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A COOPERATIVE GAME OF INFORMATION TRADING:
THE CORE AND THE NUCLEOLUS

by

Shigeo Muto[*] and Mikio Nakayama[**]

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Abstract

We study a cooperative game in which one agent has an information on a new technology and tries to sell it to n producers under the complete possibility of prohibiting its resales. We derive a necessary and sufficient condition for a nonempty core, interpret this condition, characterize the trade in the core, and examine the nucleolus and its implications when the core is empty.

[*]Faculty of Economics, Tohoku University, Japan.

[**]Faculty of Economics, Toyama University, Japan, and currently visiting at Department of Managerial Economics and Decision Sciences, Northwestern Univeristy, Evanston, Illinois 60201.

1. Introduction

In this paper we study a cooperative game in which one agent tries to sell a new technology to n producers who produce the same output with the same profit. The seller is not a producer. The new technology may increase the profit of the producers who acquire it, but may decrease the profit of the producers who do not purchase it. There is also an external diseconomy such that as the number of producers who use the new technology increases, the profit of every producer may decrease. Resales of the new technology are not allowed, so that the seller can choose the number of the possessors so as to maximize a joint profit of the producers. The producers, on the other hand, can make a binding agreement not to buy the new technology if they want to do so.

Such a cooperative game model may give new insights in understanding some of the issues on information trading disclosed by noncooperative game models. For example, it is known that the profit of the buyers of the new technology generally declines relative to their preinnovational level due to the noncooperative behavior of buyers (Kamien and Tauman (1984,1986)). In cooperative models where the binding agreement not to buy the information can be made, the above conclusion will depend on whether the core of the game is empty or not.

The second point is that when the industry becomes a monopoly in the noncooperative setting such as Kamien and Tauman (1984) or Muto (1986), which occurs if the innovation is drastic in the sense of Arrow (1962), cooperative behavior might guarantee certain profit shares to the producers who otherwise had to be expelled from the market. This will also depend on the nonemptiness of the core, or other solution concepts adopted to

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determine the profit shares.

Another point would be that if we interpret the profit share imputed to the seller as the value of the technological innovation, then this is not a private value but a 'social value' reflecting the coalitional behavior of players. In this respect, Katz and Tauman (1985) provided a cooperative game in which a social value of innovation was given by the <u>Shapley value</u> of the seller. There is another cooperative solution concept, the <u>nucleolus</u>, which is also useful to determine the profit shares that may be thought of as representing a compromise among the coalitions.

Our analysis will be focused on the core and the nucleolus of the game. The results can be summarized as follows:

(1) The core is not necessarily nonempty. It is nonempty if and only if each producer can expect at least the profit E of the preinnovational level (we will express this by the inequality: $e^0 \geq E$, where e^0 denotes a certain level of profit that each producer can expect to obtain under the coalitional behavior). This condition is equivalent to saying that the <u>value</u> of the new technology to a coalition takes its maximum at the grand coalition.

(2) If the core is nonempty, the information is sold at least to one producer. If, in particular, the external diseconomy is strict on the profit of nonpossessors of the new technology, then the nonempty core indicates that every producer gets the new technology.

(3) The seller generally obtains a positive profit in the core, which is greater than or equal to the amount left when every producer obtains exactly e^0 which he could expect to obtain, and is smaller than or equal to the amount left when every producer obtains exactly the preinnovational

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profit E. However, some of the producers can be assigned a profit lower than E in the core, except when the equality: $e^0 = E$ holds; in which case the core reduces to a single point indicating that the seller is the only player obtaining a positive net profit in the game.

(4) If we assume that the underlying profit structure of the industry has a property what we call monopolistic, then the core of the game is empty. This means that monopolistic shares of profit cannot arise in our cooperative game as a coalitionally stable outcome of a trade.

(5) The nucleolus assigns to each producer the profit share which always results in an amount between e^0 and E, and to the seller all the residual.

(6) There exists a case in which the nucleolus gives a compensation to the seller for not selling the new technology at all. This occurs, in particular, when the profit structure has the monopolistic property, so that the core is empty (see (2) and (4)).

