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THE LOGIT AS A THEORETICAL MODEL OF PRODUCT DIFFERENTIATION: MARKET EQUILIBRIUM AND SOCIAL OPTIMUM

by

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Abstract

We study the implications of using the logit model to describe the demand for differentiated products. The preference foundations of the logit are discussed and the model is applied to an oligopoly situation. It is compared in this context to the frequently-used CES formulation of product differentiation and the two models are found to have very similar properties. Welfare analysis using the logit is also developed in this paper and is applied to the question of optimal vs. market equilibrium product diversity. We finally propose the logit as a useful, flexible, relevant and tractable tool for analyzing problems of differentiated product oligopoly.

Keywords: Product differentiation, logit model, optimal product diversity.

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1. Introduction

The logit model is one of the most popular discrete choice models. It can naturally be interpreted as a demand model of differentiated products: the discrete choice approach supposes each decision-maker to choose an option from a menu of competing alternatives with a common consumption aim. As the logit demand structure is used frequently in econometric studies (see for example Train (1980) Berkovec (1985) and Train (1986) for applications to the demand for automobiles) it is worthwhile analysing the *supply* response of firms facing such a demand system. Hence in looking at market equilibrium we should hope to endogenise the prices faced by consumers.

Secondly, the logit is frequently used by researchers in marketing (see for example McFadden (1980) and McFadden (1986). Insofar as some of this research is commissioned by firms (and presumably acted upon), their pricing and production decisions will be influenced by this approach. Studying the logit market equilibrium should therefore be insightful.

A third rationale for analysing the logit in an oligopoly context is provided by Perloff and Salop (1985). These authors proposed the use of probabilistic choice models to study product differentiation. Their general approach however yields relatively few predictions and results. Using a specific example (the logit) gives an intuitive feel for this class of models and their properties.

Fourthly, we shall argue that the logit has a wider applicability than the standard discrete approach might suggest, and furthermore it has appealing properties in the context of a demand system for differentiated products. It is also flexible - the logit has been generalised in many directions - as well as being tractable analytically. The logit is also amenable to welfare analysis as we show below in an application to the question of the optimal number of firms.

Hence we propose the logit as an alternative (or complementary) framework to the often-used CES formulation for analysing issues in product differentiation. The logit and the CES are compared in the penultimate section of the paper. Whilst the models do not exactly "twin", they appear to yield very similar qualitative predictions. In any given context each may have advantages or disadvantages vis-à-vis the other. These issues are discussed further in the conclusions.

2. Consumer behaviour

Initially, let us stay with the standard discrete choice derivation of the logit model (see eg. Manski and McFadden (1981) and Ben-Akiva and Lerman (1986)). Each consumer is to choose to purchase one unit of one of n+1 possible alternatives (variants of a differentiated product). Without loss of generality we let option zero be the null option of not purchasing within the group (this is discussed further at the end of section IV). There are $H \geqslant 1$ different consumer types. The conditional indirect utility of a consumer of type h purchasing option i at price p_i is given by[2]

$$U_{ih} = \alpha_{ih} - p_i + \mu\varepsilon_{ih};\; i = 0,1,\ldots n;\; h = 1,\ldots H. \qquad (1)$$

where ε_{ih} is a random variable normalised to zero mean and unit variance. Hence $\mu > 0$ measures the relative importance of this term. The term α_{ih} represents observable characteristics and may further be split (if desired) into consumer attributes (such as income) and pure product attributes (such as mileage per gallon) - see Amemiya (1981) for example. The term $\alpha_{ih} - p_i$ is frequently called the measured utility. The random term represents unobservable taste differences, unobservable product attributes etc. (see eg. Ben Akiva and Lerman (1986)); the fact that the manner in which consumers evaluate the large number of differing characteristics inherent in any product is not systematic and varies across individuals.

Each agent chooses the option i for which the conditional utility is greatest. Now, assuming that the ε_{ih} are identically independently Gumbel distributed yields the logit model (Manski and McFadden (1981)). The probability - as modelled by the firm - of consumer type h purchasing option i is given by:

$$\mathbb{P}_{ih} = \frac{\exp(\alpha_{ih}-p_i)/\mu}{\sum_{j=0}^{n} \exp(\alpha_{jh}-p_j)/\mu}; \quad i=0,1,\ldots n, \quad h=1\ldots H \qquad (2)$$

Two limit cases are worthy of note. For $\mu \to 0$, consumer choice is perfectly predictable - the option of which $(\alpha_{jh}-p_j)$ is greatest is chosen by all consumers of type h. Secondly, for $\mu \to \infty$, $\mathbb{P}_{ih} \to \frac{1}{n+1}$, and differences in product and consumer attributes have no predictive power. The demand functions (2) have two important derivative properties[3] used below:

$$\frac{\partial \mathbb{P}_{ih}}{\partial p_i} = \frac{\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)}{\mu} \qquad i=0,1;\ldots n. \qquad (3)$$

and

$$\frac{\partial \mathbb{P}_{ih}}{\partial p_j} = \frac{\mathbb{P}_{ih}\ \mathbb{P}_{ih}}{\mu} \qquad i,j=0,1\ldots n,\ j \neq i. \qquad (4)$$

Hence products are symmetric substitutes.

3. Costs and Profits

We suppose there are n firms, each producing a single different product i=1...n. The outside option i=0 is assumed to yield a utility $V_{oh} \equiv \alpha_{oh}-p_o$. Production for each of the n firms entails constant marginal costs, c_i, i=1...n.

Let there be g(h) consumers of type h, h=1...H. Then we can write firm i's expected profit as

$$\pi_i = (p_i-c_i) \sum_{h=1}^{H} g(h)\,\mathbb{P}_{ih}; \qquad i=1\ldots n. \qquad (5)$$

Firms are assumed to be risk neutral[4].

The equilibrium concept we consider is Bertrand-Nash:prices are the strategy variables.

4. Equilibrium existence

We now investigate the conditions under which a pure strategy Bertrand-Nash equilibrium exists for a fixed number of firms, n, given the profit functions (5) and the demand system (2). For all case of practical interest, without loss of generality (and to economise on notation) we can set g(h)=1, h=1...H. This simply entails counting identical types separately. The profit derivative is

$$\frac{d\P_i}{dp_i} = \sum_{h=1}^{H} \mathbb{P}_{ih} + (p_i - c_i) \sum_{h=1}^{H} \mathbb{P}_{ih}(\mathbb{P}_{ih}-1)/\mu; \; i=1...n \qquad (6)$$

Now, noting that $\mathbb{P}_{ih} \in (0,1)$ for all finite prices, the derivative (6) is positive when evaluated at $p_i = c_i$, at which point profit is zero. Furthermore the profit function is twice continuously differentiable and $\lim_{p_i\to\infty} \P_i = 0$ (by l'Hôpital's rule). Hence a maximum in $\P_i$ is characterised by (6) identically zero, or

$$\frac{p_i - c_i}{\mu} = \frac{\sum_{h=1}^{H} \mathbb{P}_{ih}}{\sum_{h=1}^{H} \mathbb{P}_{ih}(1-\mathbb{P}_{ih})} > 0 \; ; \; i=1...n. \qquad (7)$$

It is shown in Appendix 1 that any solution to (7) is a local maximum if and only if

$$\sum_{h=1}^{H} \sum_{k=1}^{H} \mathbb{P}_{ih} \mathbb{P}_{ik} \{(\mathbb{P}_{ik} - \mathbb{P}_{ih})^2 - (1-\mathbb{P}_{ih})\} < 0 \quad i=1...n \qquad (8)$$

at that solution. For instance, if all consumers are of the same type then $\mathbb{P}_{ik} = \mathbb{P}_{ih}$ for all h,k=1...H (as $\alpha_{ik} = \alpha_{ih}$) and (8) is guaranteed to hold. In this case then the profit functions are necessarily strictly quasi-concave in own price (although note that they are not concave). Equilibrium existence then follows from a standard fixed point argument (see Friedman (1986) for example). Hence *there exists a pure strategy price equilibrium when all consumers are of the same type.* It is further shown in Appendix 2 that this equilibrium is unique.

