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OPTIMAL PRODUCTION GROWTH FOR THE MACHINE LOADING PROBLEM

by

V. Balachandran*

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* Associate Professor, Graduate School of Management, Northwestern University,Evanston, Illinois

ABSTRACT

This paper investigates a production growth logistics system for the machine loading problem (generalized transportation model), with a linear cost structure and minimum levels on total machine hours (resources) and product types (demands). An algorithm is provided for tracing the production growth path of this system, viz. in determining the optimal machine loading schedule of machines for product types, when the volumes of (i) total machine hours and (ii) the total amount of product types are increased either individually for each total or simultaneously for both. Extensions of this methodology, when (i) the costs of production are convex and piecewise linear, and (ii) when the costs are non-convex due to quantity discounts and (iii) when there are upper bounds for productions are also discussed. Finally, a "goal-programming" production growth model where the specified demands are treated as just goals and not as absolute quantities to be satisfied is also considered.

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V. Balachandran
Northwestern University

1. INTRODUCTION

Growth path logistics system and the prespecified market growth rate in the context of a "transportation problem" was analyzed by Srinivasan and Thompson via their rim operator theory [14]. The same has also been discussed utilizing the network formulation and converting into an uncapacitated transfortation problem by Fong and Rao [10]. These are general extensions and characterizations of earlier work on such logistics systems modelled as a "transportation problem" [8,9]. However, nothing has been reported into the literature on such logistics systems for the "machine loading problem" [1,7,10] characterized as a generalized transportation problem or the anlogous "networks with gains" [13]. This paper is expected to fill this gap in considering a logistics system modelled as a "generalized transportation problem" for the machine loading problem where one minimizes the total production costs of a homogeneous good given by different product types from a set of machines. Specifically, we will consider the "growth path" of such a system, i.e. given a planning horizon, how the optimal total cost, the machine hours available treated as sources, product types required considered as demands, vary continuously as a path, when the total volume in the system, both with respect to total machine hours available and total amounts of each product types and their totals increase subject to certain lower bounds on each machine type's available hours and each product type's demand. These patterns of changes are essential for optimal production planning for both short and long term projections of the entire system relative to facilities, financial commitments, growth of the firm, manpower needs, etc.

Such an investigation is needed in the context of production-marketing and logistics system. For example, consider a firm that plans to expand so that it has to ascertain which machine type capacities have to be increased and which additional product types as demands should be sought after. It is reasonable to assume that the firm wishes to maintain at least its current levels of production and demands. The cost of production of a product type from a specified machine involves also other manpower, transportation, promotion and other related costs and the revenue from selling the same. It is assumed that the net costs (or revenues) are <u>proportional</u> to the amounts concerned, to enable us to obtain a net per unit cost (or negative per unit profit) for each machine type-product type combination. For such a case, the growth path will provide the effect on the optimal solution as the total volume of both types mentioned is increased. Further, job shops that are committed, comprising of the machine types involved and the product types produced by these machines with the associated personnel would benefit to decide on the additional workforce required in overtime and hiring or redistribution of work schedule decisions.

2. THE MACHINE LOADING PROBLEM AND AN OPERATOR THEORY FOR OPTIMAL PRODUCTION PATH DECISIONS

The following provides the mathematical formulation of the "lower bounded and constrained total volume machine loading problem."

(1) Let $\mathcal{I} = \{1,2,\ldots,i,\ldots,m\}$ the set of machine types (services) and

(2) $\mathcal{J} = \{1,2,\ldots,j,\ldots,n\}$ the set of product types (demands).

For $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the following are defined

x_{ij} = amount of product type j to be produced utilizing machine type i (≥ 0)

c_{ij} = per unit cost of production of product type j using unit time of machine type i's unit time (≥ 0)

b_j = minimum amount of product j required (≥ 0)

a_i = current availability of machine time units of machine type i (≥ 0)

e_{ij} = machine utilization time of machine type i per unit of product j (> 0)

$S_i(T_j)$= Surplus amount of machine hours available of the type (demanded of product type j) (≥ 0)

$g_i(h_j)$= unit cost of production using one unit time of machine type i (or one unit of product type j) (≥ 0).

With these known values, the following problem (P) is defined.

(3) $$\text{Minimize } Z = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} e_{ij} x_{ij} + \sum_{i \in \mathcal{I}} g_i(a_i+S_i) + \sum_{j \in \mathcal{J}} h_j(b_j+T_j)$$

subject to the following constraints:

(4) $\sum_{j \in \mathcal{J}} e_{ij}\, x_{ij} \leq a_i + S_i$ for $i \in \mathcal{I}$

(5) $\sum_{i \in \mathcal{I}} x_{ij} = b_j + T_j$ for $j \in \mathcal{J}$

(6) $x_{ij} \geq 0,\ S_i \geq 0,\ T_j \geq 0$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$

where x_{ij}, S_i and T_j are the decision variables.

<u>REMARK 1</u>. We can generalize this problem in the framework of network with gains [13] which can also accommodate a coefficient f_{ij} in constraint set (5) for each x_{ij}. For such a case one could define $x'_{ij} = f_{ij}\, x_{ij}$ so that new constants $c'_{ij} = c_{ij}/f_{ij}$ and $e'_{ij} = e_{ij}/f_{ij}$ change this problem to have the same structure as in (3)-(6). Thus without any loss of generality we consider problem P where one always ensured unity as coefficients of x_{ij} in constraint set (5). Further, we may have upperbounds for x_{ij} and the problem can be handled by the method provided by Balachandran and Thompson [1].

