Discussion Paper No. 746

# TURNPIKE PROPERTIES IN A FINITE HORIZON DIFFERENTIAL GAME: DYNAMIC DUOPOLY WITH STICKY PRICES

by

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October 1987

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## Introduction

In a previous paper (Fershtman and Kamien (1987)) we studied an infinite horizon model of dynamic duopolistic competition under the assumption that current price does not jump instantaneously to the price indicated by the demand function for each level of output. The evolution of the price over time is governed by a kinematic equation that specifies, for every given level of output, the change in price as a function of the gap between the current price and the price indicated by the demand function for each output level. The main objective of that paper was to investigate the relationship between the speed at which the price converges to its value on the demand function and the resultant stationary subgame perfect Markov equilibrium price.

In this paper our main purpose is to present a complete analysis of the feedback Nash equilibrium of a finite horizon linear quadratic differential game with a control constraint. In particular we are interested in the relationship between the finite horizon equilibrium strategies and the infinite horizon equilibrium strategies. We examine several "turnpike properties" and demonstrate that if we require that at the equilibirum these properties are satisfied we can reduce the feedback equilibriums set.

The infinite horizon assumption simplifies the analysis of the linear quadratic differential game for under it the equilibrium feedback strategies are autonomous decision rules that specify the control (output rate in our case) as a function of the observed state variables regardless of the time the decision is taking place. See also Driskill and McCafferty (1987) and

Reynolds (1987a,b) and for a more detailed discussion on the autonomy of the solution in an infinite horizon problem see Kamien and Schwartz (1981, p. 238). In the finite horizon case the existence of a doomsday calendar implies that the feedback equilibrium strategies are nonautonomous and this nonautonomy, of course, complicates the analysis. In both the finite and infinite horizon cases the equilibrium strategies are not unique. In the infinite horizon case we can use the asymptotic stability property to reduce the equilibria set. For the finite horizon game, we show in this paper, that one can use "turnpike properties" to reduce the equilibria set. Specifically, we show that not all the perfect Markov equilibrium satisfies the following turnpike property: for a time horizon long enough the finite horizon equilibrium output strategy stays in the neighborhood of the stationary equilibrium strategy except for some initial and final time.

Introducing price stickyness or quantity stickyness into the analysis of dynamic oligopoly introduces a time dependent structure into the model. The multiperiod oligopoly game ceases to be a game which identically repeats itself over time since the profit function at each period depends on the history of the game as well as on the players current choice of actions.[1]

Discussing the dynamic oligopolistic interaction with time dependent structure gives rise to many analytical problems. While the repetition of the one shot equilibrium is a perfect Markov equilibrium for the repeated game such a repetition is meaningless in games which are structurally linked. For many classes of such games it is yet impossible to calculate and to analyze the perfect Markov equilibrium and thus specific analytical assumptions must

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[1]Earlier works on dynamic duopoly with sticky prices includes that of Roos (1925, 1927), and Simaan and Takayama (1978). For a model of quantity stickyness see Driskill and McCafferty (1987).

be made for the sake of tractability. It is only fair to say that the differential game theory is waiting for a breakthrough in the theory of partial differential equations.

In the next section we present our model and summarize our results for the infinite horizon case. In the following section we derive the feedback equilibrium strategies and demonstrate their symmetry. Section 3 contains our turnpike results. A brief summary completes the paper.

## 1. Dynamic Duopoly with Sticky Prices

Under the sticky prices assumption, price in the market does not jump instantaneously to the price indicated by the demand function. The evolution of the price over time is governed by a kinematic equation that specifies for every given market output the changes in price as a function of the gap between the current price and the price indicated by the demand function for each level of output. Formally the evolution of price is governed by

$$\dot{p} = \frac{dp}{dt} = s[a - (u_1 + u_2) - p]; \; p(0) = p_0 \tag{1}$$

where $p(t)$ is the price at time $t$, $u_i(t)$ is the output of firm $i$ at time $t$, $a - (u_1 + u_2) = \underline{p}(t)$ is the price on the inverse demand function for the given level of output, and $0 < s \leq \infty$ denotes the speed of price adjustment. A finite $s$ implies that it takes time for the market to react to changes of quantities. As $s \to \infty$ the price converges instantaneously to the price indicated by the demand function. This can be seen by rewriting (1) as

$$p(t) = a - u_1(t) - u_2(t) - (a - u_1(0) - u_2(0) - p_0)e^{-st}$$

$$+ \int_0^t e^{-s(t-\tau)}[\dot{u}_1(\tau) + \dot{u}_2(\tau)]d\tau$$

and observing that $\lim_{s\to\infty} p(t) = a - u_1(t) - u_2(t)$.