These are the results obtained from an abstract model without an explicit formulation of the underlying economic structure of the industry. We choose to start with giving the profit structure which can be thought of as plausible in such an industry. Therefore, our model is not exactly a cooperative version of, say, Kamien and Tauman (1984) but rather is a cooperative game with its own interest. Nevertheless, these results provide new insights as to the nature of a trade of a new technology from a cooperative point of view.

2. The Profit Structure

Consider an industry where there are n producers producing the same

homogeneous output with the same technology. Assume that the profit of each producer is identical and given in terms of money. There is another agent who has an information about a new technology which may increase the profit of a producer who acquires and utilize it. The agent has no means of production, so that he can act only as a seller of the information. Hereafter we use the words information and new technology in the same meaning.

The underlying economic situation is as follows: the profit of a producer may decrease if he does not purchase the information and continues to use the 'old' technology; and the profit of $\underline{every}$ producer may also decrease by the increase in the number of $\underline{producers}$ who purchase and switch to the new technology.

By $E^P(t)$ and $E^N(t)$, we denote the monetary profit of each producer who has the information and not, respectively, when there are $t \geq 0$ producers who have acquired the information. Then, the above assumption can be summarized as follows:

$\underline{Assumption\ 1}$. $E^N(0) = E > 0$,

$$E^N(0) < E^P(1) \geq E^P(2) \geq \ldots \geq E^P(n),$$

$$E^N(0) \geq E^N(1) \geq E^N(2) \geq \ldots \geq E^N(n-1) \geq 0,$$

and $E^P(t) \geq E^N(t)$ for all $t = 1, 2, \ldots, n-1$.

Notice that $E^P(0)$ and $E^N(n)$ are not defined. However, for convenience, we put $E^P(0) = E^N(n) = 0$. We call the collection $\{E^P(t), E^N(t) \mid t = 0, 1, 2, \ldots, n\}$ satisfying Assumption 1 the $\underline{profit}$ $\underline{structure}$ of the industry.

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3. The Game.

Let $N^0 = \{0\} \cup N$ be the set of players, where 0 is the seller of the information and $N = \{1,2,\ldots,n\}$ is the set of all producers. Any nonempty subset of N^0 will be called a coalition. By S^0, we always denote a coalition that contains 0, i.e., $S^0 = \{0\} \cup S$, $S \subseteq N$.

Given the profit structure $\{E^P(t), E^N(t) \mid t=0,1,2,\ldots,n\}$, coalition S^0 seeks to maximize its joint profit of production if a trade of the information is to be carried out. Resales of the information can be completely prohibited, so that S^0 can determine the profit independently of the complementary coalition $N^0 - S^0$. On the other hand, any coalition $S \subseteq N$ cannot acquire the information, and the worst conceivable state would be when all the other producers not in S acquire the information.

Let v denote a function that assigns to every coalition its maximal joint profit. Then,

$$v(S^0) = \max \{tE^P(t) + (s-t)E^N(t) \mid t \in \{0,1,2,\ldots,s\} \}$$
$$\text{for all } S^0 \subseteq N^0.$$

and

$$v(S) = sE^N(n-s) \qquad \text{for all } S \subseteq N,$$

where and hereafter, s denotes the number of producers in S^0 and S. Thus, the information trading in N^0 is represented by the cooperative game (N^0, v) in characteristic function form. Notice that the number of the producers possessing the information in S^0 is not necessarily s. We define $t(s)$ to be the number of the producers possessing the information in S^0 if

$$t(s) = \min \{r \mid rE^P(r)+(s-r)E^N(r)=v(S^0), r \in \{0,\ldots,s\} \}.$$

That is, we assume that the information is shared by the smallest number of producers in case there are more than one numbers that attain $v(S^0)$. Before

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completing the description of the game, we state here a straightforward
property of v which will be used later.

Lemma 3.1. $v(N^0) \geq v(S^0) + (n-s)E^N(t(s))$, for all $S^0 \subseteq N^0$.

