Discussion Paper No. 747

SYMMETRY AND EXTENSIONS OF ARROW'S THEOREM

by

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October 1987

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Abstract

By emphasizing the symmetry of certain set theoretic conditions, shown to be associated with Arrow's Impossibility Theorem, a characterization of "kinds of axioms" is obtained. More precisely, if the defining properties of a model satisfies these conditions, then the model must have a conclusion much like that of Arrow's theorem. Because the conditions are described in set theoretic terms, the applicability of these results extends beyond the usual setting of complete, binary, transitive rankings to space of utility functions, probability distributions, etc. In this manner, not only can new extensions of Arrow's theorem be obtained, but it is shown how the same "kinds of axioms" applies to, say, problems about the aggregate excess demand function, the Hurwicz-Schmeidler dictatorial result about Pareto optimal, Nash equilibria, the Gibbard-Satterthwaite theorem about manipulability, etc.

Stimulated by Arrow's seminal work [1], social choice has become an active research area. There are lists of axioms forcing impossibility statements, conditions admitting possibility assertions, and the Gibbard [3] - Satterthwaite [19] theorem about manipulation. (An excellent survey is Sen [20].) What is missing from the literature is a simple, unifying mathematical explanation - one that with a single argument can subsume several seemingly different conclusions, one that easily permits extensions of classical theorems and the derivation of new results, and one that captures the elusive frontier between possibility and impossibility statements. A step toward such a description is given here. The idea is to shift emphasis from *what particular set of axioms* yield possibility or impossibility conclusions, to *what kinds of axioms* cause these results. This approach is illustrated by showing how Arrow's Theorem, several other social choice results, a statistical paradox about contingency tables, the Hurwicz-Schmeidler study of optimal Nash equilibria, certain questions about economic allocation procedures, and conclusions from several other disciplines are all closely related. This assertion may be surprising if only because the examples come from different disciplines where the sets of underlying assumptions or axioms may have little to do with each other. What unifies these models is that while the assumptions and axioms differ, they are all of the *same combinatoric kind*; consequently, these models have related properties. For instance, by characterizing *what kinds of axioms* give rise to an Arrow-type theorem, as I do here, results from different literatures can be unified and extended in several directions.

My presentation has a geometric flavor where the goal is to create an easily used, versatile technique. The idea is this. Often, aggregation models from social choice, economics, probability, and other areas are described in terms of the requirements we want the system to satisfy; e.g., the independence conditions from social choice. But, are these conditions self-contradictory? To investigate this issue we might examine all logical, combinatoric possibilities. It turns out that, for several models, the combinatoric analysis of the axioms involve related arguments. This suggests characterizing "*kinds of axioms*" in terms of the associated combinatoric analysis. This program is started here; I characterize the kind of axioms that are related to Arrow's theorem. To do so, I introduce a geometric representation that I call the *binary overlap principle*. It is based the geometry of certain sets - the "level sets" of the imposed conditions.

We now know why social aggregation procedures have difficulties. An aggregation process maps a domain onto a much smaller range, so the problems and paradoxes are created by the "squashed overflow". In an earlier paper [16] (also see [15]), I demonstrated that this explains the paradoxes for several classes of social choice, voting, and probability models. To prove my assertion, I embedded "discrete models" into classes of smooth mappings. Then, the existence and the creation of new paradoxes are obtained with calculus techniques. But certain discrete problems, such as Arrow's theorem, cannot be handled in this manner. So, the results given here can be viewed as extending the discussion of [16]. Indeed, one can show that the overlap principle corresponds to the rank conditions of [16].

A secondary theme for this paper comes from economics. Sen [20,p.1074] points out that "*Economists did not .. take much notice of this [social choice] literature, or of the problem studied in them, until the "informational crisis" sent them searching for other methods.*" One way to study information is with the mechanisms introduced by L. Hurwicz [6]; an approach that has proved to be a convenient formulation to analyze incentive problems and organizational design. A central issue is to understand the relationship between an allocation process and the associated mechanisms. For smooth mechanisms, we have answers; in [9,17,18] geometric tools are created that characterize all possible "message mechanisms" associated with a given "smooth allocation procedures". But, because this characterization is based on the *level sets* of certain smooth functions, the techniques do not extend to discrete allocation processes - indeed, the discrete problem remains open. (Some partial results are in [8].) However, as S. Reiter [13] recognized, social choice models are discrete examples of Hurwicz's "one shot" mechanisms. So, in this spirit, a secondary objective of this paper is to use the analysis of social choice models to understand what kind of mathematics is needed for the mechanism design of discrete systems. It turns out that the "level set" approach still applies where the differential geometric techniques developed to analyze the level sets for smooth allocation procedures are replaced with an algebraic group theoretic analysis.

The emergence of these algebraic structures reinforces my belief [15] that they explain the difficulties common to social choice and other discrete decision and allocation problems. (This runs against Sen's comment [20,p.1078], "*.. but - beware - no 'group theory' is involved!*") These algebraic symmetries - the wreath product of certain permutation groups - play a critical role in the development of the overlap principle; indeed, a complete characterization of other classes of "kinds of axioms" relies on these structures. However, I decided to suppress these complicated, algebraic symmetry structures in order to focus attention on the overlap principle and to make the paper easier to read. (A brief introduction to the wreath product is in [15].)

In Section 2, the basic concepts used in this paper are introduced with a two voter, three candidate formulation of Arrow's theorem. In Section 3, the ideas are abstracted into the *overlap principle.* The flexibility of the overlap principle is illustrated by obtaining simple proofs of several known social choice results as well as to derive some new, and some whimsical ones. In this manner, the connection among several well known social choice results along with problems from statistics, economics, and game theory becomes immediate. Because the emphasis of the overlap principle is on *how the imposed properties or axioms* divide information into equivalence classes, rather than on what particular information used (e.g., complete, binary, transitive rankings), extensions are immediate. To illustrate how implicitly defined overlap conditions arise, a new proof of the Gibbard - Satterthwaite Theorem as well as the Hurwicz-Schmeidler theorem [10] about Pareto optimal Nash equilibria are given. Some extensions of the overlap principle as well as a description of the frontier between possibility and impossibility conclusions are given in Section 4. Section 5 contains the proofs of the major theorems.

2. A SIMPLE EXAMPLE

The ideas of this paper can be demonstrated with a geometric proof of Arrow's theorem for a two voter, three candidate process. To do this, we need a geometric representation for the complete, binary, transitive rankings of the candidates $\{c_1, c_2, c_3\}$. Starting with an equilateral triangle, identify each vertex with a candidate. (See Figure 1.) In this triangle, define a binary relationship in terms of the proximity of a point to a vertex. Thus, a point p corresponds to the ranking $c_1 \succ c_2$ if and only if p is closer to vertex c_1 than to vertex c_2. This relationship divides the equilateral triangle into the regions displayed in Figure 1. The open regions - the smallest triangles - correspond to strict rankings without "indifference" among the candidates, while the line segments and the baricentric point correspond to rankings with indifference. For instance, region A corresponds to the ranking $c_1 \succ c_2 \succ c_3$, while the line segment between regions C and D represents $c_3 \succ c_1 = c_2$. Let P(1,2,3) denote the 3! open

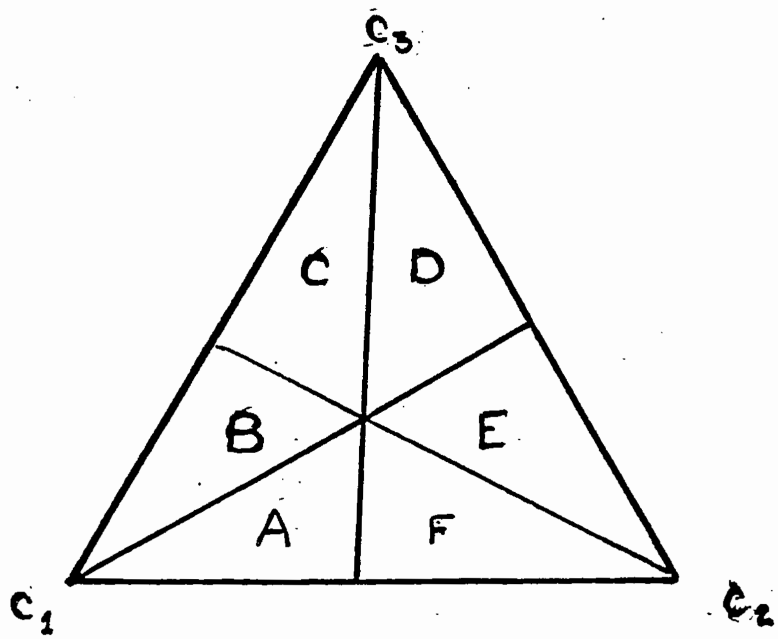

Figure 1

regions where the rankings do not admit indifference. Let P(i,j) denote the two equivalence classes of rankings in P(1,2,3) where $c_i > c_j$ and where $c_j > c_i$. Consequently, P(1,2) = {{A,B,C}, {D,E,F}}. Geometrically, these two equivalence classes are the two right triangles in Figure 1 separated by the line $c_1 = c_2$. In general, the two sets in P(i,j) are represented by the two right triangles separated by the indifference line $c_i = c_j$. I'll show how Arrow's theorem is a consequence of the geometric positioning of these sets of right triangles.

In a two voter, three candidate context without indifference, a social welfare function is a mapping

2.1 F: P(1,2,3) x P(1,2,3) -----> P(1,2,3).

The cartesian product represents the two voters' possible rankings. The standard Arrowian conditions are replaced with the following requirements.

1. The usual Pareto condition forces all outcomes to be admitted. I require only that **F is onto.**

2. The IIA condition states that for each i and j, the relative ranking of c_i and c_j depends only on the voters' relative rankings of these candidates. This is equivalent to requiring **for each choice of i, j, that**

2.2 F: P(i,j) x P(i,j) ----> P(i,j).

3. If the first voter is a dictator for F, then F can be represented by a mapping depending only on the first variable. Replace the "no dictator" axiom with the condition that **F cannot be represented by a function of a single variable.**

Theorem 1. There does not exist a mapping of the form given by Equation 2.1 that satisfies conditions 1, 2, and 3. If a mapping given by Eq. 2.1 satisfies 1 and 2, then it can be represented by a function of a single variable that is generated either by mapping each relationship $c_i > c_j$ to itself (a dictator), or by mapping each relationship $c_i > c_j$ to $c_j > c_i$ (an anti-dictator).

Arrow's theorem is an immediate consequence. An ealier version of this result is in Saari [14], and a portion of it was restated in a axiomatic form in Kim and Rouch [12]. See Sen [20] for added discussion and references.

Outline of the proof. Assume that the theorem is false because such an F exists. By (3), there are situations where each voter, by changing rankings, can alter the outcome. According to (2), if the new ranking interchanges the relative ranking of c_i and c_j, then it is because the voter changed her relative ranking of these two alternatives. In fact, from (2), this same P(i,j) change in F occurs whenever 1) she makes this change in the relative rankings and 2) the other voter keeps his same ranking of this pair.

This argument reduces the analysis to how F changes the relative rankings of pairs of candidates. (Thus, the rest of the proof relies on the positioning of the right triangles in Figure 1.) Because of (3) and symmetry, assume without loss of generality that there are situations where voter 1 can alter the relative ranking of c_1 and c_2 and there are situations where voter 2 can alter the relative ranking of c_2 and c_3. Namely, if voter 2 has a specified ranking of c_1 and c_2, then as voter 1 varies her rankings between the right triangles representing $c_1 > c_2$ and $c_2 > c_1$, so does the image of F (but not necessarily in the same direction.) If the specified ranking for voter 2 is $c_1 > c_2$, then let him vary between regions A and B; otherwise, let him vary between D and E. In either situation, voter 2 has fixed P(1,2) and fixed P(1,3) rankings while retaining the freedom to change his P(2,3) ranking. A similar analysis holds for voter 1. In order for voter 2 to change the P(2,3) outcome, voter 1 may need to have a specific ranking of this pair. If it is the right triangle $c_2 > c_3$, let her vary between A and F; if it is $c_3 > c_2$, then restrict her to C and D. Again, voter 1 can change her P(1,2) ranking while keeping her P(2,3) and P(1,3) rankings fixed.

As these voters vary in their assigned regions, the P(1,2) and P(2,3) images of F (the group outcome) change independent of each other. Thus, there are situations where the P(1,2) outcome is the right triangle corresponding to $c_1 > c_2$, while the P(2,3) outcome is the right triangle corresponding to $c_2 > c_3$. These two triangles intersect in region A - $c_1 > c_2 > c_3$ - which forces the binary ranking of $c_1 > c_3$. On the other hand, there are situations where the two "triangle" outcomes are $c_2 > c_1$ and $c_3 > c_2$. The intersection of these triangles is region D, which requires $c_3 > c_1$. Consequently, *even though both voters have fixed* P(1,3) *rankings, the group ranking of these two alternatives, given by the image of F, changes.* This contradicts (2), and the first part of the theorem is proved.

The second part of the theorem also follows from the geometric positioning of the right triangles. Obviously, a dictator or an anti-dictator can be defined, so we only need to show that no other mapping exists. Without loss of generality, assume there is a mapping $g: P(1,2,3) \rightarrow P(1,2,3)$ that satisfies (1) and (2), that preserves the P(1,2) ranking, but reverses the P(2,3) ranking. This forces the images of $c_1 > c_2$ and of $c_3 > c_2$ to be the two right triangles containing A. Indeed, the intersection of these two triangles is precisely A - $c_1 > c_2 > c_3$. Because this intersection defines the relative ranking $c_1 > c_3$, the definition of g over P(1,2) and P(2,3) uniquely determines g: P(1,3) --> P(1,3). More precisely, the g preimage of $c_1 > c_3$ is *any* P(1,3) ranking meeting the intersection of the triangles for $c_1 > c_2$ and $c_3 > c_2$. In Figure 1, this intersection is {B, C}. But, B and C are in different P(1,3) classes. According to (2), this forces g to be the *constant mapping* over P(1,3) that maps both $c_1 > c_3$ and $c_3 > c_1$ to $c_1 > c_3$. This contradicts (1) and proves the second part of the theorem.