The problem P with $S_i = 0$ and $T_j = 0$ may not have a feasible solution. Thus we will introduce a slack column n+1, with $e_{i,n+1} = 1$, $c_{i,n+1} = 0$ for $\forall\ i \in \mathcal{I}$ and a fictitious machine type with abundant availability a_{m+1} and a prohibitive per unit cost of production $c_{m+1,j} = M$ (a high cost) with $e_{m+1,j} = 1$ for $\forall \in \mathcal{J}$. Further let $c_{m+1,n+1} = 0$; $e_{m+1,n+1} = 1$ and <u>no constraint exists for the newly created column n+1</u>. Let us now define the index sets $I = \mathcal{I} \cup \{m+1\}$, $J = \mathcal{J} \cup \{n+1\}$. (One may put g_{m+1} to be M and $h_{n+1} = b_{n+1} = T_{n+1} = 0$ but since that adds only a constant to the objective function (3) these are not specified here.)

Thus the resulting problem will be

(7) $$\text{Min } Z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} g_i (a_i + S_i) + \sum_{j \in J} h_j (b_j + T_j)$$

subject to

(8) $$\sum_{j \in J} e_{ij} x_{ij} = a_i + S_i \qquad \text{for } i \in I$$

(9) $$\sum_{i \in I} x_{ij} = b_j + T_j \qquad \text{for } j \in \mathcal{J}$$

(10) $$x_{ij} \geq 0; \text{ for } i \in I,\ j \in J;\ S_i \geq 0 \text{ for } i \in \mathcal{I};\ T_j \geq 0 \text{ for } j \in \mathcal{J}$$
$$\text{and } S_{m+1} = T_{n+1} = b_{n+1} = h_{n+1} = 0$$

Let

(11) $$K_M = \sum_{i \in I} \left(\sum_{j \in J} e_{ij} x_{ij} \right) \text{ and}$$

(12) $$K_P = \sum_{j \in \mathcal{J}} \left(\sum_{i \in I} x_{ij} \right) .$$

Here in (7)-(11), x_{ij}, S_i and T_j are the decision variables. Constraint (8) for every $i \in I$ states that the total time provided by machines of type i for production of all product types, should equal to the total available time units $a_i + S_i$. Since $S_i \geq 0$ from (10), the total available time is at least equal to the current level of operation a_i (the lower bound at i^{th} row). This type of statement holds for columns $j \in \mathcal{J}'$. (Note we don't have a constraint for column (n+1)). The production system given here has two different total volumes, one K_M the <u>total machine hours</u> available and K_P the <u>total amount of all products</u> required in the system. The objective function minimizes the total cost in the system, (production costs, securing additional time and demands). Though the objective function (7) can be rewritten from equations (8) and (9) as

$$(13) \qquad \text{Min} \quad Z = \sum_{i \in I} \sum_{j \in J} c'_{ij} \, x_{ij}$$

where $\qquad c'_{ij} = c_{ij} + g_i \, e_{ij} + h_j$

the form given in (7) is to be preferred since it is more suitable when the costs of producing from the machines and meeting more demands are not necessarily proportional. In such a situation we represent $g_i(\cdot)$ and $h_j(\cdot)$ as functions of (a_i+S_i) and (b_j+T_j) respectively.

Since in $(m+1)^{th}$ row, $c_{m+1,j}$ = M a high penalty cost and a_{m+1} is a very large availability, one is always assured of a feasible solution to problems (3)-(6). However if $x_{m+1,h} > 0$, this shows there is no feasible solution to the original problem. The condition $x_{m+1,j} \equiv 0$ for $j \in \mathcal{J}$ is necessary and sufficient for the existence of a feasible solution. Further $x_{m+1,n+1}$ is always > 0 (see [1]).

Summing the equations (4) for $i \in I$ and subtracting from (11) we can rewrite equation (11) by

$$(14) \qquad K_M - \sum_{i \in I} c_i = \sum_{i \in I} s_i \, .$$

From (10) where $s_i \geq 0$, it follows that

$$(15) \qquad K_M \geq \sum_{i \in I} a_i = K_a \text{ (say)} \, .$$

Similarly for the columns, from (9), (10) and (12) we have

$$(16) \qquad K_P - \sum_{j \in J} b_j = \sum_{j \in J} T_j \qquad \text{so that}$$

(17) $$K_P \geq \sum_{j \in J} b_j = K_b \quad \text{(say)}$$

(Note $b_{n+1} = T_{n+1} = 0$).

Again feasibility is always assured of the $(m+1)^{th}$ row with large a_{m+1}. However the presence of $x_{m+1,j}$ for any $j \in \mathcal{J}$ is sufficient to show that the original problem has no feasible solution. These positive $x_{m+1,j}$ provide the fact about those demands j which are not satisfied, since they use up the machine hours from a fictitious machine.

We will now transform the problem given by (7)-(12) to a Standard Generalized capacitated trasportation problem [1], with (m+2) rows (machine types) and (n+2) columns (product types). Let us add a large positive constant N to both sides of (8) and (9) and rearrange, to obtain the following:

(18) $$\sum_{j \in J} e_{ij} x_{ij} + (N-S_i) = a_i+N \qquad \text{for } i \in I, \text{ and}$$

(19) $$\sum_{i \in I} x_{ij} + (N-T_j) = b_j+N \qquad \text{for } j \in \mathcal{J}.$$

Similarly (7), (14) and (16) can be arranged respectively as

(20) $$Z = \text{Min} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} - \sum_{i \in I} g_i (N-S_i) - \sum_{j \in J} h_i (N-T_j)$$
$$+ \sum_{i \in I} g_i (N+a_i) + \sum_{j \in J} h_j (N+b_j)$$

(21) $$\sum_{i \in I} (N-S_i) = (m+1)N + \sum_{i \in I} a_i - K_M \quad \text{and}$$

(22) $$\sum_{j \in J} (N-T_j) = (n+1)N + \sum_{j \in J} b_j - K_P$$

Let the following definitions be made:

