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INFINITE HORIZON GAMES WITH PERFECT EQUILIBRIUM POINTS*

by

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Abstract

We examine the question of existence of subgame perfect equilibrium points in infinite horizon games which allow players to move simultaneously at each period. The previous literature on this question had dealt only with games of perfect information which rules out situations in which players can move simultaneously. We specify assumptions under which infinite horizon games with simultaneous moves (IHGSM) will have subgame perfect equilibrium points. We discuss two classes of games; one a class of supergames, and the other a class of games involving strategic investment, and apply our existence result to these two classes of games.

1. Introduction

The notion of subgame perfect equilibrium is a concept that has much intrinsic appeal. The basic idea is that, without commitments, behavior in a subgame can depend only on the subgame itself. Kuhn's (1953) result that any finite horizon game of perfect information admits a perfect equilibrium point in pure strategies, is by now well known. Selten (1975) extended the notion to finite horizon games with, possibly, imperfect information. These, in a sense, settled the question of existence of perfect equilibrium points for finite games.

The question that remained concerned the existence of perfect equilibrium points in games of infinite horizon. Fudenberg and Levine (1983) demonstrated the existence of perfect equilibrium in pure strategies in finite action games with perfect information and continuous payoffs. Harris (1985) showed that under rather weak assumptions one can eliminate the condition that the game be a finite action game, and demonstrated that a perfect equilibrium point exists for infinite horizon games with perfect information.

The requirement that the infinite horizon games have perfect information rules out the possibility of players moving simultaneously in each period. Hence, a large class of games cannot be analyzed using the infinite horizon games with perfect information. Therefore, one is led into investigating the question of existence of perfect equilibrium points for games in which players can move simultaneously.

In what follows we give conditions under which infinite horizon games with simultaneous moves will have perfect equilibrium points. The conditions are, of course, stronger than those imposed by Harris (1985).

Among the assumptions, the two conditions that are striking are concerned with the concavity--or more precisely, the quasiconcavity of the payoff functions and the convexity of the set of feasible histories. The conditions are used to find a fixed point. One finds at this point certain vague analogies with existence of equilibrium points of one-shot games, and it is not, therefore, difficult to see the need for assumptions.

The theorem that is shown to hold for infinite horizon games with simultaneous moves is then applied to two classes of infinite horizon games. The first is the class of repeated games. The second is a class of games which generalize the model of Fudenberg and Tirole (1983) in which firms play a game of strategic investment.

2. The Model

An infinite horizon game with simultaneous moves is a set N of players, a sequence of action spaces or strategy sets for each $i \in N$, a sequence of correspondences from infinite histories into action spaces for each $i \in N$, and payoff functions for each player over the set of possible histories.

Let X_{it} denote the action space for player i in period t. The actions that are feasible for a player $i \in N$ in period t will always be in X_{it}.

The outcome in any period t can, therefore, be represented by an element of $X_t := \otimes_{i=1}^{N} X_{it}$. Hence, the set of possible outcomes is $X: = \otimes_{t=1}^{\infty} X_t$. For any sequence of outcomes $x \in X$, we will write $(x_1, x_2, \ldots, x_t, \ldots)$.

A history up to period t - 1, which is known to all players $i \in N$ at time period t, is a finite sequence of outcomes $(x_1, x_2, \ldots, x_{t-1})$. Given the history $x \in X$, we will denote the outcomes up to time period t - 1 by

$$\ell_{t-1}(x) = (x_1, x_2, \ldots, x_{t-1})$$

and the sequence of outcomes from period t onwards we will denote by

$$\tau_t(x) = (x_t, x_{t+1}, \ldots).$$

<u>Feasible Histories</u>

Given the set of histories up to time period t - 1, the actions feasible in time period t is given by the correspondences

$$A_{it}: \ell_{t-1}(X) \to X_{it}$$

for all $i \in N$.

We will say a history $\ell_{t-1}(x)$ up to time t - 1 is <u>feasible</u> if for all $1 \leq \nu \leq t - 1$, we have

$$x_{i,\nu+1} \in A_{i,\nu+1}(\ell_\nu(x))$$

for all $i \in N$.