Thus it is evident from the above demand function that the firms face a downward sloping linear inverse demand function but the decline in price along it, as the output level increases, is retarded when s is finite.

Alternatively, one can think of a market in which consumers' utility functions depend on both current consumption and past consumption of a good, see for example Ryder and Heal (1973). Integrating (1) with $p(0) = a$ yields:

$$p(t) = a - s \int_0^t e^{-s(t-\tau)}(u_1(\tau) + u_2(\tau)d\tau$$

which implies that the current price is a function of all the time path of consumptions where recent consumption of the good has a more depressing effect on its current desirability than earlier consumption does.

Total production cost is assumed to be quadratic in ouput and identical for each firm

(2) $$c(u_i) = cu_i + \frac{1}{2} u_i^2, \ i = 1,2$$

The objective of each firm is to maximize its discounted profits.

(3) $$J^i = \int_0^T e^{-rt}[pu_i - cu_i - \frac{1}{2} u_i^2]dt, \ i = 1,2$$

subject to (1) and $u_i \geqslant 0$.

The problem now is formulated as a finite horizon linear quadratic differential game. There are two major strategy spaces that have been discussed in the differential games literature. One is the open-loop strategy space in which players choose path strategies; the other is the closed-loop or feedback strategy space in which players choose decision rule strategies.[2] In most cases, including the game we consider, the Nash equilibrium in open-loop strategies is not subgame perfect (see Fershtman (1987) for a discussion of classes of games for which the open-loop equilibrium is subgame perfect). Thus, we restrict our attention in this paper to the (subgame-perfect) feedback Nash equilibrium.

<u>Definition 1</u>: The feedback strategy space for player i is

$$S_i = \{u_i(t,p) \,|\, u_i(t,p) \text{ is continuous in } (t,p),\ u_i(t,p) \geq 0 \text{ and}$$
$$|u_i(t,p) - u_i(t,p')| \leq m(t)\,|p - p'| \text{ for some integrable } m(t) \geq 0\}.$$

<u>Definition 2</u>: A feedback Nash equilibrium is a pair of feedback strategies $(u_1^*, u_2^*) \in S_1 \times S_2$ such that

$$J^i(u_i^*, u_j^*) \geq J^i(u_i, u_j^*),\ \forall u_i \in S_i,\ i = 1,2,\ j \neq i$$

for every possible initial condition $(p_0, t_0)$.

In our previous work (Fershtman and Kamien (1987)) we investigated the above dynamic duopoly problem assuming an infinite horizon and proved the following results:

(a) The following strategies constitute an asymptotically stable feedback Nash equilibrium for the infinite horizon game

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[2]For a detailed discussion of strategy spaces in differential games see Basar and Olsder (1982).

(4) $$u_i^*(p) = \begin{cases} 0 & , \quad p \leq \hat{p} \\ (1 - sK_\infty)p + (sE_\infty - c), & p > \hat{p}, \end{cases} \qquad i = 1,2$$

where

(4a) $$K_\infty = \frac{r + 6s - \sqrt{(r + 6s)^2 - 12s^2}}{6s}$$

(4b) $$E_\infty = \frac{-asK_\infty + c - 2scK_\infty}{r - 3s^2K_\infty + 3s}$$

(4c) $$\hat{p} = \frac{c - sE_\infty}{1 - sK_\infty}$$

(b) A second equilibrium is defined by the strategies (4) but by letting

(4a') $$K_\infty = \frac{r + 6s + \sqrt{(r + 6s)^2 - 12s^2}}{6s}$$

(c) For each such equilibrium strategy there is a unique stationary equilibrium price, i.e., there is a price level $\bar{p}$ such that if a game starts at $p(0) = \bar{p}$ the equilibrium price path is $p^*(t) = \bar{p}$. This price level is given by

$$\bar{p} = \frac{a + 2(c - sE_\infty)}{2(1 - sK_\infty) + 1}$$

Notice that this stationary equilibrium price does not depend on the initial conditions of the game.