Proof. By definition,

$$v(S^0) = t(s)E^P(t(s)) + (s-t(s))E^N(t(s))$$

$$= t(s)E^P(t(s)) + (n-t(s))E^N(t(s))$$

$$- (n-t(s))E^N(t(s)) + (s-t(s))E^N(t(s))$$

$$\leq v(N^0) - (n-s)E^N(t(s)). \qquad \text{QED}$$

Based on this lemma, we define a parameter e^0 as follows:

$$e^0 = \max \{e \mid v(N^0) \geq v(S^0) + (n-s)e,$$

$$\text{for all nonempty } S^0 \subset N^0, \ S^0 \neq N^0\}.$$

The parameter e^0 is the maximum value that every coalition S^0 has to allow
each of the producers not in S^0 to obtain. We interpret e^0 as the maximum
value that every producer can _expect_ to obtain by participating the grand
coalition N^0. The parameter e^0 will play an essential role in our analysis.

4. The Core

Let $x = (x_0, x_1, \ldots, x_n)$ be an $(n+1)$-dimensional vector such that $x_0 + x_1$
$+ \ldots + x_n = v(N^0)$. We call x a _payoff vector_. If a payoff vector x
satisfies that $x_i \geq v(\{i\})$ for all $i \in N^0$, we call it an _imputation_. The
core of the game (N^0, v), denoted by C, is the set of all payoff vectors x
such that $\Sigma_{i \in S} x_i \geq v(S)$ for all $S \subseteq N^0$. For notational convenience, we
shall write x(R) instead of $\Sigma_{i \in R} x_i$ whenever x is a payoff vector and R is a

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subset of N^0.

The first result is a necessary and sufficient condition for the core to be nonempty.

Theorem 4.1 $C \neq \emptyset$ if and only if $e^0 \geq E$.

Proof. (necessity). If $x \in C$, then

$$x_0 \leq v(N^0) - v(N) = v(N^0) - nE \qquad (4.1)$$

On the other hand, by the definition of e^0, there must exist some integer r $\leq n-1$ such that for all $R^0 \subset N^0$ with $|R^0|=r+1$,

$$x(R^0) \geq v(R^0) = v(N^0) - (n-r)e^0.$$

Hence, $x(N^0-R^0) \leq (n-r)e^0$. Summing over all N^0-R^0 with $|N^0-R^0|=n-r$ we have

$$_{n-1}C_{n-r-1} \; x(N) \leq \; _nC_{n-r} \; (n-r)e^0,$$

where $_nC_r$ denotes the number of combinations which selects r out of n. Hence,

$$x_0 \geq v(N^0) - ne^0. \qquad (4.2)$$

Combining (4.1) and (4.2), we have $e^0 \geq E$.

(sufficiency). Let x be a payoff vector such that

$$x_0 = v(N^0) - ne^0, \quad x_i = e^0 \quad \text{for all } i \in N.$$

Then, by the definition of e^0,

$$x(S^0) = v(N^0) - (n-s)e^0 \geq v(S^0) \quad \text{for all } S^0 \subseteq N^0.$$

And, by Assumption 1,

$$x(S) = se^0 \geq sE \geq sE^N(n-s) = v(S) \quad \text{for all } S \subseteq N.$$

<div align="right">QED</div>

The meaning of Theorem 4.1 is intuitively clear: the core is nonempty if and only if every producer can expect to obtain at least the

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preinnovational profit E in the game. This is a precise statement of what Kamien and Tauman (1984) conjectured in the summary.

A more accurate interpretation can be obtained as follows. For each $S^0 = \{0\} \cup S$, let us call $d(S^0) = v(S^0) - sE$ the _value_ of the new technology to coalition S^0. Then the core is nonempty if and only if the value of the new technology takes its maximum at coalition N^0. Namely:

Theorem 4.2 $C \neq \emptyset$ if and only if $d(N^0) \geq d(S^0)$ for all $S^0 \subseteq N^0$.

Proof. (necessity). Suppose $d(N^0) < d(S^0)$ for some $S^0 \neq N^0$. Then we have $v(N^0) < v(S^0) + (n-s)E$. By the definition of e^0, this implies $e^0 < E$. Hence, by Theorem 4.1, $C = \emptyset$.