If a history $\ell_{t-1}(x)$ up to time period t - 1 is not feasible, then $A_{it}(\ell_{t-1}(x)) = \emptyset$ for all $i \in N$. Hence, the action correspondences $A_{it}: \ell_{t-1}(X) \to X_{it}$ is a nonempty valued correspondence only over the set of histories which are feasible up to the time period t - 1.

Let $F \subseteq X$ denote the set of infinite histories which are <u>feasible</u> in the infinite horizon game. Then, $x \in F$ satisfies the following consistency requirement. For all i:

$$x_{i,\nu+1} \in A_{i,\nu+1}(\ell_\nu(x)) \text{ for all } \nu \geq 1.$$

The set of histories which is possible given an initial feasible history $\ell_{t-1}(x)$ depends on the history $\ell_{t-1}(x)$. We will denote this by

$$G_t(\ell_{t-1}(x)) = \{y/y \in F,\ \ell_{t-1}(y) = \ell_{t-1}(x)\}.$$

Similarly, let

$$G_t^i(\ell_{t-1}(x), x_{-it}) = \{y/y \in F,\ \ell_{t-1}(y) = \ell_{t-1}(x),\ y_{-it} = x_{-it}\}$$

denote the set of feasible histories, when $\ell_{t-1}(x)$, a feasible history up to period t - 1, has been realized and all player $j \neq i$ have chosen x_{-it} a feasible action (n - 1) tuple in period t.

Strategies and Outcomes: The strategies of the players are plans describing the actions of the players contingent on any initial history. Hence, player i's strategy is a sequence of functions $\{h_{it}\}_{t=1}^{\infty}$, where

$$h_{it}: \ell_{t-1}F \rightarrow X_{it}$$

such that

$$h_{it}(\ell_{t-1}y) \in A_{it}(\ell_{t-1}y)$$

for all $y \in F$. The strategy combination for period t is then written as

$$h_t = (h_{1t}, h_{2t}, \ldots, h_{nt})$$

and

$$h = (h_1, h_2, \ldots, h_t, \ldots)$$

for a strategy combination of the entire game.

Let H denote the set of these strategies and H_i the set of strategies of player i. If $h \in H$ and $q_i \in H_i$, we will denote by $h\backslash q_i$ the strategy combination obtained by replacing player i's strategy h_i by q_i.

If a strategy combination h is used, where $h \in H$ then this will define a history $y \in F$. We will write this outcome as $[h,y]$, the history y resulting from the strategy $h \in H$. If an initial history has occurred, given by $\ell_{t-1}(x)$, and a strategy is selected for the subgame, then we will denote it by $h|_{\ell_{t-1}(x)}$ and the subsequent history by $\tau_t(x)$. The outcome in the subgame generated by the history $\ell_{t-1}(x)$ will then be written as $[h|_{\ell_{t-1}(x)}, \tau_t(x)]$ where $\tau_t(x)$ is the history generated by the strategy $h|_{\ell_{t-1}(x)}$ in the subgame.

<u>Payoffs</u>: The payoff function u^i of player i is a function defined on the set of feasible histories F, $u^i: F \to \mathbb{R}$

<u>The Game</u>: The infinite horizon game can now be completely described by:

$$[\{X_{it},A_{it}\}_{t=1}^{\infty},u^{i}]_{i\in N}$$

where X_{it} is the set containing the action spaces of player i in period t for all possible histories up to time t; A_{it} is the correspondence which describes the actions that are possible given the histories up to time t - 1, and u^i is the payoff function defined on the set of feasible histories.

3. Infinite Horizon Games with Simultaneous Moves

The infinite horizon game described above is one in which at time t the players know the history up to time t - 1, and make their choices conditional on this history. At period t, however, the players may make their choices without knowing the choice of the other players at period t. Hence, the infinite horizon game is one in which players move simultaneously in each period t, and their information is the past history up to period t - 1. Hence, in these games, the information that players have is about the past history and this is common to all the players.

In particular, a repeated game with complete information is an infinite horizon game with simultaneous moves. However, because the action spaces of the players can be different in each period, an infinite horizon game with simultaneous moves has a much more general structure than the repeated games in which the action spaces are kept fixed over time.