(d) Only the equilibrium specified in (a) satisfies the global asymptotic stability property. This property implies that regardless of the initial value of the state variables (price in our case) the equilibrium path converges to the stationary equilibrium. It was investigated for capital accumulation growth models (see, for example, Brock and Scheinkman (1976)) and for capital accumulation games (see Fershtman and Muller (1986)).

(e) Letting the speed of adjustment goes to infinity, the price stickyness disappears. We showed the the static Cournot equilibrium price is the asymptotic limit of the open loop Nash equilibrium, which is not subgame perfect, while the stable closed-loop Nash equilibrium price converges to a value below it.

From the above results it is evident that in the infinite horizon case the equilibrium strategies are time autonomous. $K_\infty$ and $E_\infty$ are constants that depend on the parameters of the problem, $u_i^*(p)$ is a linear function of p that prescribe an output rate for every p independent of t, and the critical price $\hat{p}$ is also time autonomous. In the finite horizon case the existence of a doomsday calendar implies that the equilibrium strategies are nonautonomous. The output rate depends on the date t as well as on the price and the critical price $\hat{p}$ is now a function of t.

## 2. The Finite Horizon Feedback Nash Equilibrium

Our first task is to find the equilibrium strategies and the resultant equilibrium price trajectory.

Theorem 1: Consider the following strategies

$$(5) \qquad u_i^*(t,p) = \begin{cases} (1 - sK(t))p - c + sE(t) & \text{if} \quad p \geq \hat{p}(t) \\ 0 & \text{if} \quad p < \hat{p}(t) \end{cases}$$

where

$$(6) \qquad K(t) = \frac{\alpha_1(1 - e^{(\alpha_1-\alpha_2)3s^2(T-t)})}{1 - \frac{\alpha_1}{\alpha_2} e^{(\alpha_1-\alpha_2)3s^2(T - t)}}$$

(7) $$E(t) = -\int_t^T e^{\int_t^\tau [3s^2K(\xi)-3s-r]d\xi}[saK(\tau) + 2scK(\tau) - c]d\tau$$

and

(8) $$\hat{p}(t) = \frac{c - sE(t)}{1 - sK(t)}$$

where $\alpha_1$ and $\alpha_2$ are the two solutions of the quadratic equation $3s^2K - (6s + r)K + 1 = 0$. Then $(u_1^*,u_2^*)$ constitute a symmetric feedback Nash equilibria for the above finite horizon game.

<u>Proof</u>: Using the value function approach (see Starr and Ho (1969)) the feedback equilibrium strategies $(u_1^*(t,p),u_2^*(t,p))$ must satisfy at every t the following Hamilton-Jacobi-Bellman equations.

(9) $$-V_t^i(t,p) + rV^i(t,p) = \max_{u_i \geq 0} \{(p - c_i)u_i - \frac{1}{2} u_i^2 + sV_p^i(t,p)[a - p - (u_i + u_j)]\}, \quad i,j = 1,2, \ j \neq i$$

where $V^i(t,p)$ is the value for firm i of the game that starts at time t at the price p, $V_p^i = \partial V^i/\partial p$, and $V_t^i = \partial V^i/\partial t$.

Notice that the right side of (9) is concave with respect to $u_i$. Assume for the moment that there is an interior solution to this maximization problem.[3] In this case $u_i^*$ that maximizes this expression is given by

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[3]We ignore at this stage the constraint $u_i \geq 0$. We will elaborate on the implications of this constraint and on the modification that should be made in the solution later on.