(sufficiency). We have $v(N^0) \geq v(S^0) + (n-s)E$ for all $S^0 \subseteq N^0$, which means $e^0 \geq E$. Hence, by Theorem 4.1, $C \neq \emptyset$. QED

The following proposition states the relations between the nonemptiness of the core and the number $t(n)$ of the producers who actually acqiure the information.

Proposition 4.3 (i) $C \neq \emptyset$ implies $t(n) \neq 0$.

(ii) Assume that $E > E^N(1)$. Then $C \neq \emptyset$ implies $t(n) = n$.

Proof. (i) By Theorem 4.1, we have $e^0 \geq E$. Hence, by the definition of e^0, $v(N^0) \geq v(S^0) + (n-s)E$ for all $S^0 \neq N^0$. Therefore, if $t(n) = 0$ we must have $v(N^0) = nE$, so that $sE \geq v(S^0)$ for all $S^0 \neq N^0$. But, this is a contradiction because for $S^0 = \{0,1\}$ we have

$$v(\{0,1\}) = \max \{E^P(1), E\} = E^P(1) > E$$

by Assumption 1. Hence $t(n) \neq 0$.

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(ii) Suppose on the contrary that $t(n) < n$. Then, if $x \in C$ and $i \in N$, we have

$$x_i \le v(N^0) - v(N^0 - \{i\})$$

$$\le t(n)E^P(t(n)) + (n-t(n))E^N(t(n))$$

$$- t(n)E^P(t(n)) - (n-1-t(n))E^N(t(n))$$

$$= E^N(t(n)).$$

Summing over all $i \in N$, we have

$$x(N) \le nE^N(t(n)).$$

On the other hand, the fact that $x \in C$ and the assumption that $E > E^N(1)$ imply

$$x(N) \ge v(N) = nE > nE^N(1).$$

Hence, $E^N(t(n)) > E^N(1)$, which implies that $t(n) = 0$ by Assumption 1. But, this contradicts (i), so that we must have $t(n) = n$.   QED

The profit structure with $E > E^N(1)$ simply says that the profit of a producer who does not acquire the information will be strictly lower than the preinnovational level. In this case, every producer obtains the information as an outcome of the trade in the core. Anyway, there is at least one producer who obtains the information if the core is nonempty.

We can now partially characterize the imputations in the core.

Theorem 4.4   Assume that $e^0 \ge E$, and let

$$K = \{x \in \mathbb{R}^{n+1} \mid x(N^0) = v(N^0), \ E \le x_i \le e^0 \ \text{for all} \ i \in N \}.$$

Then: (i) $K \subseteq C$.

(ii) $x \in C$ implies $v(N^0) - ne^0 \le x_0 \le v(N^0) - nE$.

(iii)   If $e^0 = E$, then $C = K = \{(v(N^0) - nE, E, \ldots, E)\}$.

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<u>Proof</u>.   (i)   If $x \in K$, then for all $S^0 \subseteq N^0$,

$$x(N^0 - S^0) \leq (n-s)e^0.$$

Hence, by the definition of e^0, we obtain

$$x(S^0) \geq v(N^0) - (n-s)e^0 \geq v(S^0) \quad \text{for all } S^0 \subseteq N^0.$$

For all $S \subseteq N$, we have

$$x(S) \geq sE \geq sE^N(n-s) = v(S).$$

(ii)   This follows from the inequalities (4.1) and (4.2).

(iii)   Let $x \in C$.   Then,

$$x_0 = v(N^0) - ne^0 = v(N^0) - nE \qquad\qquad (4.3)$$

Then, since (4.2) now holds in equality, there must exist a nonnegative integer $r \leq n-1$ such that for all $R^0 \subset N^0$ with $|R^0| = r+1$, $x(R^0) = v(R^0)$. By (i) of Proposition 4.3, we have $r \neq 0$.   Otherwise, we would have $x_0 = 0 = v(N^0) - nE$ by (4.3), so that the contradiction that $t(n) = 0$.   Hence, $R^0 \neq \{0\}$.   Take any $i \in R^0$, $i \neq 0$ and $j \in N^0 - R^0$.   Recall that $R^0 \neq N^0$.   Then,

$$x((R^0 - \{i\}) \cup \{j\}) = x(R^0) - x_i + x_j = v(R^0).$$

Hence $x_i = x_j$.   Repeating this procedure we eventually have

$$x_1 = x_2 = \ldots = x_n = (v(N^0) - x_0)/n = E.$$

Hence $C \subseteq K$.   The converse follows from (i).   QED

Property (i) says that the 'normal' imputations are contained in the core if it is nonempty.   But, of course, the inclusion may be proper. Property (ii) bounds the profit share to the seller.   When $e^0 = E$, the core reduces to a single point.   Notice, in this case, that $x_0 > 0$ because $t(n) \neq 0$, so that the seller is the only player in the game who obtains a positive net profit.