A formal definition of an infinite horizon game with simultaneous moves (IHGSM) is given below:

Definition 3.1: An infinite horizon game with simultaneous moves (IHGSM) is an infinite horizon game $[\{X_{it}, A_{it}\}_{t=1}^{\infty}, u^i]_{i \in N}$ such that for some finite history $\ell_t(x) \in \ell_t(F)$, the action spaces $A_{it+1}(\ell_t(x))$ of at least two players is not a Singleton.

That is, at some point for some history at least two players will have to move simultaneously. This rules out games with perfect information, where players cannot move simultaneously, so that only one player has an action space which is not a Singleton.

Definition 3.2: A strategy n-tuple $h^* \in H$ of the infinite horizon game is a subgame perfect equilibrium point of the game, if for every time $t \geq 1$, and feasible history $\ell_{t-1}(x) \in \ell_{t-1}(F)$ up to time $t - 1$, the strategy n-tuple $h^*|_{\ell_{t-1}}(x)$ for the subgame generated by $\ell_{t-1}(x)$ is an equilibrium point for the subgame.

That is, for every $t \geq 1$, and history $\ell_{t-1}(x)$,

$$u^i[\ell_{t-1}(x), \tau_t(x)] \geq u^i[\ell_{t-1}(x), \tau_t(y)]$$

for all

$$\tau_t(y) \in [\tau_t(z)/\tau_t(z) \text{ is generated by } (h^*/q_i)|_{\ell_{t-1}(x)}]$$

for all $i \in N$.

Hence, a subgame perfect equilibrium strategy is an equilibrium on any subgame of the infinite horizon game.

4. Existence of Subgame Perfect Equilibrium For Infinite Horizon Games with Simultaneous Moves

The existence result that will be given will require certain assumptions on the primitives of the infinite horizon game with simultaneous moves.

Assumption 1: X_{it} is a compact convex subset of a Euclidean space for all i and t.

Assumption 2: X is given the product topology.

Assumption 3: The correspondences $A_{it}: \ell_{t-1}(F) \to X_{it}$ are upper semicontinuous, closed-valued, and convex-valued correspondence for all i and t.

Assumption 4: The payoff functions $u^i: F \to \mathbb{R}$ are quasiconcave on F.

Assumption 5: The correspondences $A_{it}: \ell_{t-1}(F) \to X_{it}$ all satisfy the following convexity assumption. Let

$$\ell_{t-1}(z) = \alpha\ell_{t-1}(x_1) + (1 - \alpha)\ell_{t-1}(x_2) \in \ell_{t-1}(F).$$

Then, for any

$$x_{it}^1 \in A_{it}(\ell_{t-1}(x_1)) \text{ and}$$
$$x_{it}^2 \in A_{it}(\ell_{t-1}(z))$$

we have

$$z_{it} = \alpha x^1_{it} + (1 - \alpha)x^2_{it} \in A_{it}(\ell_{t-1}(z)).$$

Assumption 5 could be interpreted in two possible ways. The first interpretation has been commonplace in the literature and involves using behavior strategies over the tree. The second interpretation is one in which the pure strategies or histories lie in some kind of convex set. The second interpretation, of course, severely limits the nature of the game tree, but allows one to think in terms of pure strategy equilibria.

Assumptions 1-5 give us the following structure for the set F of feasible histories of the infinite horizon game.

Lemma 4.1: F is a closed, convex subset of the product space X.

Proof: Let $\{X_\nu\}_{\nu=1}^{\infty}$ be a sequence of histories in F converging to $x^* \in X$. We want to show that x^* is in F. Suppose $x^* \notin F$, then it is not feasible, so that there must be $t \geq 1$ and an $i \in N$ such that

$$x^*_{it} \notin A_{it}(\ell_{t-1}(x^*)).$$

This implies that there exists an open set G containing $A_{it}(\ell_{t-1}(x^*))$ such that $x^*_{it} \notin G$. But since $A_{it}\colon \ell_{t-1}(F) \to X_{it}$ is upper semicontinuous and closed valued and $\ell_{t-1}(x_\nu) \to \ell_{t-1}(x^*)$ in $\ell_{t-1}(X)$, $A_{it}(\ell_{t-1}(x_\nu)) \subseteq G$ for all ν sufficiently large, so that, in particular, $\{x^\nu_{it}\}_{\nu=1}^{\infty}$ does not converge to x^*_{it} This gives us a contradiction. Hence, F must be closed.