Consider now the case when consumers are of different types (that is, $\alpha_{ih} \neq \alpha_{ik}$ for some h,k= 1...H). From (8), a stronger sufficiency condition is

$$2(\mathbb{P}_{ik}-\mathbb{P}_{ih})^2 - (2-\mathbb{P}_{ih}-\mathbb{P}_{ik}) < 0; \text{ for all } h,k=1...H \quad (8')$$

This in turn is guaranteed to be satisfied if

$$|\mathbb{P}_{ih}-\mathbb{P}_{ik}| < 1/2; \quad h,k=1...H \quad (8'')$$

Thus equilibrium existence is guaranteed unless there is a large difference in consumer types (leading to large differences in purchase probabilities and refuting (8")). The rôle of μ is important here. The larger is μ the smaller is the relative influence exerted by the α_{ih} terms which differentiate types. This increases the likelihood (8") will hold, and for μ large enough it will necessarily hold. A similar rôle is played by μ in the extended Hotelling model of spatial competition. There existence is guaranteed for μ large enough, but there is no pure strategy equilibrium for μ sufficiently small (de Palma et al. (1985)).

To illustrate what may go awry here, let us consider the limit case $\mu=0$. Now each consumer will purchase the product for which $\alpha_{ih}-p_i$ is greatest (see (1)). In the event of a tie, demand is presumed (by continuity) to be split equally among tying firms. Let there be two consumer types in numbers N_A and N_B, and two firms, i=1,2. Furthermore, let there be no outside option ($V_0 \to -\infty$), and let us label firms and products such that $\alpha_{1A} > \alpha_{2A}$ and $\alpha_{1A}-\alpha_{2A} > \alpha_{1B}-\alpha_{2B}$.

Hence type A's have a relative preference for good 1 and the B's have a weaker relative preference - indeed, they may prefer good 2. The demand addressed to firm 1 is then given by

$$D_1 = \begin{cases} 0 & \text{for } \alpha_{1A}-p_1 < \alpha_{2A}-p_2 \\ N_A/2 & \text{for } \alpha_{1A}-p_1 = \alpha_{2A}-p_2 \text{ and } \alpha_{1B}-p_1 < \alpha_{2B}-p_2 \\ N_A & \text{for } \alpha_{1A}-p_1 > \alpha_{2A}-p_2 \text{ and } \alpha_{1B}-p_1 < \alpha_{2B}-p_2 \\ N_A+N_B/2 & \text{for } \alpha_{1A}-p_1 > \alpha_{2A}-p_2 \text{ and } \alpha_{1B}-p_1 = \alpha_{2B}-p_2 \\ N_A+N_B & \text{for } \alpha_{1B}-p_1 > \alpha_{2B}-p_2 \end{cases} \tag{9}$$

and $D_2 = N_A+N_B-D_1$. This demand function is illustrated in figure 1.

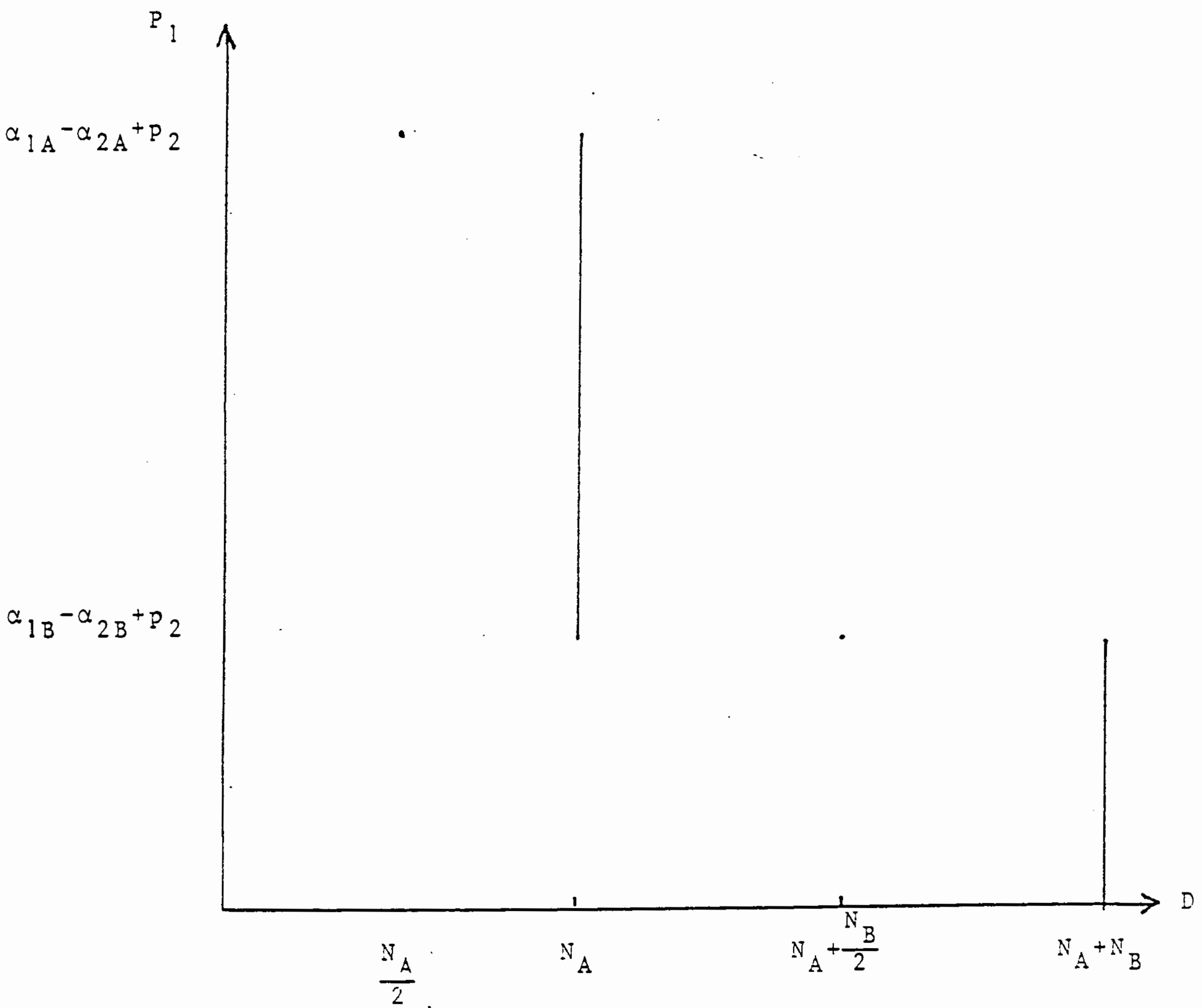


Figure 1. The demand function for the case $\mu=0$ and two consumer types.