(23) $x_{i,n+2} = N-S_i$ $e_{i,n+2} = 1$ $a_i' = a_i + N$ and $c_{i,n+2} = - g_i$ for $i \in I$

(24) $x_{m+2,j} = N-T_j$; $e_{m+2,j} = 1$ $b_j' = b_j + N$ and $c_{m+2,j} = - h_j$ for $j \in J$

(25) $a_{m+2} = (n+1)N + \sum_{j \in J} b_j - K_P$; $b_{n+2} = (m+1)N + \sum_{i \in I} a_i - K_M$

(26) $c_{m+2,n+2} = M$ (a large positive number)

(27) $Z_o = \sum_{i \in I} g_i(N+a_i) + \sum_{j \in J} h_j(N+n_j) =$ a constant

(28) $I' = I \cup \{m+2\}$, $J' = J \cup \{n+2\}$,

and $\mathcal{J}' = \mathcal{J} \cup \{n+2\}$.

With these definitions given by (23)-(28),(20)-(22), (18), (19) and (10) become

(29) Minimize $Z = \sum_{i \in I'} \sum_{j \in J'} e_{ij}\, x_{ij} + Z_o$

Subject to the following constraints:

(30) $\sum_{j \in J'} e_{ij}\, x_{ij} = a_i'$ for $i \in I'$

(31) $\sum_{i \in I'} x_{ij} = b_j'$ for $j \in \mathcal{J}'$

(32) $x_{ij} \geq 0$ for $i \in I'$, and $j \in J'$

(33) $x_{i,n+2} \leq N$ for $i \in I$

(34) $x_{m+2,j} \leq N$ for $j \in J$

REMARK 2. Since M is assumed to be large positive number, $c_{m+2,n+2} = M \Rightarrow x_{m+2,n+2} = 0$ in any optimal solution. Further the constraints $x_{m+2,j} \geq 0$ and $x_{i,n+2} \geq 0 \Rightarrow S_i \leq N$ and $T_j \leq N$ respectively which are always true by our choice of N.

Now we can think of two possible growths in the system, viz. when K_p the total volume of products increase from K_p to $K_p + \delta$. From (23)-(27), this is equivalent to studying the effects of changing a_{m+2} to $a_{m+2} - \delta$ and changing b_{n+2} to $b_{n+2} - \delta$. Since we know the current production schedule has solved the problem where all $S_i = 0$ and $T_j = 0$, we have the optimal solution x_{ij} and Z with $K_p = K_b$ and $K_M = K_a$. Then the optimal solution for any other K_M or K_p can be obtained via the "Rim Operator" theory developed by Balachandran and Thompson [1,2,3,4]. Specifically changes in K_M, starting with K_a, can be completely analyzed (since $K_M = K_a + \delta$) by utilizing the cell rim operator $\delta R^-_{m+2,n+1}$ where $\delta = K_m - K_a$. This cell rim operator provides the optimum solution of problem P^-, where the entire data of P are used except $a^-_{m+2} = a_{m+2} - \delta$, from the known optimal solution of P. (Essentially a parametric analysis when just the $(m+2)^{th}$ row value is changed from a_{m+2} to $a_{m+2} - \delta$.) Similarly for changes in K_p starting with K_b can be completely analyzed with the known optimal solution of K_b since $K_p = K_b + \delta$, by utilizing the cell rim operator $\delta R^-_{m+1,n+2}$ where, now, $\delta = K_p - K_b$. (The $n+2^{th}$ column total, changes from a_{n+2} to $a_{n+2} - \delta$.) In our earlier work on "Operator theory" [2,3,4] we have provided constructive algorithms and solution procedures for all rim operators for <u>all</u> values of $0 \leq \delta \leq \infty$. Further the simultaneous changes of both K_M and K_p also can be analyzed by the cell rim operator $\delta R^-_{m+2,n+2}$.

To start with we have an optimal solution to the "original" problem where $\sum_{i\in I} S_i = 0$ and $\sum_{j\in J} T_j = 0$ (in other words, for $K_M = K_a$ and $K_P = K_b$). Let us call this as the "original optimimum solution" to proboem (29)-(34). From (23) and (24) we know that $x_{i,n+2} = N$ for $i \in I$ and $x_{m+2,j} = N$ for $j \in J$ and $x_{m+2,n+2} = 0$ from Remark 2. Substituting these values in (29)-(34) where $K_M = K_a$ and $K_P = K_b$ we have

$$(35) \qquad \text{Min} \quad Z = \sum_{i\in I} \sum_{j\in J} c_{ij}\, x_{ij} + \underbrace{[Z_o - N(\sum_{i\in I} g_i + \sum_{j\in J} h_j)]}_{\text{constant}}$$

subject to

$$(36) \qquad \sum_{j\in J} e_{ij}\, x_{ij} = a_i \qquad \text{for } i \in I$$

$$(37) \qquad \sum_{i\in I} x_{ij} = b_j \qquad \text{for } j \in \mathcal{J} \text{ and}$$

$$(38) \qquad x_{ij} \geq 0 \qquad \text{for } i \in I \text{ and } j \in J.$$

The problem (35)-(38) is a "generalized transportation problem" and can be solved by four index algorithm [5] and the code developed by Balachandran and Thompson. (There are other procedures also available, e.g. see Balas [7], Glover and Klingman [12].) This "original optimum solution" for the problem (35)-(38) which is uncapacitated, together with the new values for $x_{i,n+2}$ and $x_{m+2,j}$ provide an optimum solution to (29)-(34). The above set of solutions for x_{ij}, $i \in I'$ and $j \in J'$ will be called hereafter as a "basic optimum solution." The following theorem provides a constructive procedure for generating such a "basic optimum solution," from the "original optimum solution" for problem (29)-(34).