Let $x_1, x_2 \in F$ be any two feasible histories. Suppose $y = \alpha x_1 + (1 - \alpha)x_2$ is not in F. Then, for some $t \geq 1$ and $i \in N$, we have

$$y_{it} \notin A_{it}[\ell_{t-1}(y)].$$

Since $\ell_{t-1}(y) = \alpha\ell_{t-1}(x_1) + (1 - \alpha)\ell_{t-1}(x_2)$, by Assumption 5 we have an immediate contradiction. []

We now begin defining some important correspondences.

Let $\Psi_t\colon \ell_{t-1}(F) \to \tau_t(F)$ be some closed valued and nonempty valued correspondence which is upper semicontinuous on $\ell_{t-1}(F)$. Let

$$w_{it}(\ell_{t-1}(x),\ x_{-it},\ z_{it})$$
$$:= \min\{u^i(y)/\ell_t(y) = (\ell_{t-1}(x),\ x_{-it},\ z_{it})$$

and

$$\tau_{t+1}(y) \in \Psi_{t+1}(\ell_{t-1}(x),\ x_{-it},\ z_{it})\}.$$

Therefore, $w_{it}(\ell_{t-1}(x), x_{-it})$ is the worst payoff that player i can get when the history up to period t - 1 is $\ell_{t-1}(x)$, players $j \neq i$ has chosen $x_{-it} \in X_{-it}$ and the subsequent play of the game is restricted to $\Psi_t(\ell_{t-1}(x),\ x_{-it},\ z_{it})$. Hence, one can think of $\Psi_t(\ell_{t-1}(x),\ x_{-it},\ z_{it})$ as being the set of perfect equilibrium paths of the subgame generated by the history $\ell_{t-1}(x)$, the choices x_{-it} of players $j \neq i$ and the choice z_{it} of player i, if such perfect equilibrium paths exist, and that punishments for deviations must be restricted to such perfect equilibrium paths.

Hence, for a given correspondence $\Psi_t\colon \ell_{t-1}(F) \to \tau_t(F)$ we can think of the punishment function $w_{it}\colon \ell_{t-1}(F) \otimes X_t \to \mathbb{R}$.

Now, we define

$$b_{it}(\ell_{t-1}(x),x_{-it}) = \sup\{w_{it}(\ell_{t-1}(x),x_{-it},z_{it})/z_{it} \in A_{it}(\ell_{t-1}(x))\}.$$

Thus, $b_{it}\colon \ell_{t-1}(F) \otimes X_{-it} \to \mathbb{R}$ denotes the best payoff that player i can guarantee himself when players $j \neq i$ have chosen x_{-it}, by making the appropriate choice in period t, knowing that henceforth he will be punished with the worst <u>feasible</u> punishment in the resulting subgame that ensues from period t + 1 onwards.

The following two results give us some properties of the two functions. We first define a convenient notation. For any history $\ell_{t-1}(x) \in \ell_{t-1}(F)$, define

$$A_t(\ell_{t-1}(x)) := \otimes_{i=1}^{n} A_{it}(\ell_{t-1}(x)).$$

Hence, $A_t(\ell_{t-1}(x))$ is the set of feasible outcomes in period t if the history up to period t - 1 is $\ell_{t-1}(x)$. From the assumptions, the correspondence

$$A_t\colon \ell_{t-1}(F) \to X_t$$

is upper semicontinuous, closed valued and convex valued.

<u>Lemma 4.2</u>: The function $w_{it}\colon grA_t \to \mathbb{R}$ is continuous on graph of A_t.

Proof: Since $A_t: \ell_{t-1}(F) \to X_t$ is upper semicontinuous and closed valued, graph A_t is a closed set.