The proofs of both parts of the theorem depend upon the symmetry properties of the simplex as captured by positioning of the right triangles in the three equivalence classes P(i,j). Critical to this analysis is that the geometry of the *image space is restrictive*; e.g., for each triangle from P(1,2) there is one from P(2,3) where their intersection is in *only one* triangle from P(1,3). Moreover, this holds for each triangle in P(1,3). This restrictive effect on the image fixed the images of F to obtain the contradiction. Similarly, in the second part of the theorem, these image restrictions limited the options for g. The second critical element is that the geometry admits *flexibility of movement in the domain.* For each triangle from P(1,2) there is one from P(2,3) where their intersection meets *both* triangles from P(1,3). This was used in both parts of the theorem to allow a voter to alter his rankings of one pair while retaining fixed rankings for the other two. Added flexibility occurs if at least two voters affect the outcome. The proof of the theorem exploits this contradictory interplay between restrictions (in the range) and the flexibility (in the domain) admitted by the overlapping geometry. The fact that this geometry was derived from binary, transitive, ordinal rankings is incidental. Consequently, the essence of Arrow's theorem extends to a surprisingly wide realm of situations. Indeed, whenever a set of axioms can be described with a similar geometric representation, the same conclusions result. In other words, the kinds of axioms that lead to an Arrow-like theorem can be characterized by emphasizing the appropriate geometric - set theoretic conditions of overlapping regions.

3. THE OVERLAP PRINCIPLE

In this section, an overlap principle is introduced and illustrated with several examples. The examples are selected to show why the *same* basic argument proves and extends several classical theorems and to suggest other uses of the main theorem.

Notation: Let $|A|$ denote the cardinality of set A. If $A = \{A_1,..,A_n\}$ and $B = \{B_1,..,B_m\}$ are collection of sets, let $A \cap B = \{A_j \cap B_k : 1 \leq j \leq n,\ 1 \leq k \leq m\}$.

Let $D = D_1 x..xD_N$ be the cartesian product of the $N \geq 2$ sets D_k, let R be a given set, and let

3.1 $F: D \dashrightarrow R$

be given. The sets D_j replace the domain sets P(1,2,3) from Section 2. There is no restriction on the choice of D_j -- it could be a set of binary, transitive rankings, probability measures, spaces of admissible strategies, function spaces of utility functions, or anything else. Indeed, the choice of D_j could even differ from agent to agent where, say, D_1 is a set of transitive rankings, D_2 is a set of probability measures, etc. The critical aspect is not what information is represented by D_j, but how the information is divided into equivalence sets. Replacing the division of P(1,2,3) into the subsets P(i,j) is the division of each set D_k into the *informational equivalence classes* $I^j(k) = \{I^j(k,1), I^j(k,2)\}$, j=1,2,3. The superscript j indices the three "independence conditions" while k identifies the voter or agent. The cartesian product $I^j = X_k\ I^j(k)$ replaces $\{P(i,s)\}^N$ in the independence condition Eq. 2.2.

Although I^j replaces the "independence" or IIA conditions of Arrow's theorem, these sets can be modified to include models with interdependency among voters' rankings or agents' actions. Such interdependency can be viewed as defining E, a proper subset of D. If E is given, then the sets I^j are restricted to E. More precisely, $I^j(k)$, k=1,..,N, is defined by $I^j \cap E$. For instance, $E=\{(c_1>c_2)^N, (c_2>c_1)^N\}$ requires *all* voters to have the same relative ranking of the candidates c_1 and c_2. With such an E, $I^1 = P(1,2)^N \cap E = E$ while $I^2 = P(2,3^N) \cap E = P(2,3)^N$. This E models the Pareto condition illustrated in Corollary 2.6.

The range, R, can be any set where the critical aspect is its subdivision into other equivalence sets. Let this subdivision be given by $R^j = \{R^j{}_1, R^j{}_2,..,R^j{}_k\}$, $k \geq 2$, j=1,2,3. The sets, R^j, replace the earlier subdivision of the range P(1,2,3) into the three classes of two sets, P(i,j).

The geometric conditions that provide the interplay between the flexibility in the domain with rigidity in the range are captured by the following definition.

Definition. **The triple $\{F, \{I^j\}, \{R^j\}\}$, j=1,2,3, satisfies the *binary overlap principle* if the following four conditions hold.**

1. For each j and each k= 1,..,N, the two subsets $\{I^j(k,1), I^j(k,2)\}$ are either disjoint or equal. For each j, there is at least one choice of k where the sets are disjoint.

2. (Domain overlap) For each choice of k and for each permutation (a,b,c) of (1,2,3), there is a permutation (u,v) of (1,2) so that each of $I^a(k,1)\cap I^b(k,u)$ and $I^a(k,2)\cap I^b(k,v)$ meet both $I^c(k,1)$ *and* $I^c(k,2)$. The *restricted domain condition* is where, for each permutation (a,b,c), the domain condition is satisfied for a unique permutation (u,v). (Thus, $I^a(k,1)\cap I^b(k,u)$ does not meet both I^c classes for both choices of u.) For at least one k, the restricted domain overlap conditions are satisfied.

3. (Range overlap) Let $R^{j\prime}$ denote some pair of subsets of R^j. For each permutation (a,b,c) of (1,2,3) and for each pair of subsets, there are two subsets in $R^{a\prime}\cap R^{b\prime}$ that do not meet the same subset of R^c.

4. (Invariance) a) For j = 1,2,3, F : I^j --> R^j.

b) For at least two choices of j, the image of F meets at least two of the R^j sets.

5. If the domain independence conditions are determined by an interdependency condition E, then for at least two choices of j where the image of F is nonconstant, $I^j\cap E = I^j$.

As in Section 2, the "dictator" is replaced with the more general concept of a function of a single variable.

Definition. **Let $\pi_k:D$ --> D_k be the natural projection mapping. The mapping F:D --> R can be *represented by a function of a single variable* if there exists a choice of k and a $g_k:D_k$ --> R so that $F = g_k(\pi_k)$.**

This definition does not require F to be a function of a single variable. For instance, suppose three voters rank the three candidates c_j, j=1,2,3, with the following modification of the Borda Count. The i^{th} ranked alternative for the j^{th}

voter is assigned $(3-i)10^j$ points. The tally for each candidate determines the ordinal ranking of the alternatives and defines the mapping
$F: \{P(1,2,3)\}^3 \dashrightarrow P(1,2,3)$. Although F is a function of all three variables, it can be identified with the identity mapping (dictator) $g_3: P(1,2,3) \dashrightarrow P(1,2,3)$. If $P^{\sim}(1,2,3)$ denotes all 13 rankings depicted in Figure 1, then F can be extended to a mapping $F: (P^{\sim}(1,2,3))^3 \dashrightarrow P^{\sim}(1,2,3)$ by assuming that when the i^{th} voter is indifferent between two candidates, each of these two candidates receives the obvious average of the assigned points. This choice of F creates *sequential dictators*; if the third voter is indifferent between two candidates, then the second voter decides the group ranking between them. If both the second and third voters are indifferent between the same two candidates, then the first voter decides. (This is generalized in Theorem 3, Section 4.)

The overlap conditions capture the essence of the geometric proof of our version of Arrow's theorem. Thus, in light of the proof of Theorem 1, Theorem 2 should be expected. The formal proof in Section 5 is just an abstract version of the proof in Section 2.

Theorem 2. Assume that $F: D \dashrightarrow R$ satisfies the binary overlap principle with the sets $\{I^j\}$ and $\{R^j\}$. When F is viewed as a mapping

3.2 $\quad F: \; I^1 \cap I^2 \cap I^3 \dashrightarrow R^1 \cap R^2 \cap R^3$,

there is an index k so that F can be represented by a function of a single variable, g_k.

Suppose the image of F meets the pairs $\{R^j{}_1, R^j{}_2\}$, j=1,2,3.. There are precisely two ways to define g_k, and each is uniquely determined by whether $I^j(k,1)$ or $I^j(k,2)$ is mapped to $R^j{}_1$. The index k satisfies the restricted domain condition and all three $I^j(k)$ classes have two disjoint elements. If no such index exists, then F doesn't exist.

Theorem 2 asserts that the tensions between the flexibility in the domain and rigidity in the range extend Arrow's theorem. Moreover, a new feature emerges. If the domain of each voter admits either too much flexibility or too much rigidity, as captured by the last sentence, then such an F doesn't exist even with only one voter. For F to exist, even as a dictatorship, restrictions on the domain are required. For most social choice examples, the restricted domain conditions are satisfied, but this need not be so for examples from probability and economics.

Applications of Theorem 2

Starting with Arrow's theorem, I'll illustrate the considerable flexibility offered by Theorem 2. To underscore which overlap feature is being discussed - the nature of F, the possible definitions for the domain, or the choice of the range - examples are selected to emphasize only that feature. To start, we extend the notation in Section 2. For the n candidates, $\{c_1,..,c_n\}$, let P(1,..,n) denote the set of all n! complete, binary transitive rankings without ties of these candidates. If A is a subset of these indices, then an element of P(A) consists of the $n!/|A|!$ rankings of P(1,..,n) that preserves the relative ranking of the candidates in A. P(A) is the obvious extension of P(i,j) where its elements are the $|A|!$ disjoint subsets of P(1,..,n). The first corollary extends Theorem 1 to any (finite) number of candidates and voters.

Corollary 2.1. Let $n\geq 3$, $N\geq 2$, and $F: (P(1,..,n))^N \longrightarrow P(1,..,n)$ be given. Suppose F is onto and that for each pair (i,j), F satisfies the independence condition $F: (P(i,j))^N \longrightarrow P(i,j)$. F can be represented by a function of a single variable that corresponds to either a dictator or to an anti-dictator.

Proof. Start with $I^1(k) = R^1 = P(1,2)$, $I^2(k) = R^2 = P(2,3)$, and $I^3(k) = R^3 = P(1,3)$. The overlap conditions are satisfied, so F is represented by a function of one variable on the domain $P(1,2)^N \cap P(2,3)^N \cap P(1,3)^N$. Next, let $I^1(k) = R^1 = P(1,2)$, $I^2(k) = R^2 = P(2,4)$ and $I^3(k) = R^3 = P(1,4)$. It follows from Theorem 2 that F can be represented by a function of a single variable over $P(1,2)^N \cap P(2,4)^N \cap P(1,4)^N$. Both of these domains include $P(1,2)^N$, so in both cases the same voter is the dictator or the anti-dictator. The proof is completed with the obvious induction argument.

The distinction between whether a dictator or an anti-dictator reigns can be determined with a montonicity condition, such as a pareto condition, on some pair or even by specifying the image of a single point.

Corollary 2.2. a. Suppose in addition to the assumptions in Corollary 2.1, it is known that $F((c_1 \succ c_2 \succ \ldots \succ c_n)^N)$ is in the P(1,n) class corresponding to $c_1 \succ c_n$. The function F can be represented by a dictator.

b. Let p be a profile in $P(1,..n)^N$. If the assumptions of Corollary 2.1

are satisfied and F can be represented by g_k, then, for any (i,j), the P(i,j) image of F(p) determines whether k is a dictator or an anti-dictator.

These corollaries extend the standard Arrow theorem. The next corollary permits tie votes to emerge. The main feature demonstrated by Corollary 2.3 is the flexibility offered by Theorem 2 by allowing each R^j to have more than two elements. For this statement, let $P^{\sim}(1,..,n)$ be the set of all complete, transitive, binary rankings of the n alternatives, even those with ties. If A is a subset of {1,..,n}, then an element of $P^{\sim}(A)$ consists of all of the rankings in $P^{\sim}(1,..,n)$ with the same relative transitive ranking - including possible tie votes - of the candidates in A. By admitting tie votes, the concept of a dictator is weakened. So, let g_k, a *limited dictator* over P(i,j), be where g_k is either constant valued over this pair, or where $c_i > c_j$ is mapped either to $c_i > c_j$ or to $c_i \doteq c_j$. A corresponding definition defines a limited anti-dictator. So, a limited dictator may not be able to get outcomes better than, say, $c_i > c_j$ and $c_i = c_j$.

Corollary 2.3. Let $n \geq 3$, $N \geq 2$, and $F: (P(1,..,n))^N \longrightarrow P^{\sim}(1,..,n)$ be given. Suppose for each pair (i,j), F satisfies the independence condition $F: (P(i,j))^N \longrightarrow P^{\sim}(i,j)$. If F is nonconstant for each pair, then F can be represented by a function of a single variable that corresponds to either a (limited) dictator or to a (limited) anti-dictator.