THEOREM 1: Let B denote the optimum basis for (35)-(38) with u_i and v_j as the dual variables (see for duals [1,2]). Let the indices r, s be defined such that

$$(39) \qquad v_{n+2} = -(u_r+g_r) = \max_{i \in I} [-(u_i+g_i)] \text{ and}$$

$$(40) \qquad u_{m+2} = -(v_s+h_s) = \max_{j \in J} [-(v_j+h_j)].$$

Then $B' = B + \{r,n+2)\} + \{(m+2,s)\}$ is an optimal basis to the problem (29)-(34) with the dual solutions u_i for $i \in I'$ and v_j for $j \in \mathcal{J}'$.

PROOF: First, let us prove that B' as given above is a basis. Since B is a basis for the problem (35)-(38) with (m+1) rows and (n+1) columns, this will be a "one-forest" consisting of (m+n+1) cells and mutually disconnected "one-trees" [1, 7]. Each row and column contains at least one cell. Since (r,n+2) and (m+2,s) are cells in B', they provide one cell in each $(m+2)^{th}$ row and $(n+2)^{th}$ column. Since cell (r,n+2) is unique in column (n+2) and cell (m+2,s) is unique in row (m+2) these cells, when they are joined as arcs in the one trees where r^{th} row is present and s^{th} column is present respectively, will still preserve the one tree property. Further the mutual disconnectedness existed in the "one-forest" of the original B is also preserved. Thus B' also possess the one-forest. Hence B' is a basis. If $(i,j) \in B'$, then $x_{ij} = 0$ or take its upper bounds. Let LB' and UB' be the set of cells where $x_{ij} = 0$

and those cells that take the upper bounds respectively. Let the basis structure be defined as (B', LB', UB') for the problem.

Now to prove that B' is optimal, we should show that

(41) $\quad (i,j) \in B' \Rightarrow c_{ij} = e_{ij} u_i + v_j$

(42) $\quad (i,j) \in LB' \Rightarrow c_{ij} \geq e_{ij} u_i + v_j$ and

(43) $\quad (i,j) \in UB' \Rightarrow c_{ij} \leq e_{ij} u_j + v_j$.

Since B is optimal to (35)-(38), (41) holds for the cells in B. For the cell $(r,n+2)$, $c_{r,n+2} = - g_r$ from (23). Further from (39) $- g_r = u_r + v_{n+2}$ so that $c_{r,n+2} = u_r + v_{n+2}$. Since $e_{r,n+2} \equiv 1$ from (23),

$$c_{r,n+2} = e_{r,n+2} u_r + v_{n+2}$$

thus satisfying (41). A similar argument holds for the cell $(m+2,s)$. For the cells $(i,j) \in LB$ (42) holds since B is optimal. We constructed the solution to (29)-(34) so that $x_{i,n+2} = N$ for $i \in I$ and $x_{m+2,j} = N$ for $j \in J$. Hence these cells are at their upper bounds, or $\in UB'$. For $(i,n+2) \in UB'$, with $i \in I$, from (39) we have

$$- (u_i+g_i) \leq v_{n+2} \qquad \text{so that}$$

$$c_{i,n+2} = - g_i \leq u_i + v_{n+2} \qquad (\text{since } e_{i,n+2} = 1)$$

thus satisfying (43) for such $(i,j) \in UB'$. Similar is true for cells $(m+2,j)$ for $j \in J$. Since $c_{m+2,n+2} = M$ from (26), (42) trivially holds $(i,j) \in LB$ satisfies (42), these cells together with cell $(m+2,n+2)$ make LB' and thus (42) holds for all $(i,j) \in LB'$.

Q.E.D.

These results are summarized as an algorithm given below (Algorithm A1). First we find an "original optimal solution" using our code [5] where $K_M = K_a$ and $K_P = K_b$. Later the following four options are investigated.

(i) $K_a \leq K_M \leq K_a + N$

(ii) $K_b \leq K_P \leq K_b + N$

(iii) Both variations in K_M and K_P where if
$K_M = K_a + \delta$; $K_P = K_b - \delta$ and if
$K_P = K_b + \delta$; $K_M = K_a - \delta$ and

(iv) K_M and K_P both increase at the rate of $\delta\, \alpha_M$ and $\delta\, \alpha_P$.

ALGORITHM A1. To find the production growth by solving for optimal solution of (29)-(34) for all options

(i) $K_a \leq K_M \leq K_a + N$

(ii) $K_b \leq K_P \leq K_b + N$

(iii) Changes in K_M and K_P simultaneously within the bounds

(iv) given above in (i) and (ii) respectively.

Step 1: Find the "original optimal solution"

Solve the problem given by (35)-(38) by using the "generalized stepping stone method" [1,9] (see Algorithm A4 of [1].) Let Z denote the optimum value with basis as B. Those which are non-basic, i.e. $x_{ij} = 0$, constitute the set of lower bounds called LB say. Utilizying this optimal solution to (29)-(34) as the <u>same</u> for all $i \in I$ and $j \in J$. Define $x_{i,n+2} = N$ for $i \in I$; $x_{m+2,j} = N$ for $j \in J$ and $x_{m+2,n+2} = 0$

Evaluate the duals v_{n+2} and u_{m+2} using (39) and (40) after identifying the indices r and s. Define the new basis as $B' = B + \{r,n+2)\} + \{m+2,s)\}$ as the basis for (29)-(34).

OPTION 1: Production growth when extra machine hour capacities are available:

Application of Rim Operator $\delta R^{-}_{m+2,n+1}$ for any $0 \leq \delta \leq N$.

Step 2: Let the iteration count $K = 1$; $X_1 = X$, $B_1 = B'$; $Z_1 = Z$, $u_{i,1} = u_i$ for $i \in I'$, $v_{j,1} = v_j$ for $j \in J'$.