We now turn to the case in which the game has an empty core.   A typical

example of this occurs if the profit structure is of a particular type as follows: we say the profit structure is <u>monopolistic</u> if

$$E^N(1) = 0, \text{ and } E^P(1) \geq tE^P(t) \quad \text{for all } t = 1,\ldots,n.$$

For example, in the linear oligopoly model studied by Kamien and Tauman (1984), this profit structure is induced in case the new technology is drastic.

<u>Proposition 4.5</u> If the profit structure is monopolistic, then $C = \varnothing$.

<u>Proof</u>. By assumption, we have

$$d(N^0) = \max \{E^P(1), nE\} - nE = \max \{E^P(1) - nE, 0\}$$
$$< E^P(1) - E = v(\{0,1\}) - E = d(\{0,1\}).$$

Hence, by Theorem 4.2, we have $C = \varnothing$.   QED

Under the monopolistic profit structure, the industry would become a monopoly if producers acted noncooperatively. Proposition 4.5 states that in this case coalitional behavior of players prevents any imputation from being the stable outcome as far as the core is concerned.

5. The Nucleolus

So far, we have considered the core to see what imputations can be the stable outcomes of the game. We now turn to the question: what imputation should then be the outcome, especially when the core is empty? While in general this is a difficult question to answer, we shall examine the nucleolus and its implications in the cooperative trade of information. For other recent economic applications of the nucleolus, see Legros (1987).

The nucleolus is a cooperative solution concept which exists uniquely and is contained in the nonempty core (Schmeidler (1969)). Intuitively, the nucleolus is the outcome of the game that represents a compromise among coalitions such that the largest 'dissatisfaction' of all coalitions is thereby minimised. For the formal definition, see Schmeidler (1969). We only state below the property that is needed in the proof.

Let x be an imputation of the game (N^0, v), let $v(R) - x(R)$ be called an excess of coalition $R \subseteq N^0$ given x, and let $\theta(x)$ be the 2^{n-1}-dimensional vector obtained by arranging the excesses of all coalitions in the nonincreasing order. Then, the nucleolus has the following property: x is the nucleolus if and only if for all imputations $y \neq x$,

$$\theta_h(x) < \theta_h(y), \text{ where } h = \min \{ j \in \{1, 2, \ldots, 2^{n-1}\} \mid \theta_j(x) \neq \theta_j(y) \}.$$

Thus, the nucleolus is the imputation that minimizes the vector $\theta(\bullet)$ in the lexicographical order.

To obtain the nucleolus of our game, we need some notations. For any real number e, let $f(e)$ and $g(e)$ be defined as follows:

$$f(e) = \max \{ v(S^0) - v(N^0) + (n-s)e \mid \varnothing \neq S^0 \subset N^0, S^0 \neq N^0 \},$$

$$g(e) = \max \{ v(S) - se \mid \varnothing \neq S \subseteq N \}.$$

These functions give maximal excesses to coalitions S^0 and S, respectively, against the payoff vector $(v(N^0) - ne, e, \ldots, e)$.

Theorem 5.1   Let x^* be the payoff vector defined by

$$x^*_0 = v(N^0) - ne^*, \text{ and } x^*_i = e^* \text{ for all } i \in N,$$

where e^* satisfies $f(e^*) = g(e^*)$. Then e^* is uniquely determined between e^0 and E, and x^* is the nucleolus of the game (N^0, v).

Proof.   See Appendix.

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The nucleolus x^* of the game (N^0, v) is thus the imputation that balances the maximal dissatisfactions between the coalitions containing the seller and not. The profit share of each producer is always between e^0 and E, and the seller always obtains a positive profit share. When the core is empty so that $e^0 < E$, the producers compromise with the profit less than the preinnovational level E.