Let $z \in grA_t$, and if $\{z^k\}_{k=1}^{\infty}$ is any sequence that converges to z, we have that for any open set G containing $\Psi_{t+1}(z)$, there exists a k_0 such that $\Psi_{t+1}(z^k) \subseteq G$ for all $k \geq k_0$, by the upper semicontinuity of the correspondence $\Psi_{t+1}: \ell_t(F) \to \tau_{t+1}(F)$. This will imply that $w_{it}(z^k) \to w_{it}(z)$. Since z was arbitrary, it follows that the function is continuous on grA_t. []

This result shows that for any history $\ell_{t-1}(x)$ and $x_{-it} \in A_{it}(\ell_{t-1}(x))$, the function w_{it} is continuous over the choices that player i makes. We now investigate a property of the function b_{it}. Let

$$A_{-it}(\ell_{t-1}(x) := \otimes_{j \neq i} A_{jt}(\ell_{t-1}(x)).$$

We can then talk about the correspondence

$$A_{-it}: \ell_{t-1}(F) \to X_{-it}$$

and its graph. The set grA_{-it} is a closed set since the correspondences $A_{it}: \ell_{t-1}(F) \to X_{it}$ are u.s.c. and closed valued. The function $b_{it}: \ell_{t-1}(F) \otimes X_{-it} \to \mathbb{R}$ is actually defined over grA_{-it}. We now examine a continuity property of this function.

<u>Lemma 4.3</u>: The function $b_{it}: grA_{-it} \to \mathbb{R}$ is continuous.

<u>Proof</u>: By definition,

$$b_{it}(\ell_{t-1}(x),x_{-it}) = \sup\{w_{it}(\ell_{t-1}(x),x_{-it},z_{it})/z_{it} \in A_{it}(\ell_{t-1}(x)).$$

Now, consider the set

$$\{w_{it}(\ell_{t-1}(x),x_{-it},y_{it})/y_{it} \in A_{it}(\ell_{t-1}(x))\} \subseteq \mathbb{R}.$$

From the continuity of $w_{it}\colon grA_t \to \mathbb{R}$, it follows that the correspondence defined by

$$W_{it}\colon grA_{-it} \to \mathbb{R}$$

as $W_{it}(\ell_{t-1}(x),x_{-it}) = \{w_{it}(\ell_{t-1}(x),x_{-it},y_{it})/y_{it} \in A_{it}(\ell_{t-1}(x))\}$

is upper semicontinuous, since $A_{it}\colon \ell_{t-1}(F) \to X_{it}$ is an upper semicontinuous correspondence. Hence, for any point $(\ell_{t-1}(x), x_{-it}) \in grA_{-it}$, and a sequence $\{\ell_{t-1}(x^k),x^k_{-it}\}_{k=1}^{\infty}$ converging to $(\ell_{t-1}(x),x_{-it})$ for any open set G containing $W_{it}(\ell_{t-1}(x),x_{-it})$, there exists a k_0 such that for all $k \geq k_0$, $W_{it}(\ell_{t-1}(x^k),x^k_{-it}) \subseteq G$.

Therefore, $b_{it}(\ell_{t-1}(x^k),x^k_{-it})$ converges to $b_{it}(\ell_{t-1}(x), x_{-it})$. Since $\ell_{t-1}(x), x_{-it})$ was arbitrary, if follows that $b_{it}\colon grA_{-it} \to \mathbb{R}$ is continuous. []

In order that after the history $\ell_{t-1}(x)$, a player i does not have an incentive to deviate, a history y must occur which gives the player at least as much as he could guarantee himself. Hence, the history y must be such that $u^i(y) \geq b_{it}(\ell_{t-1}(x), y_{-it})$, where the other players are conforming to the history y.

For every history $\nu = \ell_{t-1}(x) \in \ell_{t-1}(F)$, we define the correspondence

$$E^{\nu}_{it}: X_{-it} \rightarrow X_{it}$$

as follows:

$$E^{\nu}_{it}(x_{-it}) := \{y_{it} / \exists\ y \in gr\Psi_{t+1} \text{ s.t. } \ell_t(y) = (\ell_{t-1}(x), x_{-it}, y_{it})\}$$

such that $u^i(y) \geq b_{it}(\ell_{t-1}(x), x_{-it})\}$.

