bar{y}]$ on the real line. We conclude this section with an example in the continuum case in which the partnership problem is solved.

Intuitively, the formula in Theorem 1 for the dimension of the affine space of solutions to the first order problem suggests that there should exist an infinite dimensional space of solutions for a generic problem in the continuum case. This can be proven with the following procedure for reducing the first order problem in the continuum case to a linear system of the form that is described in the proof of Theorem 1.

As before, let $\hat{a}$ denote the efficient profile of inputs. The first order problem is to find sharing rules $s_1(\cdot),\ldots,s_m(\cdot)$ on $[\underline{y},\bar{y}]$ such that

$$\int_{\underline{y}}^{\bar{y}} s_i(y)f_i(y,\hat{a})dy = q_i(\hat{a}) \tag{4.0}$$

for all $1 \leqslant i \leqslant m$, and

$$\sum_{i=1}^{m} s_i(y) = y \tag{4.1}$$

for all $y \in [\underline{y},\bar{y}]$. We consider sharing rules of the form $s_i(y) = t_i(y) + c_i(y)$, where $t_1(y),\ldots,t_m(y)$ are any (integrable) functions that satisfy $\sum_{i=1}^{m} t_i(y) = y$, and $c_1(y),\ldots,c_m(y)$ are unspecified functions that we shall determine. When these sharing rules are substituted into (4.0) and (4.1), the first order problem is reduced to finding $c_1(y),\ldots,c_m(y)$ such that for $1 \leq i \leq m$,

$$\int_{\underline{y}}^{\bar{y}} c_i(y) f_i(y,a)dy = q_i^*, \tag{4.2}$$

and for $y \in [\underline{y},\bar{y}]$,

$$\sum_{i=1}^{m} c_i(y) = 0, \tag{4.3}$$

where (by definition)

$$q_i^* \equiv q_i(\hat{a}) - \int_{\underline{y}}^{\bar{y}} t_i(y) f_i(y,\hat{a})dy.$$

There are now several ways that (4.2) and (4.3) can be reduced to a system of the form in (2.7). One approach is to first partition $[\underline{y},\bar{y}]$ into a finite number (greater than two) of sets, and then to let $c_1(y),\ldots,c_m(y)$ be unknown simple functions that are constant over each set in this partition. Equations (4.2) and (4.3) can then be regarded as a system of linear equations in the values that these functions assume; different systems can be obtained by varying the choice of the subsets of $[\underline{y},\bar{y}]$ over which each of the functions is constant. The systems that are obtained in this fashion differ from (2.7) only in their right-hand side, which does not play a role in the proof of the existence of solutions in the generic case. An alternate approach is to let each $c_i(y)$ be a polynomial in y of degree $k \geq 2$ with unknown coefficients. Equations (4.2) and (4.3) define a linear system in these coefficients that is of the same form as (2.7). This latter approach can be used to compute smooth solutions to the first order problem. Each of these procedures is illustrated in the example at the end of this section.

Theorem 2 has the following extension to the continuum case: if $F_i(\cdot)$ is negative on $(\underline{y},\bar{y})$, and $s_1(\cdot),\ldots,s_m(\cdot)$ are piecewise C^1 functions with left- and right-hand limits at all points in $[\underline{y},\bar{y}]$ that solve the first order problem, then at least two of these sharing rules are nonincreasing over some subintervals of $[\underline{y},\bar{y}]$. This is shown by a calculation that resembles the proof of Theorem 2. Let $y_0 = \underline{y}$, $y_{n+1} = \bar{y}$, and let $y_1 < y_2 <\ldots< y_n$ denote the points in $[\underline{y},\bar{y}]$ at which the functions $s_1(\cdot),\ldots s_m(\cdot)$ may fail to be C^1. The marginal expected return to the ith partner at the efficient inputs is