The number of producers who acquire the information as an outcome of the game was indicated by $t(n)$. When the core is empty and $t(n)=0$, in particular, a positive profit share to the seller would imply a compensation for not selling the information. This occurs, for example, if the profit structure is monopolistic.

Proposition 5.2   Assume that the profit structure is monopolistic.

(i)   If $E^P(1) > nE$, then $t(n) = 1$, $e^* = nE/(2n-1)$, and

$$x^*_0 = E^P(1) - (n/(2n-1))nE > 0.$$

(ii)   If $E^P(1) \leq nE$, then $t(n) = 0$, $e^* = (2nE - E^P(1))/(2n-1)$, and

$$x^*_0 = (n/(2n-1))(E^P(1) - E) > 0.$$

Proof.   The monopolistic profit structure implies that

$$v(S^0) = \max \{E^P(1), \, sE\} \quad \text{if } S^0 \neq \{0\}.$$

Hence, the assertion on $t(n)$ is straightforward in each case. Also, x^*_0 is a matter of calculation. By Theorem 5.1 we need only check if $f(e^*) = g(e^*)$ in each case. Notice, in both cases, that we have $v(S) = 0$ if $S \neq N$ and $v(N) = nE$, and that a simple calculation yields $E > e^*$. Hence,

$$g(e^*) = n(E - e^*) = v(N) - ne^*.$$

(i) $f(e^*) = \max \{v(S^0) - se^* + ne^* - v(N^0) \mid S^0 \neq N^0\}$

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$$= \max \{E^P(1) - se^* + ne^* - E^P(1) \mid 1 \leq s \leq n-1\}$$

$$= (n-1)e^*$$

$$= (n-1)(n/(2n-1))E$$

$$= ((2n-1)E - nE)n/(2n-1)$$

$$= n(E - e^*)$$

$$= g(e^*).$$

(ii) $f(e^*) = \max \{ \max \{sE - se^* + ne^* - v(N^0) \mid 1 \leq s \leq n-1\},$

$$E^P(1) - e^* + ne^* - v(N^0)\}$$

$$= \max \{(n-1)(E - e^*) + ne^* - nE, \; E^P(1) - e^* + ne^* - nE\}$$

$$= \max \{(E^P(1) - E)(n-1)/(2n-1) + ne^* - nE,$$

$$(E^P(1) - E)2n/(2n-1) + ne^* - nE\}$$

$$= (E^P(1) - E)2n/(2n-1) + ne^* - nE$$

$$= (2E^P(1) - 2E + 2nE - E^P(1) - 2nE + E)n/(2n-1)$$

$$= (E^P(1) - E)n/(2n-1)$$

$$= n(E - (2nE - E^P(1))/(2n-1))$$

$$= n(E - e^*)$$

$$= g(e^*). \hspace{4cm} \text{QED}$$

In case (i), the information will be sold to one of the producers, and the monopolistic profit is shared by the seller and all of the producers. It is easily verified that the profit share of the seller is greater than that of each producer.

In case (ii), the information cannot be sold in this market; instead, the seller is compensated by the amount x^*_0 for not selling the information at all. If $E^P(1) \geq 3E$, then the compensation exceeds the profit share to each producer.

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6. Concluding Remarks

We have analized how a new technology would be traded if coalitional behavior of players is allowed. We found that the existence of a nonempty core depends upon whether or not the profit structure allows each of the buyers (producers) to obtain the preinnovational profit as an outcome of the trade, or equivalently, whether or not the value of the new technology to a coalition is maximal at the coalition of all players. In the extreme case in which the profit structure does not exhibit a strict externality, Theorem 4.2 immediately indicates that the core is nonempty; and, in the case in which it is monopolistic the core is empty.

This last conclusion would become different if we assumed that the seller could also produce the output. The seller would then assure himself at least the profit $E^P(1)$, and the core would be nonempty under the monopolistic profit structure. This case was considered as an example of a certain type of cooperative games called 'Big Boss Games' in Muto et. al. (1987).