<u>Lemma 4.4</u>: For all $t \geq 1$, and history $\ell_{t-1}(x) \in \ell_{t-1}(F)$, the correspondences

$$E^{\nu}_{it}: A_{-it}(\ell_{t-1}(x)) \rightarrow A_{it}(\ell_{t-1}(x))$$

is nonempty valued, convex valued, compact valued, and upper semicontinuous.

<u>Proof</u>: For each history $\ell_{t-1}(x)$ and $x_{-it} \in A_{-it}(\ell_{t-1}(x))$, consider $x_{it} \in A_{it}(\ell_{t-1}(x))$ such that $u^i(y) \geq u^i(z)$ for all z: $\ell_t(z) = (\ell_{t-1}(x), x_{-it}, z_{it})$. Then, clearly, $x_{it} \in E_{it}(x_{-it})$. This shows that the correspondence is nonempty valued.

From Assumption 5 and the quasiconcavity of the utility functions over histories, it follows that $E^{\nu}_{it}: A_{-it}(\ell_{t-1}(x) \to A_{it}(x))$ is convex valued.

Now, let $\bar{x}_{it}$ be a limit point of $E_{it}(x_{-it})$. Suppose $\bar{x}_{it} \notin E_{it}(x_{-it})$. Then for all $y \in \mathrm{gr}\Psi_{t+1}$ such that $\ell_t(y) = (\ell_{t-1}(x), \bar{x}_{it}, x_{-it})$, we have

$$(*) \qquad u^i(y) < b_{it}(\ell_{t-1}(x), x_{-it}) \ldots$$

Since, for all $y \in \mathrm{gr}\Psi_{t+1}$ such that $\ell_t(y) = (\ell_{t-1}(x), x_{-it}, \bar{x}_{it})$, we have

$$u^i(y) < b_{it}(\ell_{t-1}(x), x_{-it})$$

therefore, there is an open set $v \supseteq \{y \in \mathrm{gr}\Psi_{t+1} / \ell_t(y) = (\ell_{t-1}(x), x_{-it}, \bar{x}_{it})\}$ such that $u^i(y) < b_{it}(\ell_{t-1}(x), x_{-it})$ for all $y \in V$. By the upper semicontinuity of $\Psi_{t+1}: \ell_t(F) \to \tau_{t+1}(F)$, it now follows that there exists $x_{it} \in E^{\nu}_{it}(x_{-it})$ which will satisfy (*). This gives us a contradiction. Hence, $E^{\nu}_{it}: A_{-it}(\ell_{t-1}(x)) \to A_{it}(\ell_{t-1}(x))$ is closed valued.

Suppose $E^{\nu}_{it}: A_{-it}(\ell_{t-1}(x)) \to A_{it}(\ell_t(x))$ is not upper semicontinuous. Then there exists a point $\bar{x}_{-it} \in A_{-it}(\ell_{t-1}(x))$ and a neighborhood G of $E^{\nu}_{it}(\bar{x}_{-it})$ such that for a sequence $\{x^k_{-it}\}^{\infty}_{k=1}$ where $x^k_{-it} \to \bar{x}_{-it}$, there exists $y^k_{it} \in E_{it}(x^k_{-it})$ such that $y^k_{it} \notin G$.

That is, for all $y^k \in \{y \in \mathrm{gr}\Psi_{t+1} / \ell_t(y) = (\ell_{t-1}(x), x^k_{-it}, y^k_{it})\}$, we have

$$(+) \qquad u^i(y^k) < b_{it}(\ell_{t-1}(x), \bar{x}_{-it}) \ldots$$

Since $\Psi_{t+1}: \ell_t(F) \to \tau_{t+1}(F)$ is compact valued, it follows that

$$\Psi_{t+1}(\ell_{t-1}(x),x^k_{-it},y^k_{it})$$

is a compact set. Hence, from (+) it follows that for every k there exists an ε_k for which