Proof. This corollary is proved with the same kind of induction argument used in the proof of Corollary 2.1. So, we only need to show that the new range, satisfies the range overlap conditions. Start with $R^1 = P^{\sim}(1,2)$, $R^2 = P^{\sim}(2,3)$, and $R^3 = P^{\sim}(1,3)$. We know that the strict rankings given by P(i,j) satisfy the range overlap conditions. So, it suffices to consider a pair with strict ranking and another pair with indifference. The set $\{c_1 > c_2,\ c_1 = c_2\} \cap \{c_2 > c_3,\ c_3 > c_2\}$ contains $\{c_1 > c_2 > c_3\}$ and $\{c_1 = c_2 > c_3\}$. Each of these sets are in different $P^{\sim}(1,3)$ sets. (See Figure 1.) Likewise, the intersection $\{c_1 > c_2,\ c_1 = c_2\} \cap \{c_2 > c_3,\ c_3 = c_2\}$ contains $\{c_1 = c_2 > c_3\}$ and $\{c_1 = c_2 = c_3\}$; each is in a different $P^{\sim}(1,3)$ set. Thus, the range overlap conditions are satisfied. By symmetry, the same conclusion holds for any triplet of indices. This completes the proof.

Corollary 2.3 admits many possibilities ranging from a dictator to a limited dictator where $c_i > c_j$ is mapped to itself iff $i<j$; otherwise it is mapped to $c_i = c_j$. If n=3, then the image of F consists of the four rankings $\{c_1>c_2>c_3, c_1=c_2>c_3, c_1>c_2=c_3, c_1=c_2=c_3\}$. By selectively relaxing the nonconstancy condition on F, all sorts of other situations emerge with different fiefdoms. For example, we could have a dictator over P(1,2,3) and a limited dictator over P(3,4,5). Such a division into fiefdoms works as long as no pair of candidates are shared by competing fiefdoms.

For good reasons, the independence conditions for social choice models usually satisfy an implicit monotonicity property; e.g., the group's relative ranking of c_i and c_j are determined only by the voters' relative rankings of these same two candidates. But, does such a tacit assumption contribute to the impossibility conclusions? Why not let the j^{th} voter's relative ranking of, say, c_1 and c_2 affect the group's ranking of, say, c_2 and c_3. (Such a condition captures some of the flavor of the Hurwicz-Schmeidler "kingmaker" [10].). Corollary 2.4 proves that nothing is gained from this. Also, it shows that the relationship between the domain and range independence conditions need not satisfy the tacit monotonicity assumptions standard in the social choice literature. Indeed, the form of the independence assumptions can change with the voter. (In Corollary 2.4a, if an index has a value greater than 3, then replace it with its remainder {1,2,3} when divided by 3. For instance, 7 is replaced with 1, and 9 is replaced with 3.)

Corollary 2.4. a. Let $N \geq 2$ and $F: (P(1,2,3))^N \longrightarrow P(1,2,3)$ be given. Let $I^j(k) = P(c_{k+j-1}, c_{k+j})$, j=1,2,3, k=1,..,N. Suppose that F is onto and satisfies the independence conditions $F: I^j \longrightarrow P(j,j+1)$. There is an index s (voter s) so that F can be represented by a function of a single variable, g_s. There are only two possible ways to define g_s.

b. Let $N \geq 2$, $n \geq 3$, and $F: (P(1,..,n))^N \longrightarrow P(1,..,n)$ be given. For each k=1,..,N, let $\pi_k(-)$ be a permutation of the indices {1,..,N) for the k^{th} voter, let $I^{j,s}(k)$ be the set $P(\pi_k(j), \pi_k(s))$, and let $I^{j,s} = X_k\, I^{j,s}(k)$, j,s = 1,..,N. If F satisfies the independence conditions $F: I^{j,s} \longrightarrow P(j,s)$ where F is onto, then there is an index ß (voter ß) so that F can be represented by a function of a single variable, $g_ß$. There are only two possible ways to define $g_ß$.

Trivially, the overlap conditions are satisfied, so the corollary follows immediately from Theorem 2. *The function of one variable need not be a dictator nor an anti-dictator.* For instance, in part a, if s=2, then one of the two possible definitions has g_2 taking $c_j > c_k$ to $c_{j+1} > c_{k+1}$; so, $g(c_1 > c_2 > c_3) = c_2 > c_3 > c_1$. If the range is replaced with $P^{\sim}(1,\dots,n)$ and the nonconstancy condition of F is relaxed, all sorts of other possibilities are admitted.

The Choice of F and Quasi-dictators

The next application of Theorem 2 underscores that F need not be a mapping; e.g., it could be a correspondence where R is the power set of some other set. Secondly, it illustrates that while F must be represented as a function of one variable over the domain $I^1 \cap I^2 \cap I^3$, it need not have this representation over the full domain D.

Example. Let $N \geq 2$ and let F be a correspondence with domain $P(1,2,3,4)^N$ with values in P(1,2,3,4). Let $I^j(k) = R^j = P(j,j+1)$ for j=2,3, and equal to P(2,4) for j=3. If F satisfies the invariance conditions $F: I^j \to R^j$, j=1,2,3, then, according to Theorem 2, F can be represented by a function of one variable over the domain $I^1 \cap I^2 \cap I^3$. But, this domain imposes no restrictions on the relative ranking of c_1 and c_2. Thus, it is consistent to define such an F where the relative ranking of c_1 and c_2 is determined by, say, a majority vote. So, the relative ranking of c_2, c_3, and c_4 must be determined by a particular voter - F is represented by a function of one variable over the intersection of the equivalence classes $I^1 \cap I^2 \cap I^3$ - but majority vote applies for the ranking of $\{c_1, c_2\}$.

This example and Theorem 2 explain why nondictatorial social welfare functions so often endow some agent with considerable power. Although the specified independence conditions may not force a dictator over all of D, they may force a dictator to emerge over the sets in $I^1 \cap I^2 \cap I^3$ - he is a *quasi-dictator* over the whole domain D. An illustration of this is in a paper by Gibbard, Hylland, and Weymark [4] where they show that a related nondictatorial function exists if all of the feasible sets include c_1. As we now know from Theorem 2, this is the general situation.

Flexibility in the Choice of the Domain

Because the domain overlap conditions are specified in set theoretic terms, there is considerable freedom in the modelling. With this flexibility, we could examine some natural questions about rankings, such as those pioneered by Weymark, concerning what happens when we relax assumptions of completeness, etc. As long as the geometry defined by these new restrictions and equivalence classes of rankings satisfy the overlap conditions, the usual dictatorial conclusions apply. But, instead of showing how some of Weymarks's nice results are subsumed by Theorem 2, I will emphasize other kinds of modelling flexibility admitted by this theorem. The feature illustrated in Corollary 2.5 is that the sets $I^j(k,1)$, $I^j(k,2)$ need not be disjoint for all choices of k. This feature admits flexibility in the modelling because $I^j(k,1) = I^j(k,2)$ means that the k^{th} voter has no influence over which R^j equivalence class is selected. (This is because there is only one $I^j(k)$ component for I^j. This forces the k^{th} voter to have a constant value over this equivalence set, so he has no influence on the outcome of $F: I^j \longrightarrow R^j$.) Corollary 2.5 illustrates how such modelling can be used with Theorem 2. Part a asserts there does not exist a social welfare function where the first agent determines the group ranking of c_1 and c_2, the second agent determines the ranking of c_2 and c_3, while the third agent determines the ranking of c_1 and c_3. Part b asserts that if we want each agent to be involved with only two pairs, there is a penalty that a surjective F does not exist.

Corollary 2.5. a. Let N=3 and $F: P(1,2,3)^3 \longrightarrow P(1,2,3)$ be given. Let $I^j(k) = P(c_j, c_{j+1})$ iff k=j; otherwise let $I^j(k,1)=I^j(k,2)$. Let $R^j=P^{\sim}(c_j, c_{j+1})$. If F satisfies the independence conditions $F: I^j \longrightarrow R^j$, j=1,2,3, then F has a fixed ranking for at least two of the pairs.

b. Let N≥2 and let $F: P(1,2,3)^N \longrightarrow P(1,2,3)$ be given. Suppose for each k, one of the $I^j(k)$ equivalence class is the whole set P(1,2,3) while the other two are $I^j(k) = P(j.j+1)$. If F exists, it is constant valued for at least two of the pairs.

Proof. a. The overlap conditions are satisfied, so if F is nonconstant over two or more binaries, then F can be represented by a function of a single variable. By assumption, this is impossible. This completes the proof of part a. Part b follows from the last sentence of Theorem 2.

A standard way to obtain a possibility theorem is to restrict the domain. Corollary 2.5 shows that overly strict restrictions can reintroduce dictatorial behavior. (See Theorem 4.) For instance if the first voter can vary between only $c_1 \succ c_2 \succ c_3$ and $c_2 \succ c_1 \succ c_3$; the second voter between $c_1 \succ c_2 \succ c_3$ and $c_1 \succ c_3 \succ c_2$, and the third voter between $c_1 \succ c_3 \succ c_2$ and $c_3 \succ c_1 \succ c_2$, then Corollary 2.5 proves that this either will not avoid impossibility assertions, or F is constant over two pairs. Such a result, where certain voters are concerned only about certain outcomes, contains the spirit of Sen's theorem on liberalism [21]. In Sen's formulation, two agents have the privileged status to determine the relative ranking of certain alternatives - presumably their own - while the other alternatives are represented only through a weak pareto condition. The following version of Sen's theorem illustrates how the set E, introduced in the beginning of this section, is used.

Definition. **Let $F: P(1,..,n)^N \longrightarrow P(1,..,n)$ be given. F satisfies the *weak pareto condition* for $\{c_j, c_k\}$ if $F((c_j>c_k)^N) = c_j>c_k$ and $F((c_k>c_j)^N) = c_k>c_j$. Namely, when everyone has the same relative ranking of these two alternatives, F preserves this relative ranking.**

The weak pareto condition is not an independence condition, but, with the appropriate E set and Theorem 2, it does define an I^j set. Thus, its connection with the standard Arrow theorem becomes apparent - both results form the same kind of axioms.

Corollary 2.6. **Let $n \geq 3$, $N \geq 2$. Assume that A_1, A_2, A_3 are subsets of the indices $\{1,..,n\}$ such that $|A_j| \geq 2$ and any two of these sets have precisely one index in common. There does not exist an $F: P(1,..,n)^N \dashrightarrow P^{\sim}(1,..,n)$ such that: 1) the $P^{\sim}(A_j)$ image of F is nonconstant and it depends solely upon the j^{th} voter's rankings of the A_j candidates, j=1,2, and 2) F satisfies the weak pareto condition for the pairs of alternatives in A_3.**

If A_1 and A_2 have more than one element in common, then, an argument like that given in Corollary 2.5, shows that such an F doesn't exist. An induction argument, similar to that used in Corollary 2.1, extends this statement to a larger number of A_j sets.

Proof. Without loss of generality, assume that c_1 is the common element of A_1 and A_2, c_2 is the element in A_2 and A_3, while c_3 is in A_3 and A_1. Let E = $(c_3>c_1)^N \cup (c_1>c_3)^N$. Set E is a proper subset of $P(1,3)^N$ requiring all voters to agree about the relative ranking of these two alternatives. The following sets are defined on E. Let $I^1(1) = P(1,2)$, $I^2(2) = P(2,3)$, $I^3(j) = P(1,3)$, and all other $I^k(j)$ sets equal to D_j. The interdependency given by set E affects only the $I^3(j)$

sets - a voter's ranking must agreed with that of the other voters. The overlap principle, with set E, is satisfied, so it follows from Theorem 2 that if such an F exists, then it can be represented by a function of a single variable. Namely, the ranking of one particular voter determines the outcome of F. This contradicts the first assumption, so the theorem is proved.

Incidentally, this proof illustrates that *any* interdependency condition modelled with an E satisfying Theorem 2 is not sufficient to escape the penalties of Arrow's theorem. By examining the proofs of Theorem 2 and 4 in Section 5, one can extend the definition of E so that it is "best possible". In this manner, one can characterize the kinds of interdependency conditions that admit a possibility theorem.

So far, all of my examples are based on the geometry of P(1,..,n). This is not necessary. To illustrate, Corollaries 2.7, 2.8 show that everything extends to function spaces. The function spaces are the spaces of utility functions, and the motivating example is the model of Kalai, Mueller, and Satterthwaite [11]. Let E^c_+ be the positive orthant of a c-dimensional Euclidean space, $c \geq 2$, and let the *space of utility functions* be $U=\{u: E^c_+ \to E$: u is a smooth function, and at each point in E^c_+ the gradient of u points to the interior of $E^c_+.\}$ These utility functions are concave, monotonic, and they do not admit a satiation point.

A classical objective is to find a group utility function; to find an $F: U^N \to U$ that satisfies certain properties. If F exists, its image, u_F, defines a complete, binary, transitive relationship over E^c_+. If for $\mathbf{x} \in E^c_+$, $u_F(\mathbf{x})$ is defined in terms of $(u_1(\mathbf{x}),..,u_N(\mathbf{x}))$, then F satisfies the definition given below for pointwise binary independence where $S = E^c_+$. Indeed, by setting $S = E^c_+$ in the next definition, we recover the condition used by Kalai, Mueller, and Satterthwaite to show that such an F leads to a dictator. But, can a dictatorship be eliminated by using other choices of S, say, by requiring agreement only over some small subset of points rather than all of E^c_+? Instead of defining a ranking over all of E^c_+, how about letting the utility functions define such a ranking only over a specified set S?

Definition. Let S be a subset of E^c_+, and let P(S) be the set of all complete, binary, transitive rankings on the set S. Let $F_S: U^N \to P(S)$ be given. F_S satisfies the *pointwise, binary independence* condition over S if the following condition

holds. For for all pairs of points x_1 and x_2 from S, and for any two choices $u^j=(u_1{}^j,..,u_N{}^j)$, j=1,2, from U^N, if $u^1(x_k)=u^2(x_k)$, k=1,2, then $F_s(u^1(x_k)) = F_s(u^2(x_k))$, k=1,2.

Some restrictions need to be imposed upon the set S.