The empty core does not necessarily mean that no trade occurs. But, if no trade occurs, the nucleolus induces a compensation to the seller. This will be also the case with Shapley value, since the 'contribution' of the seller is positive at least in coalitions of the form $\{0,i\}$, $i \in N$. Such a result reflects the nature of coalitional behavior in the game.

We have assumed that resales of the information can be completely prohibited so as to obtain a well-defined characteristic function. Muto, Potters and Tijs (1986) have also developed a cooperative model, which they

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call the information market game, under the assumption of perfect patent protection. If we relax this assumption and allow resales of the information, considerations will be necessary on what kind of coalitional behavior is appropriate. Nakayama (1986) is an example of such a study, in which a bargaining set-type solution is proposed. Cooperative-game approaches in this direction would merit further study.

<u>Appendix</u>

To prove Theorem 5.1, we need some lemmas. Let us write

$$f(e) = v(S^{0e}) - s^e e + ne - v(N^0), \quad \text{and}$$

$$g(e) = v(S_e) - s_e e.$$

<u>Lemma A1</u> f is continuous, monotone increasing and satisfies $f(e^0)=0$.

<u>Proof</u>. Continuity is clear by definition. To show the monotonicity, let d>e. Then,

$$f(d) = v(S^{0d}) + (n-s^d)d - v(N^0).$$

$$\geq v(S^{0e}) + (n-s^e)d - v(N^0)$$

$$> v(S^{0e}) + (n-s^e)e - v(N^0)$$

$$= f(e).$$

That $f(e^0)=0$ follows from the definition of e^0.   QED

<u>Lemma A2</u> g is continuous, monotone decreasing and satisfie g(E)=0.

<u>Proof</u>. Continuity is clear by definition. Let d>e. Then,

$$g(e) = v(S_e) - s_e e$$

$$\geq v(S_d) - s_d e$$

$$> v(S_d) - s_d d$$

$$= g(d).$$

Finally, by the definition of g, we have

$$g(E) = \max \{ s(E^N(n-s)-E) \mid 1 \leq s \leq n\}$$

$$= n(E^N(0)-E)$$

$$= 0. \qquad\qquad \text{QED}$$

<u>Lemma A3</u> (i) If $e^0 \geq E$, then there exists a unique e* such that

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$f(e^*)=g(e^*)$ and $e^0 \geq e^* \geq E$.

(ii) If $E>e^0$, then there exists a unique e^* such that $f(e^*)=g(e^*)$ and $E>e^*>e^0$.

<u>Proof</u>. (i) Assume that $e^0>E$. Let $h(e)=f(e)-g(e)$. Then $h(e^0)=0-g(e^0)>0$ and $h(E)=f(E)-0<0$ by Lemma A1 and Lemma A2. Since h is continuous and monotone increasing, by the mean value theorem, there exists a unique e^* such that $h(e^*)=0$ and $e^0>e^*>E$. If $e^0=E$, then $h(e^0)=f(e^0)-g(E)=0$. Hence, $e^0=e^*=E$. This proves (i). The proof of (ii) is similar, and is omitted. QED

<u>Proof of Theorem 5.1</u> Initially, we show that x^* is an imputation. If $e^0 \geq E$, then the proof of Theorem 4.1 and Lemma A3 imply $x^* \in C$, and therefore x^* is an imputation. If $E>e^0$, then $E>e^*>e^0$ by Lemma A3. Hence, for $i=0$,

$$x^*_0 = v(N^0) - ne^* > v(N^0) - nE \geq 0 = v(\{0\}).$$

To show that $x^*_i=e^* \geq v(\{i\})$ for $i \neq 0$, we first prove that $e^0 \geq E^N(n-1)$. Assume the contrary. Then, by the definition of e^0, it follows that for some $S^0 \neq N^0$,

$$v(N^0) = v(S^0) + (n-s)e^0$$
$$< v(S^0) + (n-s)E^N(n-1)$$
$$\leq v(S^0) + (n-s)E^N(t(s)),$$

which contradicts Lemma 3.1. Hence, by Lemma A3,

$$e^*>e^0 \geq E^N(n-1)=v(\{i\}) \quad \text{for all } i \in N.$$

This completes the proof that x^* is an imputation.