$$\begin{aligned}
0 &= \int_{\underline{y}}^{\bar{y}} f_i(y,\hat{a})s_i(y)dy - q_i(\hat{a}) \\
&= \left[\int_{\underline{y}}^{\bar{y}} f_i(y,\hat{a})y\,dy - q_i(\hat{a})\right] - \int_{\underline{y}}^{\bar{y}} f_i(y,\hat{a})s_{-i}(y)dy \\
&= \int_{\underline{y}}^{\bar{y}} f_i(y,\hat{a})s_{-i}(y)dy \\
&= \sum_{j=1}^{n+1} F_i(y,\hat{a})s_{-i}(y)\Big|_{y=y_{j-1}}^{y=y_j} - \int_{\underline{y}}^{\bar{y}} F_i(y,\hat{a})s'_{-i}(y)dy \\
&= \sum_{j=1}^{n} F_i(y_j,\hat{a})\left(\underline{s}_{-i}(y_j) - \bar{s}_{-i}(y_j)\right) - \int_{\underline{y}}^{\bar{y}} F_i(y,\hat{a})s'_{-i}(y)dy. \qquad (4.4)
\end{aligned}$$

In (4.4), $\underline{s}_{-i}(y_j)$ and $\bar{s}_{-i}(y_j)$ are the lower and upper limits of $s_{-i}(\cdot)$ at y_j, respectively. If $s_{-i}(\cdot)$ were an increasing function, then (4.4) clearly could not hold. The proof is then completed with the argument at the end of the proof of Theorem 2.

Finally, we note the following extension of Theorem 3 to the continuum case: if (i) $F(\cdot)$ satisfies stochastic dominance, (ii) there are only two partners, (iii) $dq_i/da_i > 0$ for both values of i, and (iv) there exist levels of output $\underline{y} < y_1 < y_2 < \bar{y}$ at which the hypotheses of Theorem 3 hold, then there exists at least a one dimensional affine space of solutions to the partnership problem. Note that (iii) is simply the standard differential hypothesis to insure that the disutility of each partner's input is a convex function. We consider sharing rules of the form $s_i(y) = y/2 + c_i(y)$, where $c_i(\cdot)$ is a step function whose discontinuities can occur only at y_1 or y_2. As in the above discussion of the first order problem, we shall solve for the values of the functions $c_1(\cdot)$ and $c_2(\cdot)$. With these sharing rules, the expected return to the ith partner given his input a_i and the -ith partner's input $\hat{a}_{-i}$ is

$$\int_{\underline{y}}^{\bar{y}} f(y,a_i,\hat{a}_{-i})s_i(y)dy - Q_i(a_i)$$

$$= \left[\int_{\underline{y}}^{\bar{y}} f(y,a_i,\hat{a}_{-i})(y/2)dy - Q_i(a_i)/2\right] + \tag{4.5}$$

$$\left[\int_{\underline{y}}^{\bar{y}} f(y,a_i,\hat{a}_{-i})c_i(y)dy - Q_i(a_i)/2\right]. \tag{4.6}$$

Since $\hat{a}$ is efficient, (4.5) is maximized at $a_i = \hat{a}_i$. It is therefore sufficient to choose $c_1(\cdot)$, $c_2(\cdot)$ such that (i) for each i, (4.6) is maximized at $a_i = \hat{a}_i$ and (ii) the budget constraint holds, i.e., $c_1(y) + c_2(y) = 0$ for all y in $[\underline{y},\bar{y}]$.

Now let $c_{i,j}$ denote the value of $c_i(\cdot)$ on $[y_{j-1},y_j]$. The first order condition for the maximization of (4.6) at $\hat{a}_i$ is

$$\left(F_i(y_1,\hat{a}),\ F_i(y_2,\hat{a})\right)(c_{i1}-c_{i2},\ c_{i2}-c_{i3}) = q_i(\hat{a}_i)/2. \tag{4.7}$$

When these equations are combined with the budget constraint, the values $c_{i1} - c_{i2}$, $c_{i2} - c_{i3}$ for $i = 1,2$ are uniquely determined; solving, one obtains $c_{11} - c_{12}$, $c_{22} - c_{23} > 0$, and $c_{12} - c_{13}$, $c_{21} - c_{22} < 0$. Together with the assumption that dq_i/da_i is positive, the argument from the proof of Theorem 3 then shows that these values determine a one parameter family of solutions to the partnership problem.