Step 3: With the solution of Step 2, apply the cell rim operator $\delta R^{-}_{m+2,n+1}$ as given by Algorithms A8, A15, of the rim operator [2,4] and identify the discrete set of "corner points," $0 = \mu_o, \mu_1, \mu_2$ $\mu_i, \ldots, \mu_t$ such that

(i) for every interval $\sum_{i=0}^{j-1} \mu_i < \delta \leq \sum_{i=0}^{j} \mu_i$ the Basis B_j doesn't change, for $j = 0$ to t and

(ii) the last interval contains N, such that

$$\sum_{i=0}^{t-1} \mu_i < N \leq \sum_{i=0}^{t} \mu_i,$$ so that

the optimal solution Z_j and the corresponding basis B_j giving $X = \{x_{ij}\}$ are completely determined for $0 \leq \delta \leq N$. This is equivalent to analyzing the production growth K_M when δ moves from K_a to $K_a + N$. The exact calculations procedures are given in Algorithms A8, A15 of [2,4]. (A detailed algorithm for the transportion type problem is given in [14].)

OPTION 2: Production Growth when extra demands for products are increasing from the current level K_b to $K_b + N$

Application of cell Rim Operator $\delta\, R^{-}_{m+1,n\_2}$ for any $0 \leq \underline{\delta}$ N.

Steps (2) and (3) are similar to those of option (1) except we apply cell rim operator $\delta\, R^{-}_{m+1,n+2}$ and get a new set of "corner points" μ_i. This growth path yields the study of K_P when δ moves from K_b to $K_b + N$.

OPTION 3: Simultaneous variation in K_M and K_P such that

(i) when $K_M = K_a + \delta$, $K_P = K_P - \delta$ or

(ii) when $K_P = K_P + \delta$; $K_M = K_a - \delta$.

Application of cell rim operator $\delta\, R^{\mp}_{m+2,n+2}$ for any $0 \leq \delta \leq N$, - or + as (i) or (ii) is required.

Again Steps (2) and (3) are similar to those of option (i) except we now apply cell rim operator $\delta\, R^{\mp}_{m+2,n+2}$ and obtain a new set of "corner points" and their associated Basis B_i', Z value and solution $X = \{x_{ij}\}$ depending upon whether we want to analyze (i) or (ii) as the case may be so that

(i) when $K_a \leq K_M \leq K_a + N$, then $K_b \geq K_P \geq K_b - N$ or

(ii) when $K_b \leq K_P \leq K_b + N$, then $K_a \geq K_M \geq K_a - N$.

OPTION 4: Simultaneous increase in K_M and K_P in the directions (Steps) of α_M and α_P such that $K_M = K_a + \alpha_M\delta$ and $K_P = K_b + \alpha_P\delta$.

Here we increase both K_a and K_P so that there is growth on both volumes in the system (viz. in machine capacity as well as required demands). This is achieved by applying area rim operator

$\delta\ R^A$ such that the $(m+2)^{th}$ row total is <u>decreased</u> by $\delta\alpha_M$ <u>and</u> $(n+2)^{th}$ column total is <u>decreased</u> by $\delta\alpha_P$. This can be achieved by using algorithms A8, A15 of [2,4]. Once again Steps (2) and (3) are the same except this area rim operator application and one can get the "corner points" μ_i's such that the basis B_i, associated Z value and solution $X = \{x_{ij}\}$ can be obtained systematically, such that, the simultaneous increase of

$$K_a \leq K_M \leq K_a + N \quad \text{and}$$

$$K_b \leq K_P \leq K_b + N$$

can be investigated via area rim operators [2].

<u>REMARK 3</u>: It is to be noted that each cell rim operator is independent (see [2]), so that for Option 4, one could apply Option 1 and to the resultant problem can apply Option 2, or vice versa, so that results for simultaneous growths can be obtained. This avoids area rim operator application. Thus a rectangular array of "corner points" can be obtained. For instance the first interval $\mu_o < \delta \leq \mu_1$ is obtained utilizing $\delta\ R^-_{m+2,n+1}$ cell rim operator giving the new basis, Z value and solution X. To this resultant problem one can apply Option (2) in its entirety so that $K_b \leq K_P \leq K_P + N$ is investigated for $K_a \leq K_M \leq K_a + \mu_1$. Then we go to the next interval for $\delta\ R^-_{m+2,n+1}$ such that analysis for $K_a + \mu_1 \leq K_M \leq K_a + \mu_1 + \mu_2$ is obtained and the entire Option (2) is once again obtained and so on until we cover the entire range of $K_a \leq K_M \leq K_a + N$.

REMARK 4: For any K_M (or K_P) such that $K_a \leq K_M \leq K_a + N$ we know the value of $\delta = K_M - K_a$ (or $K_P - K_b$) and an index t, such that

(44) $$\sum_{i=0}^{t-1} \mu_i < \delta \leq \sum_{i=0}^{t} \mu_i \quad \text{and}$$

(45) $$\lambda = \delta - \sum_{i=0}^{t-1} \mu_i \ .$$

The rim operator Algorithm A8 and A15 [2,4] provides a procedure of updating x_{ij} for the cells in the two-tree such that

(46) $$x_{ij}\ (\text{new}) = x_{ij,t} + m_{ij,t}\ \lambda$$

where $x_{ij,t}$ are the solution $X = \{x_{ij}\}$ at the t^{th} interval, $m_{ij,t}$ the multipliers obtained from Algorithm A8, A15 [2,4] which are non-zero of the cells in the two-tree and λ defined in (45). The associated total cost Z will be

(47) $$Z\ (\text{new}) = Z_t - \lambda\ (u_{m+2,t} + v_{n+1,t})$$

where $u_{i,t}$ and $v_{j,t}$ are the duals obtained for i^{th} row, and j^{th} column for the t^{th} interval. (Note that $e_{m+2,j} \equiv 1$ for $(m+2)^{th}$ row and $v_{n+1,t} \equiv 0$ for all t since there is no constraint in $(n+1)^{th}$ column.)