Supposing production costs to be zero, it is straightforward to show (see Appendix 3) that no equilibrium in pure price strategies exists for $(N_A+N_B)(\alpha_{1B}-\alpha_{2B}) < N_A(\alpha_{1A}-\alpha_{2A})$. As previously noted, introducing further consumer heterogeneity into the model via the parameter μ will serve to smooth the discontinuities in the profit functions and to increase the likelihood of finding an equilibrium.

Moving away now from the special example, when an equilibrium exists, price exceeds marginal cost (see (7)) regardless of the level of cost, as long as the number of firms is finite and $\mu > 0$. Rather than proceed with the generalised asymmetric case, it is instructive in getting an intuitive feel of the properties of the model to consider the special case when there is but one consumer type and product qualities are equal across firms; that is

$$\alpha_{ih} \equiv \alpha \quad \text{for all} \quad i=1,\ldots n, \quad \text{and} \quad h=1,\ldots H.$$

Let N be the total number of consumers. It is incidentally noteworthy that equilibrium with symmetric product qualities is the outcome of a game where product qualities are endogenously chosen by firms, given a marginal production cost that is an increasing and strictly convex function of quality (see Anderson and de Palma (1986)), and a single consumer type. We retain the possibility of consuming the outside option, with measured utility, V_o. For $V_o \to -\infty$, the model reverts to the case where all consumers purchase only from the firms producing the differentiated products. This is essentially the case analysed by Perloff and Salop (1985). For $V_o > -\infty$, the demand for the differentiated products exhibits elasticity in aggregate.

Henceforth in the paper we discuss the symmetric model outlined above. Before the detailed analysis, we first discuss alternative preference foundations of the logit approach.

5. Alternative Preference Foundations

Under the assumption of a single consumer type and equal product qualities, the logit demand function is given by

$$D_i = N\mathbb{P}_i = \frac{N\exp\{(\alpha-p_i)/\mu\}}{\sum_{j=1}^{n} \exp\{(\alpha-p_j)/\mu\}+\exp\{(V_o/\mu)\}} \quad i=1,\ldots n \qquad (10)$$

This model was derived in section 2 from a discrete choice model of individual consumer behaviour.

However, the demand system (10) may also be generated from a *representative consumer approach* (see Spence (1976) and Dixit and Stiglitz (1977) for examples of this approach in the context of differentiated products). Suppose the representative consumer has an *indirect utility function* of the form

$$\Lambda = N\mu\ell n\{\sum_{i=1}^{n} \exp\{(\alpha-p_i)/\mu\} + \exp\{V_o/\mu\}\}+Y \qquad (11)$$

where Y is representative consumer income. Application of Roy's identity ($D_i = -\frac{\partial\Lambda}{\partial p_i}/\frac{\partial\Lambda}{\partial Y}$ yields exactly the demand system (10). Delving further into this approach, (11) is consistent with a *direct utility function* of an entropic form (Anderson, de Palma and Thisse (1986)). Furthermore (11) is consistent with the welfare measures derived by Small and Rosen (1981) for the logit discrete choice model. In that sense (11) is truly representative - any series of parameter changes showing an increase in welfare via the Small and Rosen approach will raise the utility of the representative consumer.

Another major approach to modelling product differentiation is the *address or characteristics* one (see for example Lancaster (1979) and Archibald, Eaton and Lipsey (1986)). It has been shown (Anderson, de Palma and Thisse (1987a)) that the logit demands (10) may be derived from such an address approach under specific assumptions on the characteristics embodied in products and on the distribution of consumer tastes across characteristics.

Hence we can provide the logit with a formidable pedigree in terms of the standard conceptual approaches to product differentiation.

6. Fixed Numbers Equilibrium

We suppose marginal production costs are constant and equal for all firms at c. It is shown in Appendix 2 that the Bertrand-Nash price equilibrium is unique and symmetric. It is given by (see (7)).

$$p^* = c + \frac{\mu}{1-\mathbb{P}^*} \qquad (12)$$

with (see (2))

$$\mathbb{P}^* = [n+\exp\{(V_o-\alpha+p^*)/\mu\}]^{-1} \qquad (13)$$

It is easy to verify that equilibrium price (p^*) output per firm ($N\mathbb{P}^*$) and profit are decreasing in both the number of firms (n) and the relative attractiveness of the outside option ($V_o-\alpha$). For the special case $V_o \to -\infty$ (when all customers purchase the differentiated products), the equilibrium price reduces to

$$p^* = c + \mu n/n-1 \qquad (14)$$

Here increasing μ, which is the heterogeneity in consumer tastes (see (1)) or else the degree of horizontal product differentiation (see Anderson, de Palma and Thisse (1987a)), raises prices and profits in this case. When V_o is finite (that is, there is outside elasticity), the effects of a change in μ are more interesting.

Two component effects can be isolated (see (1)). Firstly, for any given set of prices, increasing μ leads to more equaniminous demand - customers become more evenly distributed across goods. Secondly, demand becomes more inelastic as subjective factors in choice become more important to consumer evaluation of a given product. If $V_0-\alpha+p^* > 0$, that is, if the outside good is relatively attractive, both effects work in favour of the differentiated products and prices and profits rise necessarily for $n \geqslant 1$. For $V_0-\alpha+p^*<0$, the first effect works the opposite way. However, as shown in Anderson and de Palma (1986), in all but the monopoly case, prices and profits necessarily rise with μ. In the one firm case we do not observe the beneficial effect of reduced intra-industry competition as μ rises, and the monopolist (producing a "higher quality" product) is hampered by the fact that consumers pay less regard to quality differences. In this case it is possible that price and profit fall with μ. Finally though, even for the monopoly case, price and profit necessarily rise with μ for μ large enough.

7. Free Entry Equilibrium

Suppose entry entails a fixed cost, K. A zero profit equilibrium then requires $(p^*-c)\mathbb{P}^*N = K$. For the moment we suppose the number of firms to be a continuous variable. Denoting free-entry equilibrium values by a superscript f, it is straightforward to show, from (12) and (13)

$$\mathbb{P}^f = K/(K+N\mu); \tag{14a}$$

$$p^f = c + \mu + K/N \tag{14b}$$

The equilibrium number of firms is given explicitly as

$$n^f = 1 + 1/\tilde{K} - \exp(1+\chi+\tilde{K}) \tag{14c}$$

where $\tilde{K} \equiv K/N\mu$ and $\chi \equiv (V_0-\alpha+c)/\mu$.