Definition. A set of point, S in $E^c{}_+$, is *nonmonotonic* if for x, y $\in$ S, some component of x is larger than the corresponding component of y, and some component of y is larger than the corresponding component of x. A set S is *full* if i) there is at least one nonmonotonic pair of points in S, and ii) for each nonmonotonic pair of points, there is a third point in S so that the triplet is nonmonotonic.

It is natural to impose a monotonicity condition on F such as requiring when $u^1(\mathbf{x}) = u^2(\mathbf{x})$ and $u^1(\mathbf{y}) \geq u^2(\mathbf{y})$ that the relative rankings of **x** and **y** with $F_s(u^1)$ cannot rank **y** lower than $F_s(u^2)$. A less restrictive way is to define the j^{th} agent's independence sets for points $\{x_i, x_k\}$ as $I^{i,k}(j,1)$ = {u in U: the level set of u passing through x_i passes below x_k.} while the definition for $I^{i,k}(j,2)$ is that the level set passes above x_k. Notice that $I^{i,j}(k,1) = I^{j,i}(k,2)$. The independence condition is

3.2 for each pair of nonmonotonic points (x_i, x_j) from S, $F: I^{i,k} \text{-->} P(x_i, x_k)$.

Corollary 2.7. Let S be a full subset of $E^c{}_+$ with at least three points. Suppose $F_s: U^N \text{-->} P(S)$ satisfies the pointwise binary independence condition, the independence condition 3.2, and that F_s is not constant over at least two nonmonotonic pairs of points of S. F_s can be represented by a function of a single variable that corresponds to either a dictator or an anti-dictator.

Can a nondictatorical F_s be constructed with different kinds of economic information? For instance, the price mechanism depends, in part, on the gradients of the utility functions. The next definition permits gradients and other information to be used by replacing a point from S with a subset determined by a point in S. In this way, it describes a "general binary independence condition" that permits F_s to be defined in terms of any kind of differential information coming from u as well as the behavior of u at neighboring points. Indeed, the definition of the "B sets" even permits the ranking of two points to be based on information coming from elsewhere in E^c_+.

Definition. **Let S be a subset of E^c_+. F_s:U^N-->P(S) satisfies the *general binary independence condition* if for all finite subset of points A={x_1,..,x_t}, $x_i \in S$, and all u^1 and u^2 from U^N, the following conditions hold:**

i) There are nonempty, pairwise disjoint sets {$B^A(j,k)$}, j=1,..,N, k=1,..,t, in E^c_+ such that if x_i and x_k are nonmonotonic, then, for each j, any point from $B^A(j,i)$ and any point from $B^A(j,k)$ are nonmonotonic.

ii) For each pair (x_i,x_k) from A, if $u^1{}_j$ and $u^2{}_j$ both agree on $B^A(j,i)$ and $B^A(j,k)$, j=1,..,N, then $F_s(u^1)$ and $F_s(u^2)$ coincide on x_i and x_j.

A pointwise binary independence condition is a special case where $B^A(j,k)$ = {x_k}. Another special case would be where F_s is based on the values of u and its derivatives at a point. Here, (with a slight modification of the definition) open sets about each point in S are used to define the germ of the utility functions. The choice of $B^A(j,k)$ can vary with the point, so different types of information can be employed. For instance, at x_1 we may use the value of the utility function, and at x_2 and x_3, the gradient of the utility function. The independence conditions that replace the usual monotonicity conditions are defined in the following manner. For a triplet A={x_1,x_2,x_3}, let $I^j(k,1)$ be the set of all utility functions for the k^{th} agent that have level sets passing through $B^A(k,j)$ but below $B^A(k,j+1)$, while $I^j(2)$ are the utility functions with a level set passing through $B^A(k,j)$ but above $B^A(k,j+1)$. The independence condition is

3.3 for all triplets of nonmonotonic points F: I^j---> P(x_j,x_{j+1}).

Corollary 2.8. Let S be a full subset of E^c_+ with at least three points. Suppose that F_s:U^N-->P(S) satisfies a general binary independence condition, the independence condition 3.3, and that F_s is not constant valued over at least two nonmonotonic pairs. F_s can be represented by a function of a single variable that corresponds to either a dictator or an anti-dictator.

Corollaries 2.7, 2.8 illustrate that the problem of dictatorial behavior is not induced by what information is used, but by the division of information. In these corollaries, the dictatorial conclusions are direct consequences of an attempt to create an F_s that preserves monotonicity.

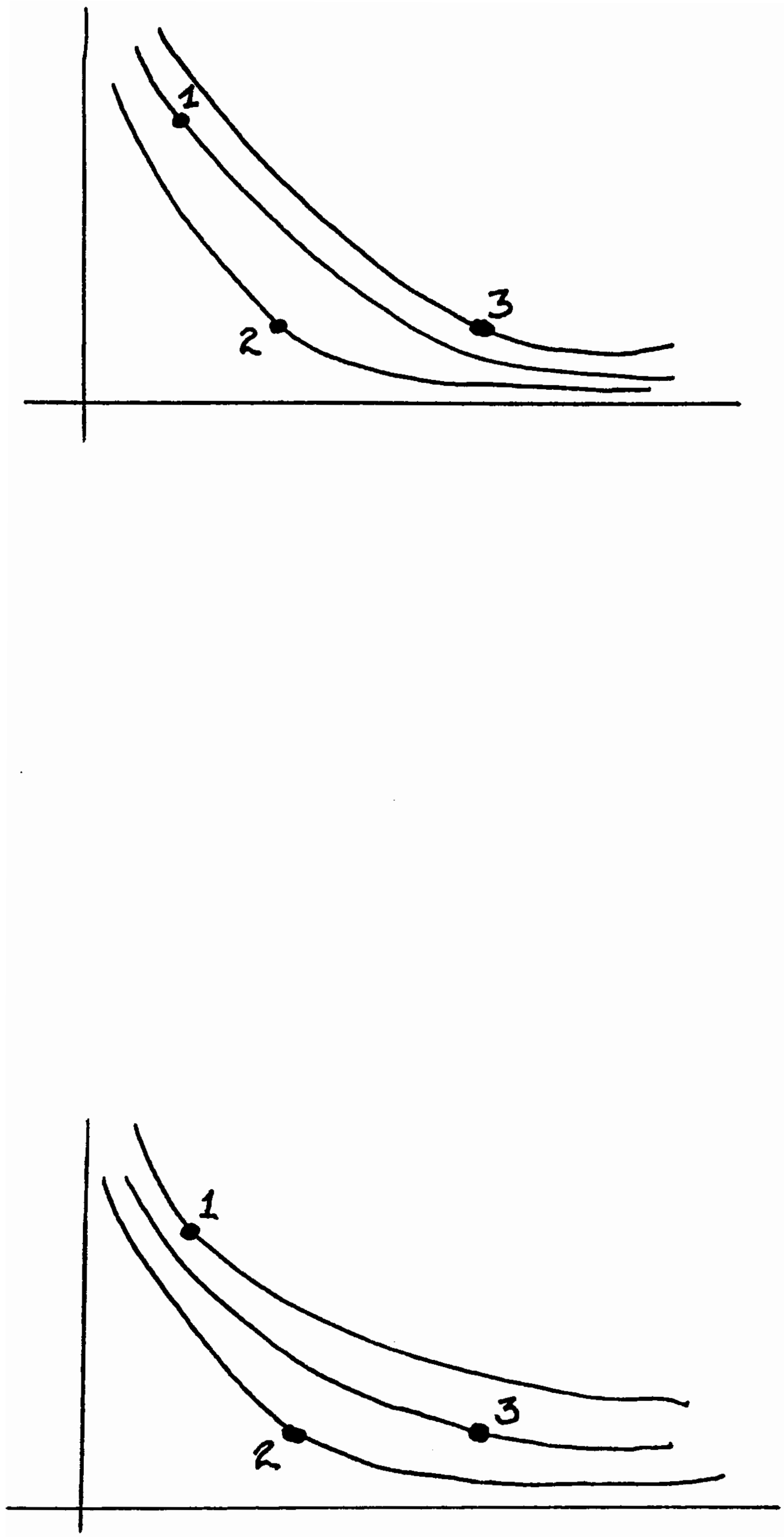

Figure 2

Outline of the Proof. As with Corollary 2.1, the proof is by induction over triplets. Let A= $\{x_1, x_2, x_3\}$ be a triplet of points that are nonmonotonic. All that needs to be proved is that the domain overlap conditions are satisfied. The proof is outlined for c=2; the extension to c>2 is immediate. The proof of the domain overlap condition is indicated in Figure 2. Because the sets $B^k(j,i)$ are nonmonotonic, with some choice of the indices, they can be arranged in a fashion similar to that given in this sketch. Now, for A = $\{x_1, x_2, x_3\}$, it is easy to see why $I^1(j,2) \cap I^2(j,1)$ meets both $I^3(j,1)$ and $I^3(j,2)$. In the first sketch, there are three level sets for the same utility function u. The first level set passes through $B^A(j,1)$, above $B^A(j,2)$, but below $B^A(j,3)$. This is possible because of the nonmonotonicity assumption. Such a u is in $I^1(j,2)$. To ensure u is in $I^2(j,1)$, the second level set passes through $B^A(j,2)$. Because c=2 and because level sets cannot cross, this level set is forced to be below both $B^A(j,1)$ and $B^A(j,3)$. There is still flexibility in the design of u to have a third level set passing through $B^A(j,3)$. Again, geometric constraints force this level set to be above $B^A(j,1)$, so it is in $I^3(j,2)$. It only remains to show there is a different utility function from $I^1(j,2) \cap I^2(j,1)$ that is in $I^3(j,1)$. This is shown in the second sketch where the first level set passing through $B^A(j,1)$ now passes above both $B^A(j,2)$ and $B^A(j,3)$. This forces the level set passing through $B^A(j,2)$ to be below $B^A(j,1)$ (for geometric reasons) and $B^A(j,3)$ (because it is in $I^2(j,1)$). These two restrictions force the level set passing through $B^A(j,3)$ to have the properties of membership for $I^A(j,1)$. Similar arguments apply to show that the domain independence conditions hold. The conclusion now follows from Theorem 2.

Some Applications to Economics

We now use the independence conditions to characterize the informational requirements of economic procedures. To see the idea, suppose we wnat to know whether we can construct a group decision procedure, based on binary comparisions, that always is immune to a Dutch Book procedure. (See, for instance, [22].) Thus, we want to know whether the ordering of the pairs is of any consequence. Can such information be combined so it always yields a transitive ranking of the alternatives? If the answer is yes, the procedure defines a social choice function that satisfies certain independence conditions, so Theorem 2 may apply. Alternatively, for a given economic procedure, we may want to determine whether certain kinds of *partial information* are adequate to capture aspects of the procedure; i.e., of what use is this partial information with respect to the procedure? If the choice of partial information defines independence conditions, Theorem 2 may apply. For instance, when analyzing a solution concept for a standard trading or exchange model among three agents, can we can recover aspects of the solution by knowing what would happen in all the possible binary trades among the three pairs of agents? Suppose externalities are introduced into a classical allocation procedure. Is the information about how *pairs* of externalities effect the classical solution of any use when considering the total effect?

To illustrate this line of thought (and to demonstrate another feature of the overlap principle), consider the aggregate excess demand function for a simple trading society with neo-classical utility functions. For a given price vector, can we obtain qualitative information about the aggregate excess demand function from the relative demand for the pairs of commondities? To be more specific, at a given price, the components of the aggregate excess demand function determine an ordinal ranking of the commodities in a natural fashion; the larger the demand for a commodity, the more favored it is. It is reasonable to expect that information about this ranking can be obtained by finding for each pair of commodities, considered at these same prices, which one is the more desired. Such a problem can be analyzed in several ways; I'll use Theorem 2. The outcome is that the information about pairs can be unreliable - even for a single agent.

Suppose the three commodities are $\{c_1, c_2, c_3\}$. The qualitative information we seek is the direction of the aggregate excess demand function. So, let the two $R^{i,j}$ classes, $\{R^{i,j}(c_i>c_j), R^{i,j}(c_j>c_i)\}$ be determined, respectively, by whether there is a positive demand for c_i or for c_j. It is easy to show that the range overlap conditions are satisfied. The domain for each agent is the set of neoclassical utility functions. The k^{th} agent is given an initial endowment $(w^k{}_1, w^k{}_2, w^k{}_3)$, $w^k{}_j>2$.. For a specified price, (p_1,p_2,p_3), $p_j>0$, the $I^{i,j}(k)$ sets are defined in the following manner: $I^{i,j}(k,c_i>c_j))$ is the set of all utility functions so that, when the remaining commodity is held fixed, the excess demand function at the price (p_i,p_j) has a net trade between 1 and 2 units in favor of c_i. If the approach of comparing binary information gives qualitative information about the aggregate excess demand function, then, for the given initial endowments and price, $F:\{U\}^N \dashrightarrow \cap\{R^{i,j}\}$ satisfies the independence condition that, for each pair (i,j), $F: I^{i,j} \dashrightarrow R^{i,j}$. Clearly, F is not determined by one agent, so the futility of such a binary approach follows from Theorem 2 once we establish that the domain overlap conditions are satisfied.