The recurrence relation for each interval K will be

(48) $$Z_{k+1} = Z_k - \mu_k\ (u_{m+2,k} + v_{n+1,k})$$

$$(\text{note, } v_{n+1,k} \equiv 0)$$

(49) $$x_{ij,k+1} = x_{ij,k} + m_{ij,k}\ \mu_k \ \text{and}$$

(50) $$u_{i,k+1} = u_{i,k} + m_{i,k}\ \Delta$$

$$v_{j,k+1} = v_{j,k} + n_{j,k}\ \Delta$$

where Λ is found in Algorithm A15, [4] and m_i, n_j are the multipliers for the duals (see Algorithm A9 of [2]).

Thus the equations (48)-(50) provide an updating recurrence relation for $(K+1)^{th}$ interval where K^{th} interval values for Z, X and u, v are known. We know the first interval value i.e., when $K = 0$.

All these discussions hold for every option.

From (47) and (48) it follows that the cost Z for any $0 \leq \delta \leq N$ is given by

$$(51) \qquad Z = Z_1 - \sum_{k=1}^{t-1} [\mu_k (u_{m+2,k} + v_{n+1,k})]$$

$$- \lambda (u_{m+2,t} + v_{n+1,t})$$

where t and λ are given by (44) and (45). Thus the rate at which the cost Z, increases when δ is increased is given by $- u_{m+2,k} - v_{n+1,k}$. This marginal cost is thus the negative sum of the duals for $(m+2)^{th}$ row and $(n+1)^{th}$ column. We also note that $u_{m+2,k} \leq 0$ and $v_{n+1,k} \equiv 0$ for the generalized transportation problem (see [1]). Thus $- u_{m+2,k} \geq 0$ and so u is a <u>non-decreasing</u> function of K for the operator $\delta R^-_{m+2,n+1}$ so that the marginal cost is non-decreasing in δ, and thus also in K_M since $K_M = K_a + \delta$.

Now, in the case of Option 2 we go to the operator $\delta R^-_{m+1,n+2}$ and equation (47)-(50) could be modified by putting m+1 for $m+2^{th}$ row and n+2 for $n+1^{th}$ column. Now since the "absorbing cell" (m+1,n+1) is always in the basis [1], $u_{m+1} \equiv 0$ and all $v_{j,t} \geq 0$. Thus, the marginal cost will be a <u>non-increasing</u> function of δ since $- v_{n+2,k} \leq 0$ for the operator $\delta R^-_{m+1,n+2}$. Thus it is non-decreasing in K_p since $K_p = K_b + \delta$.

Similar discussions can be given for Options (3) and (4) depending upon the corresponding rim operator's properties. These results lead to the following theorem.

THEOREM 2: The optimal values for the variables $x_{i,n+2}$ for $i \in I$ and $x_{m+2,j}$ for $j \in J$ as determined by Algorithm A1 are non-decreasing functions of $\delta = K_M - K_a$ for Option (1) and non-increasing functions of $\delta = K_P - K_b$ for Option (2).

PROOF: It follows from (44) and (45) that the proof of this theorem will be complete, if we show that it holds for

$$(52) \qquad \sum_{k=0}^{t'-1} \mu_k \leq \delta \leq \sum_{k=0}^{t'} \mu_k \ .$$

With λ defined in (45) for any t', $0 \leq t' \leq t$ that it holds for $0 \leq \lambda \leq \mu_t$. Since we know that $x_{ij,t+1}$ as defined in (49) is the same as that given by (46) for $\lambda = \mu_t$.

From (46), x_{ij} increases for the cells such that $m_{ij,t}$ are positive. Of course we maintain always $x_{m+2,n+2} = 0$ due to huge cost. Thus the proof will depend on showing that no cell $(i,n+2)$ for $i \in I$ or $(m+2,j)$ for $j \in J$ has a positive m_{ij}. To prove this assume the contrary that $m_{r,n+2} > 0$ where $r \in I$. Since the $r,n+2$ "two-tree" associated with the rim operator [2] has exactly two cells in column $(n+2)$, which are $(r,n+2)$ and $(m+2,n+2)$, these are the adjacent cells in the "two-tree" and since m_{ij}'s are alternately positive and negative in the adjacent cells in the two-tree both these specified cells cannot be positive. Since cell $(m+2,n+2)$ has positive multipliers m_{ij} we know $(r,n+2)$ should have a negative multiplier. A similar argument holds for $(m+2,s)$ where $s \in \mathcal{J}$. Thus the theorem holds - similar arguments hold for Option (2) as well.

3. EXTENSIONS AND GENERALIZATIONS

3.1 Capacitated Growth Paths

In certain machine loading problems certain cells x_{ij} for $i \in I$ and $j \in J$ are constrained by certain maximum capacities (upper bounds) U_{ij}, so that $0 \leq x_{ij} < U_{ij}$. In these cases, since our original Rim Operator theory was developed for the upper bounded generalized transportation problem [1,2,4], Algorithm A1 and Theorems 1 and 2 will be still true, though the Algorithm A1 has to be slightly modified to take care of U_{ij}'s.