Consider briefly the limit case $\mu \to 0$. Here the "differentiated" products become perfectly homogeneous. Under Bertrand competition, oligopoly prices are bid down to marginal costs so that at most one firm can survive in the market and all potential entrants realise this. A monopolist will set a price p^m such that $V_o = \alpha - p^m - \delta(\mu)$, with $\lim_{\mu \to 0} \delta(\mu) = 0$, hence "undercutting" the outside opportunity. All consumers then patronise the monopolist, so monopoly output tends to N with associated costs Nc+K. The condition for the market to be served is then

$$c + K/N < \alpha - V_o \tag{15}$$

It is interesting to note that the number of firms in a free entry equilibrium could exceed one for $\mu > 0$ even if (15) is not satisfied. Increases in perceived product differentiation (μ) increase the market power of producers and reduce the tendency to price undercut à la Bertrand.

8. The Social Optimum (W)

The logit model is readily convenable to welfare analysis. One question which has been posed in the context of product differentiation is the extent to which market equilibrium diverges from some welfare optimal benchmark (see for example Lancaster (1975), Salop (1979), Spence (1976) and Dixit and Stiglitz (1977)). In this section we characterise explicitly the first-best welfare optimum. The next section is devoted to the second-best solution and section 10 provides comparison of these benchmarks with market equilibrium.

The welfare function for the first-best problem is simply the sum of producer surplus (profit) plus consumer surplus:

$$W = N\mu \ln\left(\sum_{i=1}^{n} \exp((\alpha - p_i)/\mu) + \exp(V_o/\mu)\right) + \frac{N \sum_{i=1}^{n} (p_i - c)\exp((\alpha - p_i)/\mu)}{\sum_{j=1}^{n} \exp((\alpha - p_j)/\mu) + \exp(V_o/\mu)} - nK \tag{16}$$

The first term is the consumer surplus attributable to products 0 through n as given by Small and Rosen (1981) or equivalently from equation (11).

It is easily (and not surprisingly) shown that marginal cost pricing is optimal. Marginal analysis then yields

$$n^W = 1/\tilde{K} - \exp \chi \tag{17a}$$

and

$$N\mathbb{P}^W = \tilde{K}N \tag{17b}$$

as the first best optimal number of firms and output per firm. It can also be shown (Anderson and de Palma (1986)) that market equilibrium provides the optimal qualities (under both first- and second-best welfare benchmarks) when these are endogenous to the model.

9. The Second-best Optimum (S)

Under the second-best optimum, the sum of consumer and producer surpluses (14), is maximised subject to the constraint that firms must at least break even. The algebra of this constrained problem is relegated to Appendix 4. The optimal values reported therein are:

$$p^S = c + \mu \tag{18a}$$

$$N\mathbb{P}^S = N\tilde{K} \tag{18b}$$

$$n^S = 1/\tilde{K} - e^{1+\chi} \tag{18c}$$

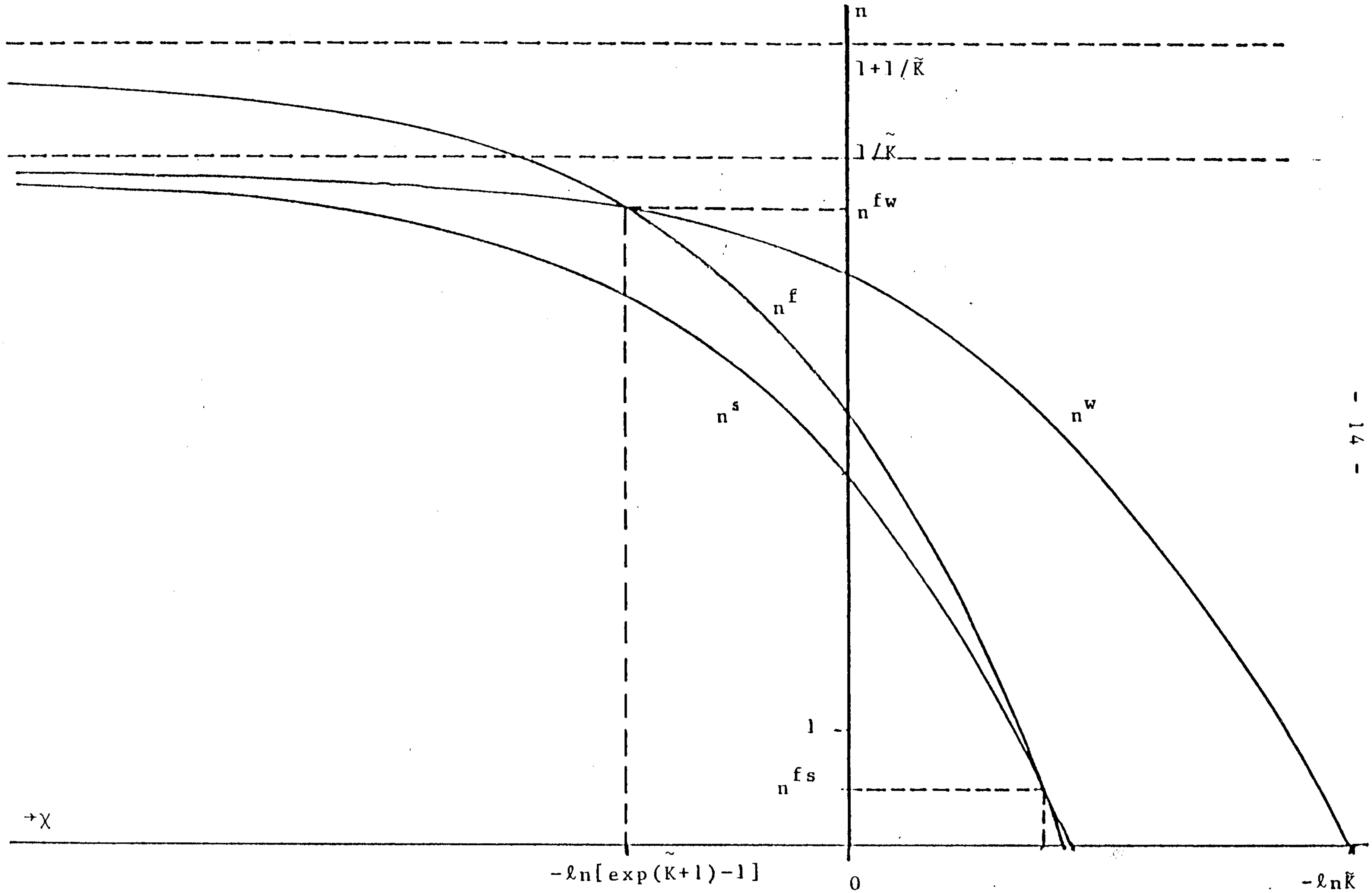
10. Equilibrium and Optimal Product Diversity

In comparison of equations (14), (17) and (18)[5] we have $\mathbb{P}^W = \mathbb{P}^S > \mathbb{P}^f$. Both social optima tend to exploit economies of scale (due to the fixed cost) beyond the level of the free-entry equilibrium. For a price comparison, $p^f > p^S > p^W = c$. The greater is the preference for variety, μ, the larger the second-best price - a larger range of products must be financed, in the presence of the profit constraint, by higher prices. To a large extent this is also true of the market solution, although in that context a larger μ also imparts more monopoly power. To shed further light on these results, we now consider the different solutions for optimal product diversity.