Example 3. We now apply the techniques of this section to a modification of the example in Section 3. As before, let $A_1 = A_2 = [0,1]$, $Q_1(a_1) = a_1^2/2$, $Q_2(a_2) = a_2^2/2$, and $y_1 = 1$, $y_2 = 2$. The range of output levels is now $[0,3]$. The output function $F(\cdot)$ is

$$F(y,a) = \begin{cases} yF(y_1,a) & \text{if} \quad y \leq 1 \\ (2-y)F(y_1,a) + (y-1)F(y_2,a) & \text{if} \quad 1 \leq y \leq 2 \\ (3-y)F(y_2,a) + (y-2) & \text{if} \quad 2 \leq y \leq 3 \end{cases}$$

where (as before)

$$F(y_1,a) = \left[(2+a_1^3-3a_1) + (1-a_2^3)\right]/6$$

$$F(y_2,a) = \left[(3-a_1^3) + (3+a_2^3-3a_2)\right]/6.$$

The expected value of y given the input profile a is

$$E(y|a) = 5/2 - F(y_1,a) - F(y_2,a).$$

Using this formula, it is easy to show that the efficient inputs are $\hat{a}_1 = \hat{a}_2 = 1/2$.

This example satisfies the sufficient conditions for extending Theorem 3 to the continuum case. We now solve the partnership problem with that procedure. When $\hat{a}_1$, $\hat{a}_2 = 1/2$ are substituted into (4.7), we obtain the linear system

$$(-3,-1)\cdot(c_{11}-c_{12},\ c_{12}-c_{13}) = 2$$

$$(-1,-3)\cdot(c_{21}-c_{22},\ c_{22}-c_{23}) = 2.$$

Adding the budget constraint $c_1(y) + c_2(y) = 0$ to this system, one can determine that $c_{11} - c_{12} = c_{22} - c_{23} = -1$ and $c_{12} - c_{13} = c_{21} - c_{22} = 1$. These four values define the following one parameter family of solutions to the first order problem (where the choice of t determines a particular solution):

$$s_1(y) = (y/2+t) + \begin{cases} 0 & \text{if} \quad y < 1 \\ 1 & \text{if} \quad 1 \leqslant y \leqslant 2 \\ 0 & \text{if} \quad 2 \leqslant y \leqslant 3 \end{cases}$$

$$s_2(y) = (y/2-t) + \begin{cases} 0 & \text{if} \quad y < 1 \\ -1 & \text{if} \quad 1 \leqslant y \leqslant 2\ . \\ 0 & \text{if} \quad 2 \leqslant y \leqslant 3 \end{cases} \qquad (4.8)$$

Each partner therefore receives a fixed fraction of the joint output, a share of the minimum output ($y_0 = 0$), plus a bonus or a penalty. Under each of these schemes, the marginal expected utility of the ith partner given the input profile a is $-(a_i-1/2)(a_i+3/2)$. Each of these solutions to the first order problem thus also solves the partnership problem.

We conclude by noting that smooth solutions to the partnership problem can also be computed in this example by using the alternative approach to the first order problem that is described earlier in this section. For i = 1, 2, let $c_i(y) = x_i y^2 + z_i y$. When the first order problem is solved in this form, we find that $x_1 = -1$, $z_1 = 3$, $x_2 = 1$, and $z_2 = -3$. These values determine the one parameter family of sharing rules

$$s_1(y) = -y^2 + 7y/2 + t,$$
$$s_2(y) = y^2 - 5y/2 - t. \tag{4.9}$$

It can be shown that the marginal expected utility of the ith partner given the inputs a is the same under each of these schemes as it is under any of the schemes in (4.8). Each of the schemes in (4.9) is therefore a solution to the partnership problem.

5. The Case of Risk Aversion.

Returning to the case in which the range of output levels is finite, we now describe how the first order problem changes when the partners may be risk averse. Under risk aversion, efficiency cannot be treated separately from Nash implementation; the efficient sharing of risk imposes $n(m-1)$ additional constraints upon the shares and the inputs. In a generic problem, these additional constraints prevent the implementation of efficiency with budget-balancing sharing rules.