3.2. Convex Costs of Production

The theory discussed in Section (2) can be used when the costs of production $g(S_i)$ and $h(T_j)$ are general convex functions that can be approximated by a convex piecewise linear functions. Consider, for example, Option (1) only and only one row total a_1. If $a_1 = 100$ and per unit cost of adding a_1 are as follows

$$
\begin{aligned}
&\$5/\text{unit for} \quad 0 \leq S_1 \leq 5 \\
&\$6/\text{unit for} \quad 5 < S_1 \leq 10 \\
&\$7/\text{unit for} \quad S_1 > 10 .
\end{aligned}
$$

Then we create 3 fictitious machine types for the first, say $t = 1,2,3$; with the same c_{ij}'s for all $j \in J$, and $t = 1,2,3$, but with $c_{1,n+2} = -5$, $c_{2,n+2} = -6$ and $c_{3,n+3} = -7$. Let $S_{1,1}$, $S_{1,2}$ and $S_{1,3}$ denote the surplus machine hours of type 1 in the ranges of $0 \leq S_{11} \leq 5$, $5 < S_{12} \leq 10$, $10 < S_{13}$ so that $S_1 = S_{1,1} + S_{1,2} + S_{1,3}$. Thus we define $x_{1,n+2} = 5 - S_{1,1}$; $x_{2,n+2} = (10-5) - S_{1,2}$ and $x_{3,n+2} = N - S_{1,3}$ to get upper bounded constraints (viz. $x_{1,n+2} \leq 5$; $x_{2,n+2} \leq 5$ and $x_{3,n+2} \leq N$). Further, due to convexity, the optimum solution forces, for example $S_{1,2} > 0$ (due to $x_{2,n+2} < 5$) if and only if $S_{11} = 5$ (i.e. $x_{1,n+2} = 0$). Similarly $S_{1,3} > 0$ (i.e. $x_{3,n+2} < N$) if and only if $S_{11} = 5$ and $S_{12} = 5$. (One can use cost operators [ ] for such an analysis instead of increasing the size of the problem.)

Further in the "Transportation type problems with Quantity discounts" Balachandran and Perry [6] have provided an algorithm for an objective function which is neither convex nor concave. Thus even if $g(S_i)$ and $h(T_j)$ are non-convex (non-concave) then the above procedure could be utilized for this growth model.

Again if, the growth rates for availability of machines or the product's demands increase at prespecified rates, say α_i for S_i and/or β_j for T_j the model can be reformulated in this general frame work with new objective function and right-hand sides for $(m+2)^{th}$ row and/or $(n+2)^{th}$ column approximately modified. (Details similar to this are given in [14].)

3.3. Goal Programming Growth

It was imperative in problem (3)-(6) that the minimum demands b_j and availabilities a_i are necessarily met. On the contrary, it could be treated just as "goals" or as "expected values." Thus one may allow a <u>deficit</u> or surplus in each row or column which are non-negative. Let S_i^+ (S_i^-) and T_j^+ (T_j^-) represent the surplus (deficit) at row i and column j respectively so that $S^+(S^-)$ and $T^+(T^-)$ be their respective totals. Then we could assign a per unit reward (penalty) for the surplus (deficit) yielding the following "goal" programming [8] problem.

(53) $$\text{Min } Z = \sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij} + \sum_{i\in I} g_i (a_i + S_i^+ - S_i^-)$$

$$+ \sum_{j\in J} h_j (b_j + T_j^+ - T_j^-) + \sum_{i\in I} (s_i^- S_i^- - s_i^+ S_i^+)$$

$$+ \sum_{j\in J} (t_j^- T_j^- - t_j^+ T_j^+) - s^+ S^+ + s^- S^- - t^+ T^+ + t^- T^-$$

(where the penalty (reward) is at the rate of $t^-(t^+)$ for the <u>total</u> deficit which include priorities. Similary penalties (rewards) s^-, s^+, t_j^-, t_j^+, s_i^- and s_i^+ are defined).

Subject to the following constraints:

(54) $$\sum_{j\in J} e_{ij} x_{ij} = a_i + S_j^+ - S_j^- \quad \text{for } i \in I$$

(55) $$\sum_{i\in I} x_{ij} = b_j + T_j^+ - T_j^- \quad \text{for } j \in \mathcal{J}$$

(56) $$\sum_{i\in \mathcal{I}} S_i^+ = S^+$$

(57) $$\sum_{i\in \mathcal{I}} S_i^- = S^-$$

(58) $$\sum_{j\in \mathcal{J}} T_j^+ = T^+$$

(59) $$\sum_{j\in \mathcal{J}} T_j^- = T^- \quad \text{and}$$

(60) $$x_{ij},\ S_i^+,\ S_i^-,\ T_j^+\ \ T_j^- \geq 0 \quad \text{for } i \in I \text{ and } j \in J$$

(From (56)-(60), we have S^+, S^-, T^+, $T^- \geq 0$.)

This problem (53)-(60) can be converted to the generalized capacitated Transportation Problem due to the following.

Let $g_i^+ = g_i - s_i^+$; $g_i^- = g_i + s_i^-$

$h_j^+ = h_j - t_j^+$; $h_j^- = h_j + t_j^-$ and

$Z_o = \sum_{i \in I} g_i\, a_i + \sum_{j \in J} h_j\, b_j$ to get (53) as

$$\text{(61)} \qquad \text{Min } Z = Z_o + \sum_{i \in I} \sum_{j \in J} c_{ij}\, x_{ij} + \sum_{i \in I} (g_i^+ S_i^+ + g_i^- S_i^-)$$

$$+ \sum_{j \in J} (h_j^+ T_j^+ + h_j^- T_j^-) - s^+ S^+ + s^- S^- - t^+ T^+ + t^- T^-$$

Add N to both sides of (54) and (55) to yield

$$\text{(62)} \qquad \sum_{j \in J} e_{ij}\, x_{ij} + (N - S_i^+) + S_i^- = a_i + N \quad \text{for } i \in I$$

$$\text{(63)} \qquad \sum_{i \in I} x_{ij} + (T - T_j^+) + T_j^- = b_j + N \quad \text{for } j \in J$$