Figure 2 illustrates these solutions, n^f, n^s and n^w. All are decreasing and concave functions of $\chi \equiv (V_o-\alpha+c)/\mu$, and $n^s < n^w$. For the limit case $\chi \to -\infty$, both n^s and n^w tend to $1/\tilde{K}$ whereas n^f tends to one firm more.

Comparing second best optimum with equilibrium shows[6] $0 < n^f - n^s < 1$ for $n^f \geqslant 1$. The first best optimum may diverge significantly though.[7] If the market solution yields "too many" firms (which occurs for μ relatively small), then the difference between equilibrium and first best optimal product diversity is negligible (the one firm result). This suggests that some care should be taken in interpreting previous results in the literature which consider only the sign of (n^f-n^w) without looking at the magnitude of the difference. On the other hand, when the market solution yields "too few" firms, then the divergence between n^w and n^f may be severe. Recall that χ is a measure of the relative quality of the outside good. When the outside good is relatively unattractive (alternatively, the market provides relatively high quality), the situation is a "sort of ideal" even in a first best sense; when the market provides relatively low quality, it may provide way too few varieties.

The above analysis has supposed the number of firms to be a continuous variable. In Anderson and de Palma (1986) it is shown that the logit is also amenable to analysing the case when these numbers are constrained to be integers. Figure 3 illustrates the integer case for $\tilde{K} = .2$.



<u>Figure 2.</u> Equilibrium and Optimal Numbers of Firms

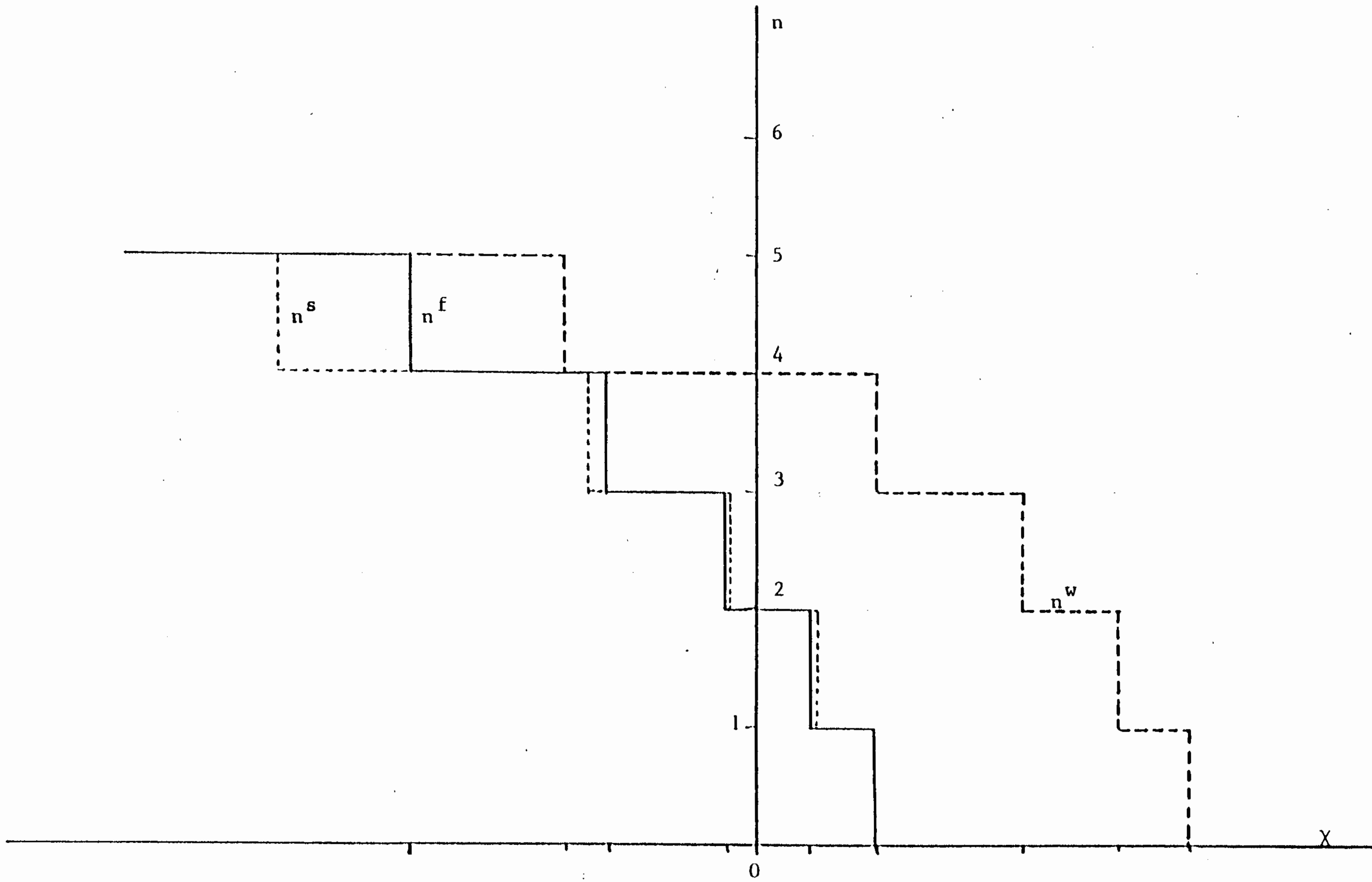


Figure 3. Equilibrium and Optimal Integer Numbers of Firms, $\tilde{K} = .2$.

11. Comparison between the logit and CES demand systems

When a specific functional form for welfare analysis in the context of differentiated products is desired, it is most frequently the CES form which is employed in the literature.[8] It is against this benchmark that we propose now to compare the CES. Algebraic derivations can be found in Appendix 5.

The CES representative consumer's utility function is assumed given by

$$U = \left(\sum_{i=1}^{n} X_i^{\rho}\right)^{1/\rho} X_o^{\alpha} \tag{19}$$

with $\rho \in (0,1)$ and $\alpha > 0$, X_o the numéraire and X_i, $i=1...n$ the variants of the differentiated product. Maximising (19) subject to the budget constraint $\sum_{i=o}^{n} p_i X_i = y$ and defining $m \equiv (1-\rho)/\rho \in (0,\infty)$ yields the demand functions

$$X_i = \tilde{y}\frac{p_i^{-1-1/m}}{\sum_{j=1}^{n} p_j^{-1/m}}\ ; \qquad i = 1...n, \tag{20}$$

so that the total spending on the differentiated products is $\tilde{y} \equiv \frac{y}{1+\alpha}$. The demand curves have slopes

$$\frac{\partial X_i}{\partial p_i} = \left(-\frac{1}{m} - 1\right)\frac{X_i}{p_i} + \left(\frac{X_i^2}{\tilde{y}m}\right) \tag{21}$$

Profit is given by $\Pi_i = (p_i - c)X_i - K$ and the profit maximising price by $X_i + (p_i - c)\partial X_i/\partial p_i = 0$. Note that a necessary condition for non negative profit is $\tilde{y} \geq K$. We assume henceforth that this condition holds.