For $1 \leqslant i \leqslant m$, the ith partner's utility function $u_i(s_i, a_i)$ is now any

C^1 function such that (i) $\partial u_i/\partial s_i > 0$, and (ii) $\partial u_i/\partial a_i < 0$. For welfare weights $\lambda_1,\ldots,\lambda_m$ (where $\sum_{i=1}^m \lambda_i = 1$) and Lagrangian multipliers $\delta_1,\ldots,\delta_n$, the Lagrangian for maximizing the weighted sum of the expected utilities subject to the budget constraint is

$$\sum_{i=1}^m \lambda_i\left[\sum_{j=1}^n u_i\left(s_i(y_j),a_i\right)f(y_j,a)\right] - \sum_{j=1}^n \delta_j\left[\sum_{i=1}^m s_i(y_j) - y_j\right].$$

Differentiating with respect to δ_j for $1 \leqslant j \leqslant n$ produces the n equations that form the budget constraint. Differentiating with respect to a_k for $1 \leqslant k \leqslant m$ leads to the m equations

$$\sum_{i=1}^m \lambda_i\left[\sum_{j=1}^n u_i\left(s_i(y_j),a_i\right)f_k(y_j,a)\right]$$
$$+ \lambda_k\left[\sum_{j=1}^n \partial u_k/\partial a_k\left(s_k(y_j),a_k\right)f(y_j,a)\right] = 0,$$

which generalize the m efficiency conditions in (2.2) of the risk neutral case. Additionally, nm equations are obtained by differentiating with respect to $s_i(y_j)$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$,

$$\lambda_i \partial u_i/\partial s_i\left(s_i(y_j),a_i\right)f(y_j,a) = \delta_j.$$

Eliminating the multipliers δ_j, these reduce to n(m-1) equations,

$$\lambda_i \partial u_i/\partial s_i\left(s_i(y_j),a_i\right)f(y_j,a) = \lambda_1 \partial u_1/\partial s_1\left(s_1(y_j),a_1\right)f(y_j,a) \tag{5.0}$$

where $2 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. The system of equations (5.0) must hold for optimal risk-sharing between the partners.

Adding the m first order incentive equations, we have (n+2)m equations in (n+2)m − 1 variables.[5] For all problems $(F,u_1,\ldots,u_m)$ in some residual subset of the set of problems that we now consider, this system is unsolvable.[6]

We now summarize the relationship between our partnership model and the principal-agent model.[7] Without discussing the principal-agent model in detail, we draw an analogy between inducing each partner and inducing a single agent to contribute the proper input. When the output is determined by his input, the agent can be given the incentive to choose the efficient input with a budget-balancing payment function; the principal simply pays the agent an appropriate amount if and only if the efficient output is observed, and punishes him with a "small" payment or penalty otherwise. A corresponding result does not hold in the partnership model because of the free rider problem. Similar results do hold for the two models when the output is uncertain. A risk neutral agent can be induced to choose the proper input with the following compensation scheme: the agent receives the entire output, but he must pay a fixed "franchise fee," no matter what he produces. If we ignore the question of whether or not the first order approach is sufficient, then Theorem 1 of this paper is an analogous result; it is only more difficult to solve for the appropriate compensation scheme in the partnership model because there are several partners that must be given an incentive to contribute their proper inputs. A risk averse agent cannot be induced to choose the proper input with a budget-balancing payment function. As in the preceding analysis of the partnership model, the problem of optimally allocating risk between the principal and the agent makes this impossible.

Footnotes

1 Holmstrom [3] considered the case in which the output is uncertain, but he did not discuss the particular problem that is the subject of our paper. For any output function in some specified class and any $\delta > 0$, he proved the existence of sharing rules that (i) sustain efficiency as a Nash equilibrium, and (ii) require an expected deficit or subsidy of no more than δ when the partners choose the efficient inputs. As he noted, however, his sharing rules may require arbitrarily large bonuses or penalties for the partners, depending upon the amount of output and the nature of the output function. It should also be noted that a stronger result is true. It is easy to show that for any choice of the output function, a Groves mechanism exists that (i) sustains efficiency as a Nash equilibrium, and (ii) balances the budget in expected value when the partners choose their efficient inputs.

2 This argument can be formalized by using the Thom Transversality Theorem [2, p. 54] to characterize the input profiles at which this linear system is degenerate.

3 The attentive reader may be concerned about difficulties that could arise when perturbation of a problem in Ψ leads to an abrupt change in the efficient point. Note that there is an open, dense subset Ψ^* of Ψ such that, for each problem in Ψ^*, the efficient point is uniquely determined; on Ψ^*, the efficient point varies continuously with the problem. By carrying out the above analysis over Ψ^*, the difficulties caused by abrupt movement of the efficient point can be avoided.

4 In particular, Theorem 2 and the above discussion of linear sharing rules show that the example of Radner et. al. [6] does not typify repeated partnership games in which the sharing rules may be designed, for the

example rests upon (i) the assumption that there are only two alternative output levels, and (ii) the use of linear sharing rules.