(10) $$u_i^*(t,p) = p(t) - c - sV_p^i(t,p),\ i = 1,2$$

with the boundary condition $V_p^i(T,p) = 0$. Substituting (10) into (9) yields

(11) $$V_t^i(t,p) - rV^i(t,p) + (p - c)(p - c - sV_p^i) - \tfrac{1}{2}(p - c - sV_p^i)^2$$
$$+ V_p^i s[a - (2p - 2c - sV_p^i - sV_p^j) - p] = 0,\ \text{for } i,j = 1,2,\ j \neq i.$$

The above equation presents a pair of partial differential equations. By solving this system and finding the value functions $(V^1(t,p), V^2(t,p))$ we can use (10) to find the equilibrium strategies. We consider the quadratic value function

(12) $$V^i(t,p) = \frac{1}{2} K_i(t)p^2 - E_i(t)p + g_i(t),\ i = 1,2.$$

Differentiating (12) with respect to t and p yield

(13) $$V_t^i(t,p) = \frac{1}{2} \dot{K}_i(t)p^2 - \dot{E}_i(t)p + \dot{g}_i(t),\quad i = 1,2$$

(14) $$V_p^i(t,p) = K_i(t)p - E_i(t)$$

Substituting (14) into (10) yields that

(15) $$u_i^*(t,p) = (1 - sK_i(t))\, p - c + sE_i(t),\ i = 1,2$$

Substituting (13) and (14) into (11) yields

$$(16)\quad \frac{1}{2}\dot{K}_i p^2 - \dot{E}_i p + \dot{g}_i - rK_i p^2/2 + rE_i p - rg_i + (p - c)(p - c - sK_i p + sE_i)$$
$$- \frac{1}{2}(p - c - sK_i p + sE_i)^2 + s(K_i p - E_i)[a - 3p + 2c + sK_i p - sE_i +$$
$$+ sK_j p - sE_j] = 0, \quad \text{for } i,j = 1,2,\ i \neq j.$$

Lemma 1. The equilibrium output strategies $u_i^*(t,p)$ are symmetric, i.e., $u_t^*(t,p) = u_2^*(t,p)$, for every $(t,p)$.

Proof. Since (16) must be satisfied for all values of p, the coefficients of $p^2$,p and the constant terms have to be zero. This implies, after some algebraic manipulation, that

$$(17)\quad \dot{K}_i + (s^2K_i + 2s^2K_j - r - 6s)K_i + 1 = 0,\ i,j = 1,2,\ i \neq j.$$

$$(18)\quad \dot{E}_i - (r + 3s - s^2K_i - s^2K_j)E_i - (a + 2c - sE_j)sK_i + c = 0,\ i,j = 1,2,\ i \neq j$$

$$(19)\quad \dot{g}_i - rg_i + .5c^2 - 2scE_i - saE_i + .5s^2E_i^2 + s^2E_iE_j = 0,\quad i,j = 1,2,\ i \neq j.$$

From (17) upon subtraction we have

$$(20)\qquad \dot{K}_1 - \dot{K}_2 + (K_1 - K_2)[s^2(K_1 + K_2) - (6s + r)] = 0$$

Let $v = K_1 + K_2$ and $w = K_1 - K_2$, which implies that $\dot{w} = \dot{K}_1 - \dot{K}_2$. Then (20) can be rewritten as

(21) $$\dot{w}(t) + w(t)[s^2v(t) - (6s + r)] = 0$$

which is a first order differential equation whose solution is

(22) $$w(t) = Ce^{\int_0^t [(6s+r)-s^2v(\tau)]d\tau}$$

where C is the constant of integration. To evaluate it we must make use of the boundary condition $V_p^i(T,p) = 0$, $i = 1,2,$, which, because it has to hold for every p, implies that $K_1(T) = K_2(T) = 0$. Thus, $w(T) = K_1(T) - K_2(T) = 0$ and the constant of integration $C = 0$. It follows, therefore, that $w(t) = 0$ and $K_1(t) = K_2(t)\ \forall\ 0 \leq t \leq T$. The same type of argument can be applied to establish that $E_1(t) = E_2(t)$, given that we have already established that $K_1(t) = K_2(t)$. Finally, it follows that $g_1(t) = g_2(t)$. Thus, $u_1^*(t,p) = u_2^*(t,p)$.

We now let $K_1(t) = K_2(t) = K(t)$, $E_1(t) = E_2(t) = E(t)$ and $g_1(t) = g_2(t) = g(t)$ and rewrite (17), (18) and (19) as

(23) $$\dot{K} = -3s^2K^2 + (6s + r)K - 1$$

(24) $$\dot{E} = (r + 3s - 3s^2K)E + (sKa + 2sKc - c)$$

(25) $$\dot{g} - rg + .5c^2 - 2sEc - sEa + 1.5s^2E^2 = 0$$

Expression (23) is a Riccati equation. Let $\alpha_1$ and $\alpha_2$ be the two solutions $[6s + r \pm \sqrt{(6s + r)^2 - 12s^2}]/6s^2$ of the quadratic equation $3s^2K^2 - (6s + r)K + 1 = 0$. Without loss of generality, let us assume that $\alpha_1 < \alpha_2$. $K(t) = \alpha_1$ and $K(t) = \alpha_2$ are both particular solutions of (23).