In a symmetric equilibrium, $X = \tilde{y}/np$. Using this in the first order condition yields an equilibrium price (for a fixed number of firms) as

$$p^* = c + cnm/(n-1) \tag{22a}$$

with associated output per firm

$$X^* = \frac{\tilde{y}}{nc}\,\frac{n-1}{(n(1+m)-1)} \tag{22b}$$

and equilibrium per firm profit

$$\Pi^* = \frac{\tilde{y}m}{(n(1+m)-1)} \tag{22c}$$

From (22c) we can directly compute n^f for the CES that is, free entry equilibrium number of firms such that profits are zero. Substitution in (22a) and (22b) then yields zero profit equilibrium price and output per firm. These values are reported in Table 1 below.

For the first best problem, utility (19) is maximised subject to the resource constraint $y = nK + c\sum_{i=1}^{n} X_i + X_o$, where the social cost of producing the numeraire is normalised to unity. For a symmetric solution, the utility function incorporating the constraint is

$$U^w = nX^\rho[y-nK-ncX]^{\alpha\rho} \tag{23}$$

Maximising (23) with respect to X and n gives the values in Table 1.

To solve the second best problem, utility (19) is maximised subject to consumer reactions (the demand functions (20))and the zero-profit constraint. At a symmetric solution, these constraints are

$$X = \tilde{y}/np \quad \text{and} \quad p = c\tilde{y}/(\tilde{y}-nK)$$

respectively. Substituting into (19) yields

$$U^s = n[\frac{\tilde{y}-nK}{nc}]^\rho (\tilde{y}\alpha)^{\alpha\rho} \tag{24}$$

Maximisation with respect to n enables us to find n^s; substitution into the constraints finds p^s and X^s. The expressions are given below.

	Fixed numbers (*)	Free entry equilibrium (f)	First best optimum (w)	Second best optimum (s)
Price	$c + \frac{cmn}{n-1}$	$\frac{c\tilde{y}(1+m)}{\tilde{y}-K}$	c	$c(1+m)$
output per firm	$\frac{\tilde{y}(n-1)}{nc[n(1+m)-1]}$	$(\frac{\tilde{y}-K}{\tilde{y}m+K})\frac{K}{c}$	K/mc	K/mc
profit per firm	$\frac{\tilde{y}m}{[n(1+m)-1]}-K$	0	$-K$	0
number of firms	n	$\frac{\tilde{y}m}{(1+m)K}+\frac{1}{1+m}$	$\frac{ym}{(1+\alpha+m)K}$	$\frac{\tilde{y}m}{(1+m)K}$

Table 1. Comparison of equilibrium and optimum for the CES representative consumer, $\tilde{y} > K$.

Let us now provide (in Table 2) the corresponding values for the logit model. To simplify, we set $V_o \to -\infty$ for the logit.

	Fixed numbers (*)	Free entry equilibrium (f)	First best optimum (w)	Second best optimum (s)
Price	$c + \frac{\mu n}{n-1}$	$c + \mu + \frac{K}{N}$	c	$c + \mu$
output per firm	$\frac{N}{n}$	$\frac{KN}{K+\mu N}$	$\frac{K}{\mu}$	$\frac{K}{\mu}$
profit per firm	$\frac{\mu N}{n-1} - K$	0	$-K$	0
number of firms	n	$1 + \frac{N\mu}{K}$	$\frac{N\mu}{K}$	$\frac{N\mu}{K}$

Table 2. Comparison of equilibrium and optimum for the logit model ($V_o \to -\infty$).

Qualitatively, there is much similarity between the two models. K plays a similar rôle in both and $\tilde{y}$ and N play roughly the same rôles. Likewise μ and $m(\equiv\frac{1-\rho}{\rho})$ act very similarly (these are the "preference for diversity" parameters).

Also, first and second best optimal outputs are equal in both models. As noted in the previous section, we find at most a one-firm difference between the entry equilibrium and constrained optimal number of firms for the logit. For the CES again $n^f > n^s$, and the difference is $0 < \rho < 1$ firms. Also, n^w exceeds n^s which is also true for the logit as long as $V_o > -\infty$ (see the previous section - note that Table 2 requires $V_o \to -\infty$). All in all, it appears that both the logit and the CES yield very similar predictions. However they do not exactly twin with each other.

Conclusions

We have presented the logit discrete choice model as a framework within which oligopolistic interaction may be analysed under product differentiation. In particular, we have developed welfare analysis for the model and indicated the flexibility of the oligopoly model. Several applied studies have used the logit demand system - hopefully now it will be possible for supply relations to be also accounted for in econometric work.

In the last part of the paper, the logit was compared to the CES representative consumer model. The latter is so far one of the few functional forms found amenable to the analysis of welfare and product differentiation (and has been extensively used in the context of international trade). We propose the logit as a viable, reasonable and tractable alternative.

The comparative static properties of the logit formulation are very close to those of the CES model. Likewise the discussion of the optimal number of firms vs. the market outcome yields the same qualitative results. A recent result by Anderson, de Palma and Thisse (1987b) may help explain the similarities between the logit and the CES representative consumer models. These authors found that the CES demand system may also be generated from a specific discrete choice model in which the distribution of preferences is Gumbel. This is exactly the distribution assumption which is used to derive the logit. However there is an important difference in the models –in casting the CES as a discrete choice model, it is assumed each consumer spends a *fixed money amount* on the good selected (see also equation (18)). By contrast, in the logit model, each consumer is assumed to buy a *single unit* of the good chosen.

The logit approach allows for considerable leeway in introducing new elements within the basic framework. Endogenous quality choice has already been mentioned. There is an emerging stream of research looking at competition in the airline industry (see e.g. Dobson and Lederer (1987)). The logit is useful (in nested form) in describing the hierarchy of decisions made by the consumer (choice of airline company, flight time, class etc). We expect similar research will extend the work of Train et al. (1987) for the telephone industry. These authors use a nested logit approach to account for the huge number of alternatives available to consumers.

The logit has already proved fruitful in the context of spatial economics (see e.g. de Palma et al. (1985), de Palma, Labbé, Thisse (1986) and de Palma, Ginsburgh, Thisse (1987)). The logit has also been extended in a dynamic context in the field of residential location (Ben Akiva and de Palma (1986)) to study transactions costs in consumer residence choice. A dynamic game with consumer memory is studied by Davidson and de Palma (1987) who analyse habit formation.

In all the examples cited above, the logit has been extended to account for the special features of the particular market studied. This paper has presented the properties of the logit as a product differentiation model in its most common form (the multinominal logit). We hope that this will be a useful guide for further analysis.