It is trivial to show that the domain overlap conditions are satisfied. Indeed, a new feature arises; the intersection of *each* $I^{1,2}$ class with *each* $I^{2,3}$ class meets both $I^{1,3}$ class. Thus, unless restrictions are imposed on the class of utility functions, *no agent satisfies the restrictive domain conditions*, so the last sentence of Theorem 2 applies. To indicate the basic ideas, I will outline why $I^{1,2}(k,c_1>c_2) \cap I^{2,3}(k,c_2>c_3)$ meets *both* $I^{1,3}$ classes. In the plane $z=w^k{}_3$ in E^3 consider the circle of radius 1.5. In this plane, the line passing through the initial endowment with normal vector (p_1,p_2) meets the circle in precisely two points. Choose the one where $c_1>w^k{}_1$. At this point, construct a level set of the utility function such that the first two components of its gradient is a positive multiple of (p_1,p_2). Use a similar construction for a level set of the utility function in $I^{2,3}(k,c_2>c_3)$. So far we've specified two level sets at two disjoint points using only partial information about the gradient. The same construction for $I^{1,3}(k)$ specifies two more points. All four points are disjoint; indeed, they do not even lie in the same plane, and only the last two are on a line passing through the initial endowment. So, it is trivial to construct a utility function with a level set satisfying the point information at the first two points and at either one (but, clearly not both) of the remaining four points. This completes the proof.

Because no agent satisfies the restrictive domain conditions, such an F doesn't exist even for one agent - such information is not reliable even to determine a single person's demand function. The same fate holds for any choice of F based on similar information. This can be illustrated with the next example that uses a kind of information often considered in economics. Suppose F is a function of the gradient of the utility function at x_j where the image of F satisfies the range conditions. Instead of using the monotonicity condition given in Corollaries 2.7, 2.8, suppose the related domain independence conditions are $I^j(k,1)$ = {u in U: the j^{th} component of the gradient of u is larger than the other two components} where $I^j(k,2)$ is the set where some other component is larger than the j^{th} component. Again, not only are the domain overlap conditions satisfied, but no agent satisfies the restricted domain conditions. Thus, such an F cannot be defined even for one agent.

Contingency Tables in Statistics

Theorem 2 can be used with issues from statistics. For instance, by treating each data point as a "voter", if follows immediately from Theorem 2 that there isn't a statistical method yielding a transitive ranking of three or more alternatives that respects binary comparisons. As another statistic question subsumed by Theorem 2, consider the problem of collapsing of contingency tables to obtain the marginal probabilities. To describe the problem, suppose a new vaccine is proposed to cure the common cold. This vaccine is to be tested in Evanston and in Ann Arbor. At each site, a test group and a control group are used and the probability of a patient regaining health is computed. Let x_E and x_A denote, respectively, the difference between these values for the two groups as measured at Evanston and at Ann Arbor. So, $x_E > 0$ means that in Evanston the new vaccine had a better success rate than the standard treatment. Finally, suppose the test results from both locations are sent to a central location and aggregated where y is the difference between probabilities of success with the vaccine and with the standard treatment. We want to compare signs of the triplet (x_E, x_A, y). *Simpson's paradox* is when the signs (+,+,-) occur; the vaccine was successful both in Evanston and Ann Arbor, but not in the aggregate.

Simpson's paradox is an annoying consequence of the combinatoric rules of conditional probabilities. Can some other measure be invented to avoid Simpson's paradox? Namely, can we find a mapping $F = (\mu_E, \mu_A, \mu)$, depending on all of the information, where the outcome assumes all sign combinations *except* (+,+,-) and (-,-,+). In this manner, the new measure avoids the pitfall of Simpson's paradox. (See [5] for some measures.) Now, we want this measure to be useful on its own at each site, so we want the sign of μ_E, μ_A, μ to depend, respectively, only on the sign of x_E. x_A, and y. This defines a binary independence condition, and it is easy to show that the range overlap conditions are satisfied. Using the results given in [16] concerning Simpson's paradox, it follows that the "one voter" satisfies the domain overlap condition, but not the restricted domain conditions. (All signs for (x_E, x_A, y) are possible.) Hence, according to the last sentence of Theorem 2, such a measure does not exist.

The Range Condition: Social Choice Functions

Theorem 2 offers flexibility in choice of the range, or outcome space. Because the range overlap conditions are in a set theoretic form, the range could be any space - a function space, a space of probability distributions, a lottery, subsets of alternatives, etc. I decided to illustrate the basic ideas with a familiar model - social choice mappings. Let the candidates $\{c_1, \ldots, c_n\}$ be given, let A_j, j=1,..,p, be a subset of these candidates, let FS = $\{A_1,..,A_p\}$ be the set of *feasible sets* of A, and let R be the set of all nonempty subsets of the candidates. A social choice correspondence, $F:FS\times\{P(1,..,n)\}^N \dashrightarrow R$, assigns to each feasible set and preference profile a nonempty subset of the feasible set. Namely, for A_j in FS and $\mathbf{x}$ in $P(1,..,n)^N$, $F(A_j,\mathbf{x})$ is a nonempty subset of A_j. F satisfies the condition of *independence of infeasible alternatives* if for A_j in FS, and if for $\mathbf{x}$ and $\mathbf{y}$ in $P(1,..,n)^N$ that agree on A_j (they are in the same $P(A_j)$ class), then $F(A_j,\mathbf{x}) = F(A_j,\mathbf{y})$. A social choice correspondence F is *strictly nonconstant* over A_j if the image of $F(A_j,-)$ has at least two disjoint, nonempty subsets. F satisfies the *choice axiom* if, for all $\mathbf{x}$, $F(A_j,\mathbf{x}) = F(\{1,..,n\},\mathbf{x})\cap A_j$. The definition of a correspondence of a single variable, a dictator, and an antidictator are the obvious ones.

The difference between a social welfare function and a social choice function is that a social welfare function determines the group ranking of the alternatives, while the social choice function selects only the set of "best" candidates. So, if a social welfare function exists, the related social choice function selects the top ranked alternative. This means that the social choice function is *realized* by the social welfare function. An important theorem by Hansson [6] specifies what kind of feasible sets permits a social welfare function can be constructed to realize a given social choice function. Thus, whenever his conditions are satisfied, there is a relationship between results for social choice and social welfare. While the conditions given below can be used to invoke Hansson's theorem, the conclusions are proved directly to illustrate how Theorem 2 includes social choice models.

Corollary 2.9. For A = $\{c_1,..,c_n\}$ where $n\geq3$, let the set of feasible sets include A and all two element subsets of A. Let F be a social choice correspondence that satisfies the condition of independence of infeasible alternatives, the choice axiom, and is strictly nonconstant over the pairs of alternatives. F can be represented by a function of a single variable.

Of course, Corollary 2.9 could be modified to obtain a result with the flavor of Corollary 2.4 and some of the other statements.

Proof. As with Corollary 2.1, the proof is by induction. Choose three candidates, say $\{c_1, c_2, c_3\}$. Assume that $A_j = \{c_j, c_{j+1}\}$. Let $I^j(k) = P(j,j+1)$, and define R^j to be $\{\{c_j, A_j'\}, \{c_{j+1}, A_j'\}\}$ where A_j' is the complement of A_k in A. (So, if n=3, $R^1 = \{\{c_1, c_3\}, \{c_2, c_3\}\}$.) It follows from the condition of independence of infeasible alternatives and the choice axiom that $F: I^j \dashrightarrow R^j$. It is easy to show that the sets R^k satisfy the range overlap conditions. For instance, again in n=3, $R^1 \cap R^2 = \{\{c_1, c_3\}, \{c_2, c_3\}\} \cap \{\{c_2, c_1\}, \{c_3, c_1\}\} = \{\{c_3\}, \{c_1\}, \{c_1, c_3\}, \{c_2, c_3\}\}$. The first two sets are in different R^3 subsets. From this, the conclusion follows from Theorem 2.

Gibbard - Satterthwaite

As a last illustration of Theorem 2, I'll give a proof of the Gibbard - Satterthwaite theorem that differs from the standard method depending on the distribution of power. For simplicity of exposition, restrict attention to three alternatives. (In much the same manner as described for the earlier corollaries, the results extend to all values of $n \geq 3$.) Recall that if $A = \{c_1, c_2, c_3\}$ is the set of candidates, then a *voting scheme* is a function $F: \{P(1,2,3)\}^N \dashrightarrow A$. For $\mathbf{x}_j \in P(1,2,3)$, $c_i >_j c_k$ iff this is the relative ranking of the two candidates in $\mathbf{x}_j$. A voting scheme is *manipulable* iff there exists $\mathbf{x}, \mathbf{y} \in \{P(1,2,3)\}^N$ that differ only in the j^{th} component and $F(\mathbf{y}) >_j F(\mathbf{x})$. The j^{th} component for $\mathbf{y}$, $\mathbf{y}_j$, represents the j^{th} voter's misrepresentation of his true ranking. We say that j manipulates F at $\mathbf{x}$ with $\mathbf{y}_j$.

Corollary 2.10 (The Gibbard-Satterthwaite Theorem). Let F be a voting scheme from $(P(1,2,3)\}^N$ to $\{c_1, c_2, c_3\}$ where the range of F is onto. F is either dictatorial or manipulable.

Proof. Assume F is not manipulable; we show it is dictatorial. For the pair $A_{i,j} = \{c_i, c_j\}$, let $R^{i,j} = \{\{c_i, A'_{i,j}\}, \{c_j, A'_{i,j}\}\}$. For instance, $R^{1,2} = \{\{c_1, c_3\}, \{c_2, c_3\}\}$. The range overlap conditions are satisfied. Corollary 2.10 follows from Theorem 2 once we show that the $I^{i,j}(k)$ sets are $P(i,j)$. This proof illustrates how these implicitly defined independence conditions are extracted from "level set" and monotonicity properties of F. To emphasize the ideas, the proof is divided into three lemmas. First, note that $F^{-1}(c_j) \neq \emptyset$ for all j.

Lemma 1. If $F(\mathbf{x}) = c_j$, and if $\mathbf{x} = (\mathbf{x}_1, .., \mathbf{x}_N)$ varies only in the k^{th} component where this variable, $\mathbf{y}_k$, is in the same $P(i,j)$ class, then F remains in the same $R^{i,j}$ class. If when $\mathbf{y}_k$ changes $P(i,j)$ classes, the image of F changes $R^{i,j}$ classes, then the change is monotonic; e.g., if $\mathbf{y}_k$ moves from $P(c_j > c_i)$ to

$P(c_i > c_j)$, then the image of F moves from $\{c_j, c_k\}$ to $\{c_i, c_k\}$.

Proof. Without loss of generality, let k = 1. Suppose the first part of the lemma is false because the image of F changes $R^{i,j}$ classes when this voter changes to $\mathbf{y}_1'$ where both $\mathbf{x}_1$ and $\mathbf{y}_1'$ are in the same P(1,2) class. If this voter's relative ranking is $c_i > c_j$, he can manipulate the outcome of F at $\mathbf{x}$ with $\mathbf{y}_1'$; otherwise he can manipulate the outcome of F at $(\mathbf{y}_1', \mathbf{x}_2, \ldots, \mathbf{x}_N)$ with $\mathbf{x}_1$. Both contradict the assumption that F is not manipulable. Similarly, if changing the P(i,j) classes has the reversed effect on the image, then either one way, of the other, the first agent can manipulate the outcome. If this agent's relative ranking is $c_i > c_j$, then F is manipulated at $\mathbf{x}$ with $\mathbf{y}_1$; otherwise F is manipulated at $(\mathbf{y}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ via $\mathbf{x}_1$.

Definition. The change of a ranking $\mathbf{x}_i$ to $\mathbf{y}_i$ is called a *level set change with respect to c_j* iff for each choice of k, $c_k > c_j$ in $\mathbf{x}_i$ iff the same relative ranking holds in $\mathbf{y}_i$.

In other words, in a level set change, all of the candidates ranked above c_j in $\mathbf{x}_i$ are also ranked above c_j in $\mathbf{y}_i$ and vice versa. So, c_j remains at the same level and all candidates originally above (below) remain above (below). For instance, $c_1 > c_2 > c_3$ and $c_2 > c_1 > c_3$ are level set changes with respect to c_3, but not with respect to c_j, j=1,2..

Lemma 2. If $F(\mathbf{x}) = c_j$, and $\mathbf{y}$ differs from $\mathbf{x}$ only in the k^{th} voters ranking which is a level set change with respect to c_j, then $F(\mathbf{y}) = c_j$.

Proof. Assume the lemma is false, and that $F(\mathbf{y}) = c_i$. Because the k^{th} agent made a level set change, this agent's relative ranking of c_i and c_j remains the same. Thus, this voter can either manipulate F at $\mathbf{x}$ with $\mathbf{y}$-$\mathbf{x}$ or at $\mathbf{y}$ with $\mathbf{x}$-$\mathbf{y}$.

Lemma 3. For each i and j, $F: I^{i,j} \to R^{i,j}$.

Proof. If this lemma were false, there would be a profile $\mathbf{x}$ where $F(\mathbf{x}) = c_1$, and a profile, $\mathbf{y}$, in the same $P(1,2)^N$ class as $\mathbf{x}$, where $F(\mathbf{y}) = c_2$. Because we can go from $\mathbf{x}$ to $\mathbf{y}$ with a series of individual ranking changes in the same P(1,2) class, it follows from Lemma 1 that there is an intermediate profile, $\mathbf{z}$, in the same $P(1,2)^N$ class, such that $F(\mathbf{z}) = c_3$. First, assume that all rankings in $\mathbf{x}$ with $c_1 > c_3$ have the ranking $c_1 > c_3 > c_2$ or $c_2 > c_1 > c_3$. If this isn't so, it can be achieved with c_1 level set changes. According to Lemma 1, if P(1,2) invariant changes alter the outcome to c_3, it is due to P(1,3) changes for a subset of these voters; let $V^{1,3}{}'$ be the indices of these voters, and let $\mathbf{z}'$ be the new profile. (Notice, these are c_2 level set changes where $c_1 > c_3$ becomes $c_3 > c_1$.) Now, to change the image from c_3 to c_2, certain voters keep their rankings in the same P(1,2) class, but they must change P(2,3) from $c_3 \succ c_2$ to $c_2 \succ c_3$. Let $V^{2,3}$ be the indices of these voters and let **y** be the profile. We can assume that $V^{1,2}$ and $V^{2,3}$ are disjoint. This is because the voters with the **x** ranking of $c_2 \succ c_1 \succ c_3$ have the wrong P(2,3) ranking to make this change. For the other voters in $V^{1,2}$, the P(1,3) change results in $c_1 \succ c_3$. If this doesn't change the F image to c_1 (the only possibility), then this voter wasn't needed in $V^{1,2}$. If it does, then, according to Lemma 1, the next P(2,3) change cannot change the outcome to c_2.