Again (56) and (58) can be written as

$$\text{(64)} \qquad \sum_{i \in \mathcal{I}} (N - S_i^+) + S^+ = mN \text{ and}$$

$$\text{(65)} \qquad \sum_{j \in \mathcal{J}} (N - T_j^+) + T^+ = nN\ .$$

Further (57) and (59) can be viewed as

$$\text{(66)} \qquad \sum_{i \in \mathcal{I}} S_i^- + (N - S^-) = N \text{ and}$$

$$\text{(67)} \qquad \sum_{j \in \mathcal{J}} T_j^- + (N - T^-) = N$$

Recalling K_M from (11) and K_P from (12) (the total volumes) we add (11), (64) and (65) and subtract from it the <u>sum</u> of (62) over i, to get

(68) $$S^{+} + (N-S^{-}) = K_M + N - \sum_{i \in I} a_i$$

Similarly we get for columns,

(69) $$T^{+} + (N-T^{-}) = K_P + N - \sum_{j \in \mathcal{J}} b_j$$

Let $Z_o' = Z_o + \sum_{i \in I} g_i^{+} N + \sum_{j \in J} h_j^{+} N + s^{-}N + t^{-}N$; (60) becomes

(70) $$\text{Min } Z = Z_o' + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} - \sum_{i \in I} g_i^{+} (N-S_i^{+})$$

$$+ \sum_{i \in I} g_i^{-} S_i^{-} - \sum_{j \in J} h_j^{+} (N-T_j^{+}) + \sum_{j \in J} h_j^{-} T_j^{-}$$

$$- s^{+} S^{+} - s^{-} (N-S^{-}) - t^{+} T^{+} - t^{-} (N-T^{-}).$$

Since S_i^{+}, S^{-}, T_j^{+}, T^{-} are defined to be ≥ 0 we have

(71) $$(N-S_i^{+}) \leq N \text{ for } i \in I, \; (N-T_j^{+}) \leq N \text{ for } j \in J$$

$$(N-S^{-}) \leq N \quad \text{and} \quad (N-T^{-}) \leq N.$$

Thus (62) to (70) yield a (m+4) row (n+4) column capacitated generalized transportation problem as given in Figure 1. Again Algorithm 1 with very minor modifications can be used after solving with $K_M = K_a$ and $K_P = K_b$ and choosing options of relevant interest and applying proper cell rim operators $\delta\, R^{+}_{m+4,n+1}$ or $\delta\, R^{+}_{m+1,n+4}$ as the case may be.

FIGURE 1

Mch. types → / Product types ↓	1 2 ··· j --- n	Slack Column n+1	n+2	n+3	n+4	Machine hours Available	Equation #
1 2 ⋮ i ⋮ m	$\begin{bmatrix} x_{ij} \\ c_{ij} \\ e_{ij} \end{bmatrix}$	$x_{i,n+1}$ 0 1	* $\begin{bmatrix} N-S_i^+ \\ -g_i^+ \\ 1 \end{bmatrix}$	$\begin{bmatrix} S_i^- \\ g_i^- \\ 1 \end{bmatrix}$		a_1+N --- a_i+N --- a_m+N	(62)
m+1 Fict. Mchne.	$\begin{bmatrix} X_{m+1,j} \\ M \\ 1 \end{bmatrix}$					N+N	
m+2	* $\begin{bmatrix} N-T_j^+ \\ -h_j^+ \\ 1 \end{bmatrix}$				$\begin{bmatrix} T^+ \\ -t^+ \\ 1 \end{bmatrix}$	nN	(65)
m+3	$\begin{bmatrix} T_j^- \\ h_j^- \\ 1 \end{bmatrix}$				* $\begin{bmatrix} N-T \\ -t^- \\ 1 \end{bmatrix}$	N	(67)
n+4			$\begin{bmatrix} S^+ \\ -s^+ \\ 1 \end{bmatrix}$	* $\begin{bmatrix} N-S \\ -s^- \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	K_M+N $-\Sigma\, a_i$	(68)
Demands	$b_1+N, \ldots, b_n+N$	No Constraint	mN	N	K_p+N $-\Sigma\, b_j$		
Equa. #	(63)		(64)	(65)	(69)		

Legend $\begin{bmatrix} \text{Var. Name} \\ \text{Unit Cost} \\ \text{Multiplier} \end{bmatrix}$

Cross hatched blocks denote that corresponding X_{ij} should be made zero by setting c_{ij} = M (a large penalty)

* denotes that the upper bounds for these X_{ij} are N

REFERENCES

[1] Balachandran, V. and G. L.Thompson, "An Operator Theory of Parametric Programming for the Generalized Transportation Problem - I, Basic Theory," Management Science Research Report 293, Carnegie-Mellon University, Pittsburgh, Pa.

[2] ______________________________, "An Operator Theory of Parametric Programming for the Generalized Transportation Problem - II, Rim, Cost and Bound Operators," Management Science Research Report 294, Carnegie-Mellon University, Pittsburgh, Pa.

[3] ______________________________, "An Operator Theory of Parametric Programming for the Generalized Transportation Problem - III, Weight Operators," Management Science Research Report 297, Carnegie-Mellon University, Pittsburgh, Pa.

[4] ______________________________, "An Operator Theory of Parametric Programming for the Generalized Transportation Problem - IV, Global Operators," Management Science Research Report 305, Carnegie-Mellon University, Pittsburgh, Pa.

[5] ______________________________, "Four index Algorithm for the Generalized Transportation Problem," Management Science Research Report 310, Carnegie-Mellon University, Pittsburgh, Pa.

[6] Balachandran, V. and A. Perry, "Transportation Type Problems with Quantity Discounts," Discussion Paper, The Ceneter for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, Illinois