Appendix 1. Equilibrium existence for the general model

The second order condition is

$$\frac{d^2 \P_i}{dp_i^2} = 2 \sum_{h=1}^{H} \frac{\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)}{\mu} + \frac{p_i - c_i}{\mu} \sum_{h=1}^{H} (2\mathbb{P}_{ih}-1) \frac{\mathbb{P}_{ih}(\mathbb{P}_{ih}-1)}{\mu}$$

Evaluating this expression at $d\P_i/dp_i=0$ yields (using (7)):

$$\operatorname{sgn} \frac{d^2\P_i}{dp_i^2} = \operatorname{sgn}\{2 \sum_{h=1}^{H} \mathbb{P}_{ih}(\mathbb{P}_{ih}-1) . \sum_{k=1}^{H} \mathbb{P}_{ik}(1-\mathbb{P}_{ik})$$

$$+ \sum_{k=1}^{H} \mathbb{P}_{ik} . \sum_{h=1}^{H} (2\mathbb{P}_{ih}-1)(\mathbb{P}_{ih}^2-\mathbb{P}_{ih})\}$$

We therefore wish to show

$$-2(\sum_h \mathbb{P}_{ih}^2)^2 - 2(\sum_h \mathbb{P}_{ih})^2 + 4(\sum_h \mathbb{P}_{ih}^2)(\sum_h \mathbb{P}_{ih}) + 2(\sum_h \mathbb{P}_{ih})(\sum_h \mathbb{P}_{ih}^3)$$

$$-3(\sum_h \mathbb{P}_{ih})(\sum_h \mathbb{P}_{ih}^2) + (\sum_h \mathbb{P}_{ih})^2 < 0;$$

or, rewriting:

$$-2(\sum_h \mathbb{P}_{ih}^2)^2 + 2(\sum_h \mathbb{P}_{ih})(\sum_h \mathbb{P}_{ih}^3) + (\sum_h \mathbb{P}_{ih}^2)(\sum_h \mathbb{P}_{ih}) - (\sum_h \mathbb{P}_{ih})^2 < 0. (*)$$

To sign this expression, note that

a) $\sum_h \sum_k (\mathbb{P}_{ih}-\mathbb{P}_{ik})^2 \mathbb{P}_{ih}\mathbb{P}_{ik} = 2(\sum_h \mathbb{P}_{ih}^3)(\sum_k \mathbb{P}_{ik}) - 2(\sum_h \mathbb{P}_{ih}^2)(\sum_k \mathbb{P}_{ik}^2)$

b) $\sum_h \sum_k \mathbb{P}_{ih}\mathbb{P}_{ik}(\mathbb{P}_{ih}-1) = (\sum_h \mathbb{P}_{ih}^2)(\sum_k \mathbb{P}_{ik}) - \sum_h \mathbb{P}_{ih} \sum_k \mathbb{P}_{ik}$

From (a) and (b), the desired condition (*) can be rewritten

$$\sum_h \sum_k \mathbb{P}_{ih}\mathbb{P}_{ik}\{(\mathbb{P}_{ik}-\mathbb{P}_{ih})^2 - (1-\mathbb{P}_{ih})\} < 0$$

which is the condition given in the text. □

If this condition holds at any extremum (when the first order condition is zero) then the profit function is necessarily strictly quasiconcave.

Appendix 2. Uniqueness of Equilibrium

Let us show uniqueness for the case when there is but one consumer class. The proof does though cover the cases where the α_i's and c_i's are not necessarily the same for all firms. From (6) the first-order conditions are now (given N consumers):

$$\frac{d\Pi_i}{dp_i} = N\mathbb{P}_i(1 + \frac{p_i - c_i}{\mu}(\mathbb{P}_i - 1)) = 0, \quad i=1\ldots n$$

which is the implicit form of the best-reply function. From the analysis of the previous section we know this function has a unique solution $p_i^{br}(p_1\ldots p_{i-1}, p_{i+1}\ldots p_n)$. Now let us show the best-reply function is a contraction, that is $p_i^{br}(.)$ has the property

$$\sum_{\substack{j=1\\j\neq i}}^{n} \left|\frac{\partial p_i^{br}}{\partial p_j}\right| < 1.$$

Once this is shown, uniqueness of equilibrium follows directly (see e.g. Friedman (1986)). Note first that, evaluating along the best reply function:

a) $$\frac{1}{N}\frac{\partial^2\Pi_i}{\partial p_i^2} = \mathbb{P}_i((\frac{\mathbb{P}_i - 1}{\mu}) + (\frac{p_i - c_i}{\mu})\frac{\mathbb{P}_i(\mathbb{P}_i - 1)}{\mu}) < 0$$

b) $$\frac{1}{N}\frac{\partial^2\Pi_i}{\partial p_i \partial p_j} = \mathbb{P}_i^2\mathbb{P}_j(\frac{p_i - c_i}{\mu^2}) > 0, \quad i\neq j, i,j=1\ldots n$$

Now, $\dfrac{\partial p_i^{br}}{\partial p_j} = -(\dfrac{\partial^2\Pi_i/\partial p_i\partial p_j}{\partial^2\Pi_i/\partial p_i^2})$ is positive by (a) and (b) so that the desired contraction property is (after some manipulation) $\sum_{j=1}^{n}\partial^2\Pi_i/\partial p_i\partial p_j < 0$. Note that this is the dominant diagonal property. For the logit model above (using (a) and (b), this condition is

$$\sum_{j=1}^{n}\frac{\partial^2\Pi_i}{\partial p_i\partial p_j} = \mathbb{P}_i((\frac{\mathbb{P}_i - 1}{\mu}) + (\frac{p_i - c_i}{\mu^2})\mathbb{P}_i(\sum_{j=1}^{n}\mathbb{P}_j - 1)); \quad i=1\ldots n$$

which is ensured negative as desired.

□

Let us now show that for $c_i = c_j$ and $\alpha_i = \alpha_j$, $i,j=1 \ldots n$, the unique equilibrium is symmetric. This is true as there is a unique solution p^* to (12) and (13) (as $d\mathbb{P}^*/dp^* < 0$ in (12)).

Appendix 3

We here find conditions for existence of pure price strategy Nash equilibrium for the demand system (9) where firms have zero production costs.

We first show there can be no equilibrium with $D_1=N_A$. Firm 1's best price in this range would be $p_1=p_2+(\alpha_{1A}-\alpha_{2A})-\delta$, with δ an arbirarily small positive constant[9]. However, 2's best price in this range is $p_2=p_1+(\alpha_{2B}-\alpha_{1B})-\delta$. Given $(\alpha_{1A}-\alpha_{2A})-(\alpha_{1B}-\alpha_{2B})=k$, where k is a positive constant, these two conditions are obviously inconsistent. Now, $D_1=0$ cannot be a possible equilibrium as 1 could undercut any positive price 2 could set. Likewise for $D_1=N_A/2$.

The only other equilibrium involves $D_1=N_A+N_B$, with $p_2=0$ (by an argument à la Bertrand) and $p_1=\alpha_{1B}-\alpha_{2B}-\delta$ (1 setting the highest price that ensures the whole market). It must now be the case that 1 does not wish to raise p_1 and serve only type A consumers (at best price $p_1=\alpha_{1A}-\alpha_{2A}-\delta$). Comparing profits under these two strategies shows the equilibrium described above exists only if $(N_A+N_B)(\alpha_{1B}-\alpha_{2B}) > N_A(\alpha_{1A}-\alpha_{2A})$. Note that no equilibrium exists if the B's prefer good 2[10].

Appendix 4. The second-best problem

For the second-best optimisation problem, the zero profit condition is written as $(p_i - c)N\mathbb{P}_i = K$, or:

$$\frac{N}{K}(p_i - c)\exp(\alpha - p_i)/\mu = \sum_{j=1}^{n^s} \exp(\alpha - p_j)/\mu + \exp(V_o/\mu). \quad (*)$$

Substituting into the welfare function (16) yields

$$W^s = N\mu \ln\{\frac{N}{K}(p_i - c)\exp((\alpha - p_i)/\mu)\}.$$

Therefore at the optimum, all firms charge the same price given by $p^s = c+\mu$. The second order condition is readily verified at this solution. From (*) we then have

$$n^s = N\mu/K - \exp((\mu + V_o - \alpha + c)/\mu).$$

Again from (*), $N\mathbb{P}^s = K/\mu$.