Change **y** to **w** by using a c_2 level set change with all indices in $V^{1,2}$. According to Lemma 2, F(**y**) = F(**w**) = c_2. Profile **x** differs from **w** only for the rankings of the $V^{2,3}$ voters. So, the changes from **x** to **w** only involve P(2,3) changes in the same P(1,2) and the same P(2,3) classes. Thus, according to Lemma 1, F(**w**) is in $\{c_1, c_3\}$. This contradiction completes the proof.

Corollary 2.10 follows from Lemma 3 and Theorem 2. (Of course, we could have streamlined the proof by showing, for example, that the $V^{i,j}$ sets are singletons.) Notice that the drive for each agent to maximize the outcome of F isn't needed; we only used the associated monotonicity for F. Consequently, the essence of the Gibbard-Satterthwaite theorem extends to situations outside of strategic manipulations as well as the other extensions admitted by the flexibility in the choice of the range that is admitted by Theorem 2. Finally, it is worth noting that in [9,17], the level sets defining the message correspondences are based on integrability conditions. This involves "Lie bracket conditions"; they measure the change, or differential, of one vector field with respect to another. By examining the usual motivating examples for the Lie bracket, you will find a strong relationship with the proof of Lemma 3. This is not coincidental; such a construction occurs whenever the independence conditions for discrete models are implicitly defined; e.g., a related argument is in the last paragraph of the proof of the first part of Theorem 2. This kind of argument can be abstracted to form the discrete version of the Lie bracket condition.

4. Extensions and Possibility Theorems.

Theorem 2 can be extended by altering the domain and the range overlap conditions. There are many ways this can be done; several create interesting theories of independent interest. Rather than attempting to be complete, I'll illustrate the basic ideas with some possibility theorems. In much the same fashion other extensions, say, to quasi-transitive or acyclic rankings can be made.

The first extension is to model "indifference". The definition is based on the geometric properties illustrated in Figure 1. The key feature is that if an agent is indifferent between c_1 and c_2, then his (c_2, c_3) ranking uniquely determines his (c_1, c_3) ranking. This deprives him of the freedom to vary between P(1,3) classes that is essential to prove Theorem 2. Consequently, we should expect other voters to have a say in the outcome. This happens.

Definition. The *domain overlap conditions with indifference* for the k^{th} voter admits added sets, $I^j(k,3)$, j=1,2,3. For each j, the added set is disjoint from each of the two original I^j sets. The domain overlap condition for this new set is that, for each permutation (a,b,c) of (2,3) and s = 1,2,3, $I^a(k,3) \cap I^b(k,s)$ meets precisely one $I^c(k)$ set, $I^c(k,u)$, where s=3 iff u = 3. Furthermore, if $I^a(k,3) \cap I^b(k,u)$ meets $I^c(k,v)$, u<3, then $I^b(k,u) \cap I^c(k,v)$ meets all three of the I^a sets. Call $I^j(k,3)$ the *indifference set*.

Theorem 3 extends the version of Arrow's Theorem that admits preferences with indifference. To generalize the idea of *sequential dictators*, we need a stronger condition on the range sets.

Definition. Assume that each R^j, j=1,2,3, has two elements. Assume for each permutation (a,b,c) of (1,2,3) that there are four sets in $R^a \cap R^b$; two of them satisfy the range overlap conditions and each of the other two meet both R^c sets. Both of the latter two are of the form $R^a{}_1 \cap R^b{}_u$ and $R^a{}_2 \cap R^b{}_v$ for some permutation (u,v) of (1,2). The range sets R^j, j=1,2,3, are said to satisfy the *flexible range overlap* conditions.

Most of the choices of R^j used in this paper satisfy the flexible range overlap condition. This is true for P(i,j) as well as the sets $\{c_j, A_j'\}$. The term "flexible" refers the flexibility in the range classes similar to that admitted by the domain overlap conditions. The restriction allowing R^j to have only two sets is not necessary.

Theorem 3. Let $N \geq 2$. Let $M = \{F: D \to R$: the I^j classes satisfy the domain

overlap conditions with indifference, the R^j classes satisfy the flexible range overlap conditions, and $F: I^j \rightarrow R^j$ is non constant for at least two values of j.}

a. There is an $F \in \mathcal{M}$ that cannot be represented by a function of one variable over $I^1 \cap I^2 \cap I^3$.

b. Suppose $F \in \mathcal{M}$ is not a function of a single variable. There exists a permutation of indices, $(\beta(1), \beta(2), .., \beta(s))$, $s \leq N$, and mappings $g_{\beta(i)}: D_{\beta(i)} \rightarrow R$ with the following property. If $x_{\beta(1)} \notin I^j(\beta(1),3)$, then the R^j image of F is determined by the image of $g_{\beta(1))}$. Inductively, if $x_{\beta(\alpha)} \in I^j(\beta(\alpha),3)$, $\alpha=1,..,i<s$, but $x_{\beta(\alpha+1)} \notin I^j(\beta(\alpha+1),3)$, then the R^j image of F is determined by $g_{\beta(\alpha+1)}$. There are two possible choices for each $g_{\beta(s)}$.

An unusual example illustrating Theorem 3b is where the P(i,j) outcome of $F: \{P(1,2,3)^{\sim}\}^N \rightarrow P(1,2,3)$ is dictatorially determined by the first voter iff her ranking is not indifference. If she is indifferent about some pair P(i,j), then this ranking is determined by the second voter's P(i+1,j+1) ranking. Other examples could involve utility functions, etc.

Theorem 3 demonstrates that several voters can help determine the outcome of F when the flexibility in the domain is curtailed. This is accomplished here by *adding* sets to the admissible domain that don't satisfy the domain overlap conditions. In other words, there are situations where Arrow's theorem doesn't apply because it is *too restrictive*. A standard way to obtain possibility theorems is to create the rigidity in the domain by *subtracting* from the domain by imposing restrictions on what are admissible preferences. Theorem 4 characterizes these domain restrictions. Essentially, Theorem 4 states that the overlap principle captures the boundary between possibility and impossibility conclusions.

To motivate Theorem 4, recall that puzzling phenomena I briefly mentioned after Corollary 2.5. Certain domain restrictions permit possibility theorems. It seems reasonable to expect that with stricter restrictions, models permitting even more voter participation will result. This need not happen; the stronger restrictions can force a return to an impossibility conclusion! For instance, with axioms much like those studied by Kalai, Mueller, and Satterthwaite, Donaldson and Weymark [2] obtained a possibility theorem with an independence condition that models a form of "free disposal of goods". Yet, when they changed the independence condition in a natural but slight manner, an impossibility theorem now emerged. To see why behavior like this occurs, consider Arrow's theorem where voter ß can assume any ranking *except* $c_1 > c_2 > c_3$. According to Theorem 4, we obtain a possibility theorem. Now consider what happens if we further restrict ß's rankings to $\{c_1 > c_3 > c_2, c_3 > c_1 > c_2\}$. Now, a dictator, other than ß, is obtained. This is because with the original restriction, the $I^j(ß)$ classes are the usual $P(i,k)$ classes minus the one specified ranking. Because only this one ranking is missing, the domain overlap conditions cannot be satisfied, so as Theorem 4 asserts, a possibility conclusion holds. But, by imposing the stronger restrictions on ß's rankings, a new set of $I^j(ß)$ classes emerges. One class still is $P(1,3)$, but the two new I^j equivalence classes are singletons - the entire set. Theorem 2 applies because the stronger restrictions (which correspond to more relaxed independence conditions) create a new division of independence classes in the domain that satisfy the domain overlap condition. The next definition, which is needed for Theorem 4, captures this implicit behavior.

Definition. Suppose $\{I^j(k)$, $j=1,2,3$, satisfy the restricted domain overlap conditions for D_k. A *restriction* for the k^{th} voter is a proper subset, C_k, of $I(k) = I^1(k) \cap I^2(k) \cap I^3(k)$. A restriction C_k *implicitly defines a new set of informational equivalence classes* $\{J^j(k)\}$, $j=1,2,3$, if $x \in I^j(k,s) \cap C_k$ iff $x \in J^j(k,s) \cap C_k$, $j = 1,2,3$, $s = 1,2$.

As an example, suppose $I^{i,j}(k) = P(i,j) = \{P(c_i > c_j), P(c_j > c_i)\}$ and $C_k = \{c_1 > c_2 > c_3, c_1 > c_3 > c_2, c_3 > c_1 > c_3\}$. (These are regions {A,B,C} in Figure 1.) C_k implicitly defines the overlap classes $J^{i,j}(k) = I^{i,j}(k)$ for $(i,j) = \{(1,3), (2,3)\}$, and $J^{1,2}$ is the singleton equivalence class of the total set. In other words, because the restriction C_k forces one of the $I^{1,2}(k)$ sets to be empty, this class could be replaced with a singleton. Notice that with the restrictions, neither the original nor the implicitly defined classes satisfy the domain overlap conditions. This is because $J^{1,2}(k) \cap J^{2,3}(k) = J^{2,3}(k) \cap C_k = \{\{c_1 > c_2 > c_3\}, \{c_1 > c_3 > c_2, c_3 > c_1 > c_2\}\}$, so there aren't two sets in this intersection where each meets both $J^{1,3}(k)$ sets.

Theorem 4. Suppose the informational equivalence classes and the division of the range for given D and R satisfy the overlap principle. Assume restrictions are imposed on at least one of the voters that satisfies the restricted domain overlap conditions, say voter 1. Assume that all of the implicitly defined informational

equivalence classes generated by C_1 either fail to satisfy the overlap conditions or they have only two classes with two disjoint nonempty sets, say $J^j(1)$, $j=1,2$, where at least two of the four sets in $J^1(1) \cap J^2(1)$ are empty. There exists a function F from the restricted domain of D to R that satisfies the independence conditions $F: I^j \rightarrow R^j$, $j=1,2,3$, where F is non-constant for at least two values of j and F cannot be represented as a function of a single variable.

In other words, as long as the restrictions don't implicitly define a new class of informational equivalence classes that require, via Theorem 2, a dictatorial situation, then a non-dictatorial F exists.

Corollary 4.1. Let n=3 and $I^{i,j}(k) = R^{i,j} = P(i,j)$. If C_1, the restrictions on voter 1, are such that $C_1 \cap I^{i,j}(1,s) \neq \emptyset$ for all (i,j), s = 1,2, then there exists a mapping from this restricted domain that cannot be represented by a function of a single variable.

Example The restriction $C_1 = \{c_1 \succ c_2 \succ c_3,\ c_3 \succ c_2 \succ c_1\}$ admits a social welfare function that is not governed by an (anti) dictator. This is because each P(i,j) set meets C_1. On the other hand, the restrictions $C_1' = (c_1 \succ c_2 \succ c_3,\ c_2 \succ c_1 \succ c_3\}$ cannot avoid a dictatorial situation. This is because C_1' meets only one set in each of P(2,3) and P(1,3). As a result, both of these classes can be replaced with a singleton equivalence class of everything. The overlap conditions are satisfied and Theorem 2 holds.

Even though Theorem 4 admits a possibility conclusion, the resulting F need not be a model of participatory democracy; the remaining conditions still impose sharp restrictions on which F's are admitted. To see this, suppose restrictions are imposed only on the first voter where $C_1 = P(1,2,3) - \{c_1 \succ c_3 \succ c_2\}$ (region B in Figure 1). If F is not determined dictatorially, then the first voter must influence the outcome of at least *two* pairs. This is because if the voter has no influence over a pair, then the associated F *implicitly redefines the associated informational equivalence class* as a singleton. If this is true for two pairs, then the newly defined classes trivially satisfy the domain overlap condition, and Theorem 2 applies. Now, the constraint C_1 permits flexibility of movement in the P(1,3) and P(2,3) classes, so a variation of the argument for Theorem 1 shows that voter 1 determines the outcome of these pairs. Thus, the definition of F is thrust upon us; the first voter (anti) dictatorially determines the P(1,3) and the P(2,3) outcome. With one exceptional case, the outcome is either $P(c_3 > c_1) \cap P(c_3 > c_2) = \{C,D\}$, or $P(c_1 > c_3) \cap P(c_2 > c_3) = \{A,F\}$. In either situation, the P(1,2) outcome can be determined in any desired manner by the voters, say, with a majority vote. The one exceptional situation is when the first voter has the ranking $c_2 > c_3 > c_1$. Here the $P(1,3) \cap P(2,3)$ image is either the anti-dictatorial outcome {B}, or the dictatorial {E} - which one occurs uniquely defines how this voter determines the image of F. Thus, if the first voter has the ranking $c_2 > c_1$, he determines the P(1,2) outcome. Otherwise the P(1,2) image can be determined by a majority vote (or by any other means) of the remaining voters.

With this construction, it is easy to image other situations that could occur with the appropriate domain restrictions. For instance, situations can occur where the first voter uniquely determines the P(1,2) and P(2,3) outcomes, the second determines the P(1,4), P(1,5) outcomes, ... If this process does not uniquely determine the F outcome in P(1,..,n), then other voters can make the final determination. Such a construction results from an iterative application of Corollary 4.2.