A general solution of (23) is given by (see Ford (1955))

$$\frac{K(t) - \alpha_1}{K(t) - \alpha_2} = Ae^{-(\alpha_1-\alpha_2)3s^2t}, \tag{26}$$

where A is the constant of integration. Since $V_p^i(T,p) = 0$ for every p, this boundary condition implies that at time T, K(T) = E(T) = g(T) = 0. Evaluating (26) at t = T yields

$$\frac{\alpha_1}{\alpha_2} = Ae^{-(\alpha_1-\alpha_2)3s^2T} \tag{27}$$

Rearranging (27) yields

$$A = \frac{\alpha_1}{\alpha_2} e^{(\alpha_1-\alpha_2)3s^2T} \tag{28}$$

Substituting (28) into (26) yields that

$$K(t) - \alpha_1 = \frac{\alpha_1}{\alpha_2} e^{(\alpha_1-\alpha_2)3s^2(T-t)} (K(t) - \alpha_2) \tag{29}$$

Rearranging (29) yields equation (6).

Given K(t) we can solve (24) to find E(t). The general solution of (24) is

$$E(t) = e^{-\int_0^t [3s^2K(\xi)-r-3s]d\xi} [C + \int_0^t e^{-\int_0^\tau [r+3s-3s^2K(\xi)]d\xi} \tag{30}$$

$$[saK(\tau) + 2scK(\tau) - c]d\tau$$

Using the boundary condition E(T) = 0, we can find C, the constant of integration

(31) $$C = -\int_0^T e^{\int_0^\tau [3s^2K(\xi)-r-3s]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

Substituting (31) into (30) yields

(32) $$E(t) = -e^{-\int_0^t [3s^2K(\xi)-r-3s]d\xi}$$

$$\cdot \int_t^T e^{\int_0^\tau [3s^2K(\xi)-r-3s]d\xi} [saK(\tau) + 2scK(\tau) - c]d\tau$$

which can be rewritten as (7).

From (10) and (14) it is evident that the condition for having an interior solution to the right side of (9) is that

(33) $$p(t) \geq \frac{c - sE(t)}{1 - sK(t)}$$

If the above condition does not hold it means that p is too low and the firms should stop production, i.e., $u_i = 0$. Denote this critical price as $\hat{p}(t)$. Notice also that equation (15) satisfies the constraint $u_i \geq 0$ only if $p(t) \geq \hat{p}(t)$. Clearly for $p < \hat{p}(t)$ the quadratic value function (12) is not appropriate since it will not satisfy the Hamilton-Jacobi condition (9).

Now it only remains to define a value function for $p(t) < \hat{p}(t)$ that will satisfy (9). Clearly when $u_i = 0$ the instantaneous profit function is zero. However, the price as indicated by (1) goes up. The firms will start to produce when the price reaches the critical level $\hat{p}(t)$. Let $\hat{t}(p,t)$ be the time that it takes to reach the critical price from the price level p at time t. Now for every $p(t) < \hat{p}(t)$ let

$$(34) \qquad \tilde{V}^i(t,p) = e^{-r\hat{t}(p,t)}V^i(\hat{p}(t + \hat{t}(p,t)),\ t + \hat{t}(p,t))$$

be the value function. The intuition of this value function is straightforward. For every $p(t) < \hat{p}(t)$, instantaneous profits are zero. The firm has to wait for $\hat{t}(p,t)$ until it will start to make some profit and the value of the game starting at $\hat{p}(t)$ is already discussed. Now it only remains to prove that this value function satisfies the Hamilton-Jacobi condition.