Now, let x be any payoff vector. Then, the following Claim 1 and Claim 2 can be proved in a similar way as in Theorem 4.1:

<u>Claim 1</u>. If $x(S) \leq se^*$ for all $S \subseteq N$ such that $n-s=s^{e^*}$, then $x_0 \geq v(N^0)-$

ne*.

Note that we have $0 \leq s^{e*} \leq n-1$ by definition. Recall the meaning of the notation s^{e*} given at the beginning of the appendix.

$\underline{Claim\ 2}$.   If $x(N^0-S^0) \geq (n-s)e*$ for all $S^0 \subseteq N^0$ such that $n-s=s_{e*}$, then $x_0 \leq v(N^0)-ne*$.

Note that we have $1 \leq s_{e*} \leq n$ by definition.

If x does not satisfy the assumption in Claim 1; that is, if $x(S)>se*$ for some $S \subseteq N$ with $n-s=s^{e*}$, then

$$v(N^0-S)-x(N^0-S) > v(N^0-S)-v(N^0)+se*$$

$$= v(N^0-S)-v(N^0)+(n-(n-s))e*$$

$$= f(e*)$$

$$= g(e*).$$

This implies that x cannot be the nucleolus.

If x does not satisfy the assumption in Claim 2; that is, if $x(N^0-S^0)<(n-s)e*$ for some $S^0 \subseteq N^0$ such that $n-s=s_{e*}$, then

$$v(N^0-S^0)-x(N^0-S^0) > v(N^0-S^0)-(n-s)e*$$

$$= g(e*)$$

$$= f(e*).$$

Hence x cannot be the nucleolus.

Now, suppose that x satisfies both of the assumptions in Claim 1 and Claim 2. Then,

$$x_0 = v(N^0)-ne*,$$

and since strict inequalities cannot hold in the assumptions of Claim 1 and Claim 2, we have

$$x(S) = se* \text{ for all } S \subseteq N \text{ such that } n-s=s^{e*} \qquad (A.1)$$

and

$$x(N^0-S^0) = (n-s)e^* \text{ for all } S^0 \subseteq N^0 \text{ such that } n-s=s_{e*}. \qquad (A.2)$$

Case (i). If $s^{e*} \neq 0$ or $s_{e*} \neq n$, then at least one of (A.1) and (A.2) holds for some s such that $1 \leq s \leq n-1$. Hence, we must have

$$x_1 = \ldots = x_n = (v(N^0)-x_0)/n = e^*.$$

Therefore, in this case, x must coincide with x^*. Hence, by the nucleolus existence theorem, x^* must be the nucleolus.

Case (ii). If $s^{e*}=0$ and $s_{e*}=n$, then (A.1) and (A.2) reduce to the following single equation: $x(N) = ne^*$. In this case, we have

$$f(e^*) = v(\{0\}) + ne^* - v(N^0) = v(\{0\}) - x_0$$

$$g(e^*) = v(N) - ne^* = v(N) - x(N), \qquad \text{and}$$

$$f(e^*) = g(e^*). \qquad (A.3)$$

If for some $S \neq N_0$ we had $v(S)-x(S) > f(e^*)=g(e^*)$, then x cannot be the nucleolus. So, assume that $v(S)-x(S) \leq f(e^*)=g(e^*)$ for all $S \neq N_0$. Then, (A.3) implies that the largest excess against x is equal to the one against x^*. (Here, we ignore the excess $v(N^0)-x(N^0)$ which is always equal to 0.) Therefore we consider the second largest excess. To do this, we redefine f and g under the additional constraint that $1 \leq s \leq n-1$. Notice, then, that $1 \leq s^{e*} \leq n-1$ and $1 \leq s_{e*} \leq n-1$, and that Claim 1 and Claim 2 hold true for these s^{e*} and s_{e*}. We can now repeat the similar argument to obtain that x satisfies the assumptions in Claim 1 and Claim 2. Hence, at least one of (A.1) and (A.2) holds true for some s with $1 \leq s \leq n-1$, so that we have $x_1 = \ldots = x_n = e^*$. Hence, x^* must be the nucleolus. QED

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