Corollary 4.2. a. Let $N \geq 2$. Suppose the informational equivalence classes and the division of the range for given D and R satisfy the overlap principle. Assume restrictions, C_1, are imposed on voter 1 and that voter 1 satisfies the restricted domain overlap conditions. Suppose C_1 admits a permutation (a,b,c) of (1,2,3) so that $I^a(1) \cap I^b(1)$ contains the two sets where each meets both $I^c(1)$ sets. If F cannot be represented by a function of one variable, then the first voter determines the R^a and R^b outcome.

b. Let N=2, and suppose that restrictions C_1 and C_2 are given. Suppose for two different permutations (a(k),b(k),c(k)), that $I^{a(k))} \cap I^{b(k))}$ contains the required two sets that meet both $I^{c(k)}$ sets but the other sets in this intersection do not. If F is not a function of a single variable, then for one choice of k, the k^{th} voter determines the $R^{a(k)}$ and the $R^{b(k)}$ outcomes.

Non Standard Independence Conditions - Game Theory

I've already pointed out that the information used by each voter could change; for instance, one voter's domain could be ordinal rankings, a second voter's domain could be based on a probability distribution, while a third is given by utility functions. The next feature I will illustrate is that each independence class of each voter could represent a different type of information; the goal is to determine whether the interaction among the features are compatible. In this manner, for instance, one could examine results of the type shown by Samuelson where a transfer of initial endowments can adversely affect the final allocation. I decided to illustrate this feature by recapturing some of Hurwicz and Schmeidler's (HS) nice results about inferior Nash equilibria. (In this way we relate HS's results to Arrow's theorem.)

HS studied games, or allocation processes with a finite number of alternatives, where, for each profile, there is a Nash equilibrium which also is Pareto optimal. Such an allocation procedure is *acceptable* [10,p. 1447]. HS showed that, for two agents, an acceptable allocation function must be dictatorial, but that this same conclusion does not hold for three or more agents. Yet, they proved that a non-dictatorial solution for more than two agents requires a "kingmaker". With three players, the role of the kingmaker is to determine which of the remaining two agents is to be the dictator. Because my objectives are to illustrate Corollary 4.2, I'll show here only why the dictatorship occurs for N=2. (The proof and the comments motivating Corollary 4.1 and 4.2 suggest the reasons a kingmaker occurs.)

Consider allocation procedures with two possible outcomes, {a,b}, and two agents. The range space is *not* just the two outcomes; it is each outcome associated with how each agent honestly ranks the alternatives. For instance, typical outcomes are $\{a, a \succ_1 b, b \succ_2 a\}$, $\{a, b \succ_1 a, b \succ_2 a\}$, and $\{b, a \succ_1 b, a \succ_2 b\}$. The first outcome implies that a is the selected alternative, a is the first agent's top ranked alternative, and it is the second agent's bottom ranked alternative. The second and third outcomes do not occur because of the *pareto condition.* For instance, in the second outcome, both agents prefer an available alternative, b. This leaves 6 outcomes that do satisfy the pareto condition, and they are represented in Figure 3. In this triangle, the edge to the left represents the first voter's true ranking and defines the two R^1 classes, the edge to the right represents the second voter's true rankings and defines the two R^2 classes, while the bottom edge denotes the selected alternative and determines the two R^3 classes. Because the mapping, f, has only four variables, there are four image points, so it is not obvious whether the range overlap conditions are satisfied. By the pareto assumption, f must have an image in regions B and E. Because there is an outcome

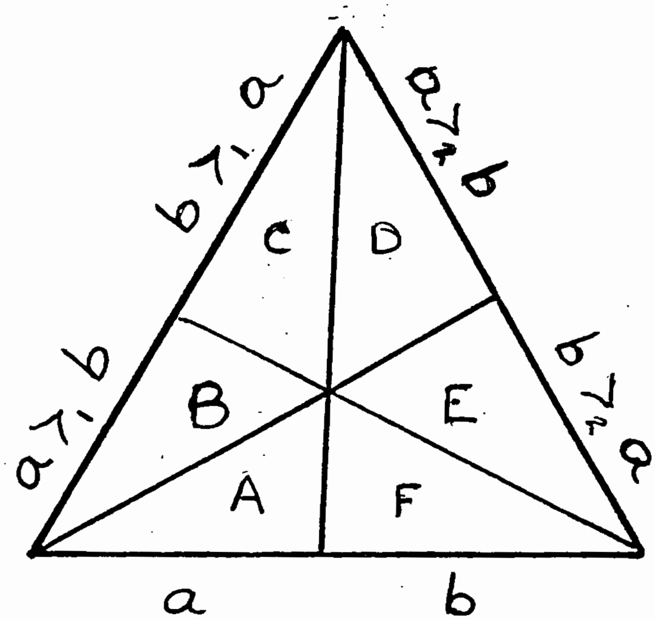

Figure 3

for each profile, there is an image point in {C, D} and in {A, F}. If the images are {A, D}, then, trivially, the first agent is a dictator. Equivalently, if they are {F, C}, then the second agent is a dictator. For either of the remaining two cases, the range overlap conditions are satisfied.

The domain for each agent, D_k, is represented by a similar triangle, but there is a slight difference in the interpretation. The bottom edge, dividing the equilateral triangle into two right triangles, corresponds to this agent's two strategies - will she state a or b is her top ranked alternative? Of course, this depends on the choice of the allocation function and on her opponent's strategy. Therefore, this axis corresponds to what appears to be her top choice based on her strategy choice. This defines the $I^3(k)$ classes. Obviously, the $I^j(j)$ classes agree with the R^j classes, j=1,2. The remaining equivalence class for each agent consists on what appears to be the true belief of the opponent. For instance, the point $(a_1, b>_1 a, b>_2 a)$ represents the first voter using a strategy to achieve a_1 when his true first choice is b, and it appears that the second agent's true first choice also is b. Such points are not admitted both under the Nash and Pareto assumptions. Thus, the representation of the triangle holds. Augment the allocation function, $f:\{a,b\}^2 \dashrightarrow \{a,b\}$, to define the mapping $F:D_1 x D_2 \dashrightarrow R$, in the natural manner. Namely, F map $I^j(j)$ to R^j, j=1,2, and f maps the strategies to the R^3 class. By construction, $F:I^j \dashrightarrow R^j$, j=1,2,3. If f is not dictatorial, we've already shown that the range overlap conditions are satisfied. The domain overlap conditions remain.

If f is not dictatorial, there are only two choices for the image set of F. Without loss of generality, assume it is {A,B,C,E}. We need to use the Nash and Pareto conditions to determine what sets are, and are not in C_k. By the Pareto condition, $\{B,E\} = \{(a_k, a>_1 b, a>_2 b), (b_k, b>_1 a, b>_2 a)\} \in C_k$. Because of the Nash condition, regions $\{C, D\} \in C_1$. It is obvious why $D \in C_1$. To see why $C = \{a_1, b>_1 a, a>_2 b) \in C_1$, note that the first voter using the strategy to get b results in a. If by changing strategy, the agent could get b, the original outcome wouldn't be a Nash equilibrium. Thus, C also is an admissible strategy. Similar arguments show that C_1 contains all of the regions *except* $(b_1, a>_1 b, b>_2 a)$ because this would change the outcome to b, and this is a personally worse outcome. Likewise, C_2 = {B,C,D,E,F}.

Based on the restrictions C_k, Corollary 4.2 holds. Consequently, either f is dictatorial, or (according to Corollary 4.1) two of the R^j classes of F are determined by one agent. Obviously, these two classes cannot be R^1 and R^2, so one of them must be R^3. This returns us to the dictatorial situation because this agent determines the {a,b} outcome.

For two agents and several alternatives, the ideas remain the same. If there are more than two agents, there are differences in the construction. Still, based on what has been shown, intuition suggests (and supporting details prove) that when a voter determines the outcome for two classes, one could determine which of the other voters prevails, and then this designated voter selects among two classes. This last voter is a dictator; the first is a HS kingmaker.

Range Overlap

If the range overlap conditions are not satisfied, flexibility is introduced into the range. A possibility theorem emerges.

Corollary 4.3. In the statement of Theorem 4, assume that the domain overlap conditions are satisfied and at least two voters satisfy the restricted domain conditions. Suppose the range overlap conditions are not satisfied because, for some permutation (a,b,c) of (1,2,3), there are *not* two sets in $R^a \cap R^b$ in different sets of R^c. There exists a mapping $F: D \to R$ satisfying the independence conditions $F: I^j \to R^j$, j=1,2,3, that cannot be represented as a function of a single variable over $I^1 \cap I^2 \cap I^3$.

Outline of the proof. Assume that both $R^1{}_1 \cap R^2{}_1$ and $R^1{}_2 \cap R^2{}_2$ meet both R^3 classes. Because the range overlap conditions are not satisfied, either both $R^1{}_1 \cap R^2{}_2$ and $R^1{}_2 \cap R^2{}_1$ are in the same R^3 class, or at least one of them meets both R^3 classes. The first cannot occur. For instance, suppose both intersections miss the $R^3{}_1$ class. That is $\{R^3{}_1 \cap R^1{}_1\} \cap R^2{}_2$ and $\{R^3{}_1 \cap R^1{}_2\} \cap R^2{}_1$ are empty. This contradicts the assumption that at least one of the $R^3{}_1 \cap R^1{}_j$ classes must meet both R^2 classes. In the latter setting, if all four sets meet the two sets in R^3, then let the first agent's ranking determine the R^1 and R^2 outcome by mapping $I^j(1,k)$ to $R^j{}_k$, j=1,2, k=1,2, and let the R^3 outcome be determined by any desired method; say a majority vote, or the second voter's ranking of this set. The remaining situation is where one of the sets, say $R^1{}_2 \cap R^2{}_1$ meets $R^3{}_1$ but not $R^3{}_2$. First, suppose there is a set in $I^1(1) \cap I^2(1)$ that meets only one of the I^3 sets. With a relabelling of the indices, we can assume that $I^1(1,2) \cap I^2(1,1)$ meets only

$I^3(1,1)$. Then, let the F be defined by having $I^j(1,k)$ mapped to $R^j{}_k$ for j=1,2, k=1,2. If the first voter's ranking is in $I^3(1,1)$, then the image is $R^3{}_1$. Otherwise, let the second voter's choice of $I^3(2,k)$ be mapped to $R^3{}_k$. Finally, suppose all sets in $I^1(1)\cap I^2(1)$ meet both $I^3(1)$ sets. The same definition of F applies.

5. PROOFS

The purpose of this section is to prove the main theorems.

Lemma 4. Let $I^j(k)$, j=1,2,3, satisfy the domain overlap condition. For each permutation (a,b,c) of (1,2,3), each the sets in $I^a(k)\cap I^b(k)$ meet at least one $I^c(k)$ sets.

Proof. Suppose false. Without loss of generality, assume that $I^1(1,1)\cap I^2(1,1)$ does not meet $I^3(1)$. Namely, $\{I^1(1,1)\cap I^2(1,1)\}\cap I^3(1,j) = \emptyset$ for j = 1,2. In turn, this means that $I^1(1,1)\cap I^3(1,j)$ can't meet $I^2(1,1)$ for j=1,2. This contradicts the domain overlap assumption.

Proof of Theorem 2. Let L^j = {k: for s $\neq$ k, there is an $x_s{}'$ in a $I^j(s)$ class so that $F(x_1{}',\ldots,x_k,\ldots x_N{}')$ changes R^j classes as x_k changes $I^j(k)$ classes}. Namely, this is a situation where when only the k^{th} voter changes classes, the R^j outcome changes. F is non-constant over at least two sets R^j, so, from the range overlap condition, for at least two choices of j, L^j is nonempty.

Suppose there are at least two indices in the union $\cup_j L^j$. Without loss of generality, assume that $1 \in L^1$ and $2 \in L^2$. For this to occur, voters 3 to N, may need to be in specific $I^j(k)$ classes, j=1,2. According to the lemma, these voters can satisfy both conditions simultaneously. Hold these domain points fixed. For voter 1 to be in L^1, $x_2{}'$ must be in a specific $I^1(2)$ class, say $I^1(2,u)$. Likewise, for 2 to be in L^2, $x_1{}'$ must be in $I^2(1,v)$ for a specific choice of v. For k=1,2, choose the $I^3(k,\beta(k))$ class so that $I^2(1,v)\cap I^3(1,\beta(1))$ meets both $I^1(1)$ classes and $I^1(2,u)\cap I^3(2,\beta(2))$ meets both $I^2(2)$ classes. According to the domain overlap conditions, this is possible.

According to the construction, as x_k changes I^k classes, the image of F changes R^k classes, k=1,2. Assume that $R^{k\prime}$ is the pair of images of F caused by this change of x_k, k=1,2. According to the construction, all outcomes in $R^{1'} \cap R^{2'}$ occur with appropriate choices of x_1 and x_2. But, according to the range overlap condition, two sets in this intersection meet different R^3 classes. This forces the R^3 image to vary even though each x_k remains in a fixed $I^3(k)$ class, k=1,..,N. This contradiction proves that each L^j has only one index, say 1.

To complete the proof, we need to show that for any choice of x_k, k=2,..,N, the R^j image of $F(x_1,..,x_N)$ depends only on which $I^j(1)$ class contains x_1. If false, then there are $\{x^{k'}\}$, $\{x_k''\}$, k≥2, so that $F(x_1,x_2',..,x_N')$ and $F(x_1,x_2'',..,x_N'')$ are in different L^j classes. By holding x_1 fixed and going through the various permutation of interchanging x_k' with x_k'', the image of F must change R^j classes. This forces an index other than 1 to be in L^j. This contradiction completes the proof.