Substituting $u_i = 0$ $i = 1,2$, into (9) yields that the condition the value function (34) must satisfy is

$$(35) \qquad -\tilde{V}^i_t(t,p) + r\tilde{V}^i(t,p) = s\tilde{V}^i_p(t,p)(a - p)$$

Using the kinematic equation (1) it is evident that if $u_i = 0$ and the price at time t is $p(t)$ then for every $\tau > t$

$$(36) \qquad p(\tau) = p(t)e^{-s(\tau-t)} + a(1 - e^{-s(\tau-t)})$$

From the definition of $\hat{t}(p,t)$ it is evident that

$$(37) \qquad \hat{p}(t + \hat{t}) = p(t)e^{-s\hat{t}} + a(1 - e^{-s\hat{t}})$$

Differentiating (37) yields that

$$(38) \qquad \frac{d\hat{t}}{dp} = \frac{-e^{-s\hat{t}}}{(a - p(t))se^{-s\hat{t}} - \dot{\hat{p}}}$$

(39) $$\frac{d\hat{t}}{dt} = \frac{\dot{\hat{p}}}{(a - p(t))e^{-s\hat{t}} - \dot{\hat{p}}}$$

Differentiating (34) with respect to p and t yield

(40) $$\tilde{V}^i_p(t,p) = e^{-r\hat{t}}\frac{d\hat{t}}{dp}[-rV^i + V^i_p\dot{\hat{p}} + V^i_t]$$

(41) $$\tilde{V}^i_t(t,p) = e^{-r\hat{t}}[-rV^i + V^i_p\dot{\hat{p}} + V^i_t]\frac{d\hat{t}}{dt} + e^{-r\hat{t}}[V^i_p\dot{\hat{p}} + V^i_t]$$

Substituting (38) and (39) in (40) and (41) it is straightforward to check that

(42) $$s\tilde{V}^i_p(t,p)(a - p) + \tilde{V}^i_t = e^{-r\hat{t}}\, rV^i$$

Since $re^{-r\hat{t}}V^i = r\tilde{V}^i$, condition (35) is satisfied and the value function $\tilde{V}_i$ satisfies the Hamilton-Jacobi condition. Q.E.D.

Notice that Theorem 1 does not define a unique pair of equilibrium strategies. Switching $\alpha_1$ and $\alpha_2$ in equation (6) will give us different formulas for K(t). Since K(t) is not uniquely defined and E(t) is a function of K(t) it is clear that E(t) is also not uniquely defined. This problem of multiplicity of equilibria occurs also in other models that involve dynamic economic interaction via linear quadratric differential games. In the next section we demonstrate that by investigating the turnpike properties of the equilibrium path we can identify one equilibrium that satisfies some desirable properties that are not satisfied by the other equilibrium

## 3. Turnpike Properties of the Feedback Equilibrium Strategies

In the capital accumulation growth literature the asymptotic properties of optimal paths are usually referred to as "turnpike properties." We borrow the turnpike terminology and discuss the asymptotic properties of the equilibrium strategies. In particular we are interested in the following three properties: (i) the equilibrium of the infinite horizon game converges to the unique stationary equilibrium regardless of the initial conditions; (ii) for a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the infinite horizon equilibrium except for some final time; (iii) for a time horizon long enough the finite horizon equilibrium stays in the neighborhood of the stationary equilibrium except for some initial and final time.

The first turnpike property is actually the asymptotic stability property that was discussed in Section 1. The second turnpike propety is proven below.

Theorem 2. For every $\varepsilon > 0$ and $T_1$ there is $T_2$ such that for every $T > T_2$ each of the two equilibrium strategies of the finite horizon game are in $\varepsilon$ neighborhoods of an infinite horizon equilibrium strategies for every $0 < t < T_1$.

Proof: We will prove the above turnpike property for just one of the equilibria. The proof for the second equilibrium will follow immediately. The equilibrium strategies of the finite horizon game are given by

$$u_i^*(t,p) = (1 - sK_T(t))p + (sE_T(t) - c), \; i = 1,2$$

Note that we deviate here from our previous notation and write $K_T(t)$ and $E_T(t)$ to emphasize that these two functions depend on the horizon of the game. For the infinite horizon game the equilibrium strategies are given by the same expression when $K_\infty$ and $E_\infty$ replace $K_T$ and $E_T$. Thus the proof will be carried out by comparing $K_T$ and $E_T$ with $K_\infty$ and $E_\infty$.