Appendix 5

In this appendix, we derive the results contained in table 1.

1. Fixed number of firms

Profit of firm i is $\P_i = (p_i - c)X_i - K$; the first order condition is:

$$\frac{\partial \P_i}{\partial p_i} = X_i[1 + (p_i - c)\{\frac{-1}{p_i}(\frac{1}{m}+1) + \frac{p_i^{-\frac{1}{m}-1}}{m\sum_j p_j^{-\frac{1}{m}}}\}] = 0$$

Looking for a symmetric solution, $X^* = \frac{\tilde{y}}{np^*}$, p^*, we obtain by substitution (22a). By substitution in $X^* = \frac{\tilde{y}}{np^*}$ we obtain (22b). Using $\P^* = (p^* - c)X^* - K$, we obtain (22c).

Let us now prove that this solution does indeed constitute an equilibrium. To do this we show that any solution to the focs is a maximum. The second order condition is

$$\frac{\partial^2 \P_i}{\partial p_i^2} = 2\,\frac{\partial X_i}{\partial p_i} + (p_i - c)\,\frac{\partial^2 X_i}{\partial p_i^2}$$

where

$$\frac{\partial^2 X_i}{\partial p_i^2} = (\frac{1}{m}+1)\,\frac{X_i}{p_i^2} + [\frac{2X_i}{\tilde{y}m} - (\frac{1}{m}+1)\,\frac{1}{p_i}]\,\frac{\partial X_i}{\partial p_i}\,.$$

Using the substitution $X_i = \frac{\tilde{y}k}{p_i}$, where $k \equiv \frac{p_i^{-1/m}}{\sum_{j=1}^{n} p_j^{-1/m}}$

and $k \in (0,1)$, and evaluating at a turning point $(\frac{\partial \P_i}{\partial p_i} = 0)$ we obtain after some manipulation:

$$\operatorname{sgn} \frac{\partial^2 \P_i}{\partial p_i^2} = \operatorname{sgn}\{2[\frac{k}{m} - (\frac{1}{m}+1)] + \frac{m}{m+1-k}\{(\frac{1}{m}+1) + [\frac{2k}{m} - (\frac{1}{m}+1)][\frac{k}{m} - (\frac{1}{m}+1)]\}\}.$$

This simplifies to

$$\operatorname{sgn} \frac{\partial^2 \Pi_i}{\partial p_i^2} = \operatorname{sgn} (1+m)(k-1) < 0.$$

Hence the profit function is strictly quasiconcave.

2. Free entry equilibrium

In this case the profit of each firm, in the symmetric equilibrium is null, thus $\Pi^f=(p^f-c)X^f-K$; with $X^f=\frac{\tilde{y}}{n^f p^f}$ we obtain the number of firms n^f for the free entry equilibrium. The price is given by $c+\frac{cmn^f}{n^f-1}$, where n^f is substituted by its value derived above. Output per firm $X^f=\frac{\tilde{y}}{n^f p^f}$ is then computing using the value of n^f and p^f derived above.

3. First best optimum

In the symmetric case, the consumer utility function (19) is

$$U = (nX^{\rho})^{1/\rho}\, X_0^{\alpha} \tag{*}$$

Introducing the resource constraint $Y = X_0+ncX+nK$ and maximising in function of X we obtain, given that $p^w = c$

$$X = \frac{y-nk}{nc(1+\alpha)} \tag{**}$$

Maximising (*) in function of n and using (**) we obtain $X^w=K/mc$ and by substitution in (**) the value of n^w. As the optimal price is equal to the marginal cost c, each firm is making a negative profit equal to -K.

4. Second best optimum

In this case, firms are making zero profit

$$Xp = Xc + K \qquad pX = cX + K$$

The budget constraint is

$$Y = npX + X_0$$

with the demand function

$$X = \frac{\tilde{y}}{np}$$

The consumer utility function is therefore in this case

$$U = \frac{(y-\tilde{y})^{\alpha}}{\tilde{y}c} \quad n^{(1/\rho-1)}(\tilde{y}-nK)$$

Maximising this expression in function of n we obtain the value of n^s. Using the zero profit condition and the demand function, we obtain

$$p = \frac{\tilde{y}c}{\tilde{y}-nK}$$

which provides the value of p^s after substitution of n^s by its value. The output per firm is then given by $X^s=\tilde{y}/n^s p^s$.

Footnotes

[1]We should also note that the logit has been applied theoretically in the context of spatial competition by de Palma et al. (1985) to restore equilibrium existence in the simple spatial oligopoly problem.

[2]Without restriction, it may be that $p_o = 0$ (there may not always be a price associated with the outside option).

[3]The logit model has sometimes been criticised for exhibiting the Independence of Irrelevant Alternatives (IIA) property, as illustrated by the famous red bus/blue bus example of Debreu (1960). However, as noted by Train (1986) (p. 22), this is no longer a problem once we allow for differences in the parameters α_{ih}.

[4]For the alternative derivations of the demand system described in section 5, profit is deterministic and it is not necessary to assume risk-neutrality.

[5]These results are collected in table form in the next section in the case $V_o \to -\infty$.

[6]Let n^{fs} denote the intersection of n^f and n^s at χ^{fs} given by (from (14c) and (18c)) $\chi^{fs} = -\ell n\,(\exp \tilde{K}-1) - 1$. Substitution shows $n^{fs} < 1$. Therefore $n^f > n^s$ for all relevant values ($n^f \geqslant 1$). From (14) and (18), $\frac{d}{d\chi}(n^f - n^s) = \frac{d}{d\chi}(-\exp(\chi+1)\ (\exp \tilde{K}-1)+1) < 0$. Hence $n^f - n^s$ is maximised at $\chi \to -\infty$. From the limit expressions, the maximum difference is one firm.

[7]Using the method described in the previous footnote, $n^f - n^w$ is maximised for $\chi \to -\infty$ with a one firm difference. On the other hand $n^w - n^f$ is maximised with n^f as small as possible. The difference at $n^f = 0$ is given by $n^w - n^f = [1-\exp(-1-\tilde{K})]/\tilde{K} - \exp(-1-\tilde{K})$, which may theoretically be very large, for small values of $\tilde{K} \equiv K/N\mu$.

[8]See for examples Blanchard and Kiyotaki (1985), Dixit and Stiglitz (1977), Feenstra and Judd (1982), Flam and Helpman (1987), Koenker and Perry (1981), Krugman (1982), Lawrence and Spiller (1983), Raubitschek (1987) and Spence (1976).

[9]A more technical argument with the same principles and conclusions could be undertaken along the lines of Lederer and Hurter (1986).

[10]Note also that when $\alpha_{1A} = \alpha_{2A} = \alpha_{1B} = \alpha_{2B}$, there is the zero price homogeneous good Bertrand equilibrium with $D_1 = D_2 = (N_A + N_B)/2$.

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