Next, we show that there are only two ways g_1 can be defined. Assume the images are $R^j{}_i$, i=1,2, and choose the indices on the range sets so that $F(I^1(1,u)) = R^1{}_u$, u=1,2, and that $R^1{}_1 \cap R^2{}_1$ is in $R^3{}_1$ but not in $R^3{}_2$. Thus, $R^1{}_1 \cap R^2{}_1 \cap R^3{}_2$ is empty. To define the $I^2(1,v)$ image, note there is a choice of v so that $I^1(1,1) \cap I^2(1,v)$ meets both $I^3(1)$ classes. Let v′ be the other index. Then, $F = g_1$ must map $I^2(1,v')$ to $R^2{}_1$. If not, then F must map $I^2(1,v)$ to $R^2{}_1$. Because $I^1(1,1) \cap I^2(1,v)$ meets both $I^3(1)$ classes, it follows from the invariance property of F that $R^1{}_1 \cap R^2{}_1$ meets both R^3 classes. This contradiction proves the assertion. The determination of the $I^3(1)$ image is done in the same fashion. Note that this proof shows that the image of g_k cannot be constant valued over any R^j. Thus, each $I^j(k)$ must have two disjoint elements.

It remains to prove the last sentence of Theorem 2. Suppose voter 1 always satisfies the domain overlap condition for all permutations of (1,2,3) and both permutations of (u,v). This means that in the argument of the preceding paragraph, there are two choices of $I^2(1,v)$, and each choice gives rise to the contradiction. Thus, F cannot be defined. Next, assume that voter 1 determines the outcome of F, but one of the $I^j(1)$ classes, say $I^3(1)$, consists of only one equivalence class. The same argument as given above shows that the R^3 outcome will vary. This creates a contradiction because $g_1(I^3(1))$ is only one R^3 class. (On the other hand, if a restriction, C_1, is imposed on $I(1) = I^1(1) \cap I^2(1) \cap I^3(1)$ that removes one of the four sets, then g_1 is well defined. If C_1 has only two sets from I^1 but each $I^j(1,s)$, j,s = 1,2, then a non-dictatorial F can be defined.)

Proof of Theorem 3. First we establish that there are choices of F that can not be expressed as a function of a single variable over the total domain. So, assume the domain and range sets are specified where the restricted domain conditions are satisfied for agents 1 and 2. Furthermore, assume that the indicing is such that $R^1{}_1 \cap R^2{}_1 \cap R^3{}_1 \neq \emptyset$. We will define an F that is a function of the two variables, x_1 and x_2 from I^j to R^j, j=1,2,3.

As shown in Theorem 2, there are only two ways to define a mapping g_k from $\{I^j(k,1), I^j(k,2)\}$ to R^j. For k = 1,2, let g_k be one of these choices. Define F in the following manner. If $x_1 \notin I^j(1,3)$, then the R^j outcome of F is given by the R^j image of g_1. If $x_1 \in I^j(1,3)$ and $x_2 \notin I^j(2,3)$, then the R^j image of F is the R^j image of g_2. If $x_1 \in I^j(1,3)$ and $x_2 \in I^j(2,3)$, then the R^j image of F is $R^j{}_1$.

It remains to show that F is well defined. If either agent 1 never is indifferent, or if when agent 1 is indifferent over all sets, agent 2 is not indifferent over any set, then there is no difficulty with the definition of F. The potential problems are on the complement of this subset of the domain. To start, suppose if agent 1 is indifferent over one set, say, she is in $I^1(1,3)$, and she isn't indifferent over one other set, say $I^2(1,u)$, $u \neq 3$. According to the domain overlap conditions, agent 1 is in $I^3(1,v)$, $v \neq 3$, and $I^2(1,u) \cap I^3(1,v)$ meets all three $I^1(1)$ classes. In turn, this forces the R^2 and R^3 images (determined by agent 1) to be such that $R^2 \cap R^3$ meets both R^1 classes. (If not, then, g_1 is not well defined for agent 1. This is because if agent 1 is in $I^2(1,u) \cap I^3(1,v)$ she still can vary between the two $I^1(1,w)$ classes, $w \neq 3$. Now, if the image contains only one R^1 class, this forces g_1 to be constant over $\{I^1(1,1), I^1(1,2)\}$ – which leads to a contradiction.) The choice of R^1 class is determined by agent 2.

The remaining situation is if both agents are indifferent over some I^j class, say I^1. The same argument given in the preceding paragraph shows that if one of the agents is not indifferent over some other I^j class, then there is flexibility in the choice of R^1 class. One has been selected. If both agents are indifferent over two I^j classes, and, hence, indifferent over all three classes, then the image is well defined. This completes the proof.

The remaining part of the part a is to show that the above construction captures the spirit of all possible choices of F. Namely, any F can be represented by a function of a single variable over the non-indifference sets. The proof of this assertion is simple if I had required F to be nonconstant over $X_k\{I^j(k,1), I^j(k,2)\}$ for at least two choices of j. Because I did not, I need to show that F can't be constant valued over I^j except when everyone but the j^{th} agent is indifferent, and then the j^{th} agent is a dictator for R^j.

Lemma 5. Let L^j = {k: for s ≠ k, there is an x_s' in $I^j(s)$ so that $F(x_1',..,x_k,..,x_N')$ changes R^j classes as x_k varies between $I^j(k,1)$ and $I^j(k,2)$}. Suppose that $|U_jL^j|>1$ and that u,v are in U_jL^j. There exists a choice of j, say j=3, so that the ranking for one of these agents, say v, need not be in $I^3(v,3)$ when u influences the R^3 outcome.

Proof. Suppose $j \in L^j$, j= 1,2, and that the lemma does not hold for these values of j. Thus, whenever j influences the R^j outcome, the other agent, k, must be in $I^j(k,3)$. Of course, the I^j rankings of agents $k \geq 3$ may be specified. According to the domain overlap conditions, the restrictions for agents 3 to N can be satisfied for both j classes. Also, by the indifference overlap conditions, agent 1 can vary between sets $I^2(1,3) \cap I^1(1,u)$, u = 1,2, while agent 2 varies between sets $I^1(2,3) \cap I^2(2,v)$, v=1,2. This forces both agents to be in $I^3(k,1) \cup I^3(k,2)$. The same argument used in the proof of Theorem 2 proves that the the R^3 outcome changes even though all voters remain in fixed I^3 classes. This proves the lemma.

To prove the theorem, assume that $j \in L^j$, j = 1, 2. Furthermore, assume that there is a profile where agent 2 need not be in $I^1(2,3)$ when agent 1 can influence the R^1 outcome. (According to Lemma 5, such profiles can be found with a relabelling of indices.) Now, suppose there is a profile where agent 1 need not be in $I^2(1,3)$ when agent 2 influences the R^2 outcome. It follows from the domain overlap conditions with indifference that whatever are the requirements on agents 3 to n, they can be simultaneously satisfied. Thus, the essence of the problem is the same as in the proof of Theorem 2, and the same contradiction is arrived at. This means that agent 1 must be indifferent (and there may be added constraints on the other agents) when agent 2 has an influence on the R^2 outcome. This means that $1 \in L^2$. The rest of the proof, to find the ordering on the indices that defines the sequential dictators, is the obvious induction and ordering argument using

Lemma 5.

Proof of Theorem 4. Suppose the restriction on the domain is imposed on agent 1. The definition of F depends on which sets are omitted from I(1). The following lemma identifies each set in this intersection in a useful manner.

Lemma 6. Assume that the three I^j sets satisfy the restricted domain condition and the domain overlap conditions, and for each choice of j, I^j consists of two disjoint classes. For each set, Z, in I(1), there is a permutation (a,b,c) of (1,2,3) so that Z is a singleton in $I^a(1) \cap I^b(1)$, but A is not a singleton in $I^a(1) \cap I^c(1)$ or in $I^b(1) \cap I^c(1)$. Index c is called the "pivotal index" for Z.

Example: For Z = B = $\{c_1 > c_3 > c_2\}$, the pivotal index corresponds to the class P(1,2). As a quick way to determine the pivotal index, notice from Figure 1 that the two regions adjacent to this ranking region, B, all lie in one of the P(1,2) classes, but this group {A,B,C} does not lie in only one P(i,j) class for any other choice of (i,j).

The proof of the lemma is much the same as that of Lemma 4. Notice that for each choice of Z, there are two permutations, but both give rise to the same pivotal index.

Assume that the restrictions are imposed on voter 1, and let Z be one of the sets that is *not* in C_1. The first assertion is that, *with a possible relabelling of the indices and with a possible change of choice of* $Z \not\in C_1$, *we can assume that* $j = 1$ *is the pivotal index for Z and that* $C_1 \cap I^j(1,s) \neq \emptyset$ *for* $j = 2, 3$, $s = 1, 2$. To see this, assume that 1 is the privotal index for Z. Now, by definition, Z is not a singleton in $I^1(1) \cap I^2(1)$ nor in $I^1(1) \cap I^3(1)$. If the other term in each intersection is in C_1, then, by use of the domain overlap conditions, it follows that the assertion is satisfied. So, suppose either one, or both intersections have no terms in C_1. If both intersections fail to meet C_1, then C_1 meets only one of the $I^1(1)$ classes, so $I^1(1)$ can be replaced with $J^1(1)$ - the singleton equivalence class of everything. If one other class fails to have C_1 meet both sets, then it too can be replaced with the singleton equivalence class.

Here the overlap conditions are trivially satisfied, so this cannot occur. Thus, C_1 meets both $I^j(1,s)$, $j = 2,3$, $s=1,2$ classes, and two of the sets in $I^j(1) \cap I^3(1)$ are not in C_1. This means that the assertion holds.

The remaining situation is that for one choice, say $I^1(1) \cap I^2(1)$, the set accompanying Z is in C_1, but in $I^1(1) \cap I^3(1)$, the set accompanying Z, Y, is not in C_1. The pivotal index for Y is 2. We already know, from this construction, that the set accompanying Y in $I^2(1) \cap I^1(1)$ is not in C_1. If the set accompanying Y in $I^2(1) \cap I^3(1)$ is not in C_1, then we are in the same situation analyzed above for Z, so the assertion holds with Y and 2 in place of Z and 1. If this set is in C_1, then we have elements of C_1 in both $I^1(1)$ and both $I^2(1)$ classes. This completes the proof of the assertion.

Choose the indices on the $I^j(1)$ classes so that before the restrictions are imposed, $I^2(1,s) \cap I^3(1,s)$ meets both $I^1(1)$ classes. Likewise, choose the indices in the range so that $R^2{}_s \cap R^3{}_s$, $s=1,2$, meets both R^1 classes. Choose the indices on $I^j(1)$ so that, before the restrictions, $\alpha = I^1(1,1) \cap I^2(1,1) \cap I^3(1,2) \neq \emptyset$ and $\beta = I^1(1,2) \cap I^2(1,2) \cap I^3(1,1) \neq \emptyset$. Define F so that the R^j image of F is $R^j{}_s$ iff x_1 is in $I^j(1,s)$, $j=2,3$, $s = 1,2$. Note that A is either α or β. If both α and β are in the restricted sets, then define the R^1 image in any desired manner based on the entries in I^1. For instance, it can be determined by which $I^1(2)$ class contains x_2, or by a majority vote of all voters, etc. If one of these sets, say β, is not in the restrictions, then let the R^1 image of F be the unique R^1 class that contains $R^2{}_2 \cap R^3{}_1$ when x_1 is in $I^1(1,2)$. When x_1 is in $I^1(1,1)$, let the R^1 image be determined in any desired manner.

To see that F is well defined over $I^1 \cap I^2 \cap I^3$, note that if x_1 is not either α or β, then it must be in $I^2(1,s) \cap I^3(1,s)$ for one choice of s. Thus, the image of F is $R^2{}_s \cap R^3{}_s$, which meets both R^1 classes. If x_1 is α or β, then the intersection of the R^2 and R^3 images uniquely defines the R^1 image. This is the definition of F. Both values are not in the domain of x_1, so this completes the proof.

Next, suppose that $I^1(1)$ consists of a single equivalence set, and $I^2(1)$ and $I^3(1)$ each have two sets. C_1 has only two sets in $I^2(1) \cap I^3(1)$, so choose the indicing so that $I^2(1,s) \cap I^3(1,s) \neq \emptyset$ for $s = 1,2$. The R^j image of F is $R^j{}_s$ iff $x_1 \in I^j(1,s)$, $j = 2,3$, $s = 1,2$, and the R^1 image is determined in any desired manner.

If the restrictions leave three sets in $I^2(1) \cap I^3(1)$, then F always can be represented as a function of one variable. This is because, as I have already shown, if F is not represented by a function of one variable, then voter 1 must have an influence on the outcome of two classes. Clearly, this must be sets R^2 and R^3. But, no matter how the R^j images of F are defined in terms of which $I^j(1)$class, j = 2,3, contains x_1, there needs is one case where the image is not $R^2{}_s \cap R^3{}_s$, s = 1,2. This forces a situation where the R^1 image is uniquely determined, and it is determined by x_1. Because $F: I^1 \dashrightarrow R^1$ and because $I^1(1)$ is a singleton, it follows that the R^1 image of F is fixed. This completes the proof.

Proof of Corollary 4.2. This is a straightforward argument using the ideas motivating the statement. As in the proof of Theorem 4, we need to have two R^j sets where the third R^α outcome is not determined. This forces the definition of F. Incidently, when C_k becomes smaller, but it still admits a non-dictatorial situation, the combinatorics usually restrict the definition of F.

Acknowledgement This research was support in part by NSF grant IST 8415384.

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