By investigating (6) it is clear that for every $\varepsilon_1 > 0$ and $T_1$ there is $T_2 > T_1$ such that $\sup_{t \leq T_1} |K_T(t) - \alpha_1| < \varepsilon_1$ for every $T > T_2$. This is true because for any given $t < T_1$, $K_T(t) \to \alpha_1$ as $T \to \infty$. Now observe that $\alpha_1$ is a stationary solution of (17) and from (4a) it is evident that $K_\infty = \alpha_1$.

The solution of $E_T(t)$ is given by (7). Using the above result, it is evident that for a given $t < T$, $E_T(t) \to E_\infty$ as $T \to \infty$. Thus for every $\varepsilon_2$ and $T_1$ there is $T_2 > T_1$ such that $\sup_{t \leq T_1} |E_T(t) - E_\infty| < \varepsilon_2$ for every $T > T_2$. Clearly, by choosing $\varepsilon_1$ and $\varepsilon_2$ to be sufficiently small, we can find $T_2$ such that for every $T > T_2$ and $0 \leq t \leq T_1$ the equilibrium strategies for the finite horizon game will be in the $\varepsilon$-neighborhood of the infinite horizon equilibrium strategies. Q.E.D.

From the above theorem it is evident that the equilibrium price path satisfies the same turnpike property.

Proposition 1: For a time horizon long enough each of the two finite horizon price equilibrium paths stay in $\varepsilon$-neighborhood of an infinite horizon price equilibrium path except for some final time.

Proof: The proof is straightforward from Theorem 2. The equilibrium price at time t is determined by the quantity strategies that have been played until time t, and since these strategies satisfy the above turnpike property it follows immediately that the equilibrium price path satisfies the same property.

Q.E.D.

The exact finite horizon price equilibrium path can be found by substituting the equilibrium strategies (5) into the kinematic equation (1) and solving the resultant differential equation using the initial condition to solve for the constant of integration. Following these steps yields that if $p_0 \geq \hat{p}(0)$ the equilibrium price path is

$$(43) \qquad p_T(t) = e^{\int_0^t s[2sK(\xi)-3]d\xi}[p_0 + \int_0^t e^{-\int_0^\tau s[2sK(\xi)-3]d\xi}[a + 2(c - sE(\tau))]sd\tau$$

If $p_0 < \hat{p}(0)$ then for $0 \leq t \leq \hat{t}(p_0,0)$

$$(44) \qquad p_T(t) = p_0 e^{-st} + a(1 - e^{-st})$$

and for $t > \hat{t}(p_0,0)$ the equilibrium price path is given by (43) when $\hat{p}(\hat{t}(p_0,0))$ is regarded as the initial price and $\hat{t}(p_0,0)$ is regarded as the initial time.

As was indicated before, there are two equilibria of the finite horizon game. Each of these equilibra "tend" to a different infinite horizon equilibrium. If we let $\alpha_1 < \alpha_2$ then the equilibrium strategies given by (5)-(8) "tend" to the asymptotically stable infinite horizon equilibrium given by (4)-(4c). Switching $\alpha_1$ and $\alpha_2$ in (6) yields, in this case, equilibrium strategies that "tend" to an infinite horizon equilibrium which is not asymptotically stable.

The asymptotic stability property of the infinite horizon equilibrium and the turnpike property discussed in Theorem 2 lead to the following turnpike property: for a time horizon long enough, the finite horizon equilibrium path stays in an $\varepsilon$-neighborhood of the stationary equilibrium except for some initial time and some final time. This property was discussed in the optimal economic growth literature (see Cass (1966)). Since for this turnpike property we need both asymptotic stability and Theorem 2 we can conclude that only the finite horizon equilibrium that tends to the asymptotically stable infinite horizon equilibrium satisfies the above turnpike property.

## Summary

We have analyzed a finite horizon differential game model of duopolistic competition through time under the supposition that prices do not adjust immediately to their level on the demand function for each level of output. We have shown that the duopolists' equilibrium strategies are symmetric and that for a sufficiently long time horizon they approach the infinite horizon equilibrium strategies arbitrarily close. They diverge from the infinite horizon equilibrium strategies as the finite horizon nears and end gaming begins. Thus, the strategies in the finite horizon model exhibit a turnpike property.

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