5 The equations consist of n budget equations, m incentive equations, and n(m-1) risk-sharing equations. The variables are the m inputs, the nm shares, and the m - 1 welfare weights.

6 A multijet transversality theorem is needed to formalize this statement, for the equations involve the values of the utility functions and their derivatives at several points (i.e., the different shares). See [2, p. 57] for details. In Theorem 1 of our paper, a generic problem is an element of some open, dense subset of the set of problems; in the risk averse case, a generic problem is in some residual subset. We obtain different results because of the differences between the utility functions in these cases. The characterization in the risk averse case is weaker because the domains of the utility functions in this case are not compact (i.e., the shares are unbounded). In the risk neutral case, the disutility of a partner's input (which is all that we use of his utility function) has a compact domain.

7 See Radner [4] for an overview of the principal-agent model. In our discussion of this model, "efficient" means Pareto optimal, relative to the utilities of the principal and the agent.

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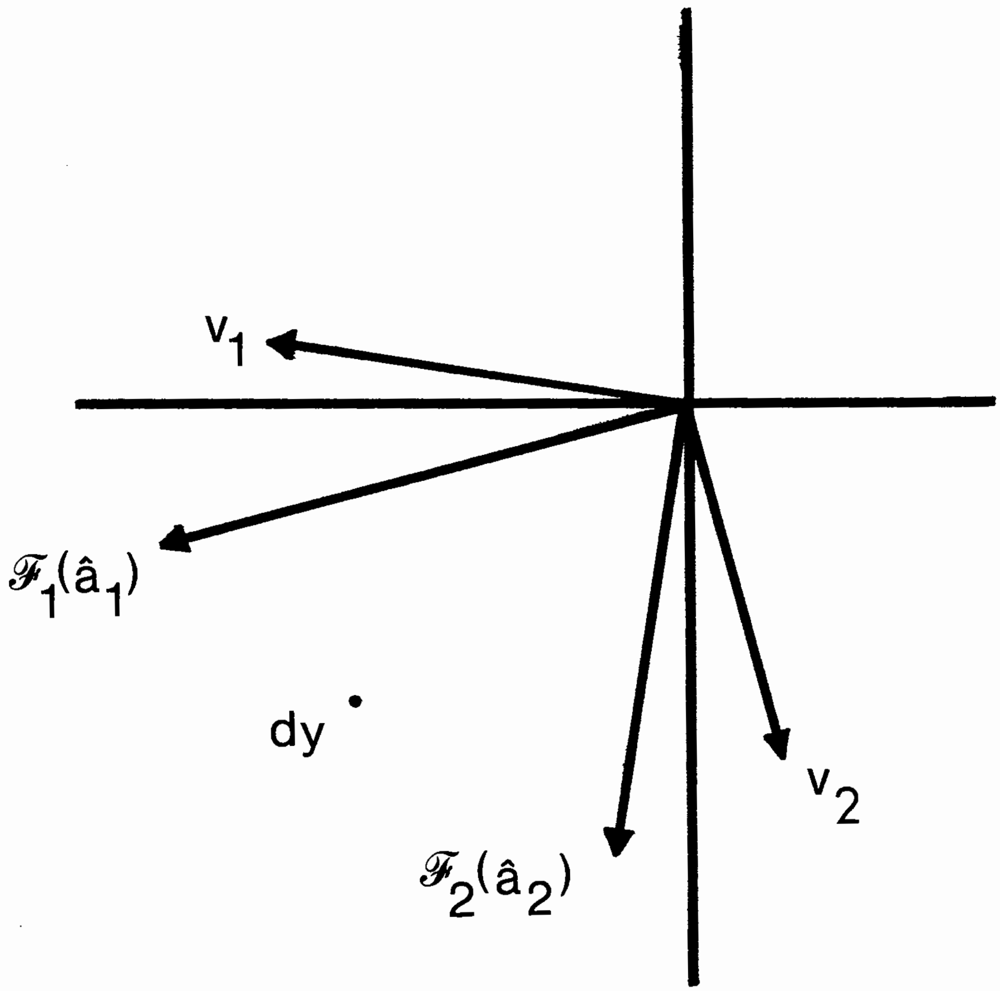
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