$$u^i(y^k) + \varepsilon^k < b_{it}(\ell_{t-1}(x),\bar{x}_{-it})$$

for every $y^k \in grG_{t+1}$ such that $\ell_t(y^k) = (\ell_{t-1}(x),x^k_{-it},y^k_{it})$. Also, since $\Psi_{t+1}: \ell_t(F) \to \tau_{t+1}(F)$ is upper semicontinuous, there exists a neighborhood V^k of y^k_{it} such that

$$u^i(y^k) + (1/2)\varepsilon^k < b_{it}(\ell_{t-1}(x),\bar{x}_{-it})$$

for every $y^k \in grG_{t+1}$ such that $\ell_t(y^k) = (\ell_{t-1}(x),x^k_{-it}, z^k_{it})$ where $z^k_{it} \in V^k$.

Cover $c\ell\{y^k_{it}/k \in N\}$ by the open sets V^k. Then the set is covered by finitely many of these sets because of compactness. Let $\mathcal{E} = \min(\mathcal{E}^k/k \in N\}$. Then, for all k, we have

$$(**) \qquad u^i(y^k) + (1/2)\mathcal{E} < b_{it}(\ell_{t-1}(x),\bar{x}_{-it}). \; . \; .$$

Now, from Lemma 4.3, because of the continuity of $b_{it}: grA_{-it} \to \mathbb{R}$, we have $b_{it}(\ell_{t-1}(x),\bar{x}^k_{-it})$ converges to $b_{it}(\ell_{t-1}(x),\bar{x}_{-it})$. Hence, there exists a $k_0 \in N$ such that for all $k \geq k_0$, there exists a $y^k \in grG_{t+1}$, $\ell_t(y^k) = (\ell_{t-1}(x),\bar{x}^k_{-it},y^k_{it})$ such that

$$u^i(y^k) \geq b_{it}(\ell_{t-1}(x), x^k_{-it}) > u^i(y^k) + (1/4)\mathcal{E}.$$

This gives a contradiction. Hence, the correspondence

$$E^{\nu}_{it}: A_{-it}(\ell_{t-1}(x)) \to A_{it}(\ell_{t-1}(x))$$

ia upper semicontinuous. []

We now prove a central result. We show that for every history $\ell_{t-1}(x)$, there exists a $y^*_t \in A_t(\ell_{t-1}(x))$ such that $y^*_{it} \in E_{it}(y^*_{-it})$ for all $i \in N$. In other words, there is an outcome in period t which rewards everyone if no one deviates from the outcome by offering everyone a payoff larger than what each can guarantee himself.

<u>Lemma 4.5</u>: For every $t \geq 1$ and history $\ell_{t-1}(x) \in \ell_{t-1}(F)$ there exists a $y^*_t \in A_t(\ell_{t-1}(x))$ such that

$$y^*_{it} \in E_{it}(y^*_{-it})$$

for all $i \in N$.

<u>Proof</u>: Since $E_{it}: A_{-it}(\ell_{t-1}(x)) \to A_{it}(\ell_{t-1}(x))$ is nonempty valued, convex valued, compact valued and upper semicontinuous, the correspondence

$$E_t: A_t(\ell_{t-1}(x)) \to A_t(\ell_{t-1}(x))$$

defined by

$$E_t(y_t) := \{E_{it}(y_{-it})\}_{i \in N}$$

is nonempty valued, convex valued, compact valued and upper semicontinuous on $A_t(\ell_{t-1}(x))$.

Hence, by Kakutani's fixed point theorem there exists a $y_t^* \in E_t(y_t^*)$.. Since $y_{it}^* \in E_{it}(y_{-it}^*)$ for all $i \in N$, we have the result. []

We have shown in Lemma 4.5 that if players are punished for deviations in period t by paths in $\Psi_{t+1}(\ell_{t-1}(x), x_{-it}, z_{it})$ where $z_{it} \in A_{it}(\ell_{t-1}(x))$ is a deviation by player i, then there is an outcome y_t^* such that players are better off if they conform, since then there exists a path which rewards all the players jointly. We required that the correspondence $\Psi_{t+1}: \ell_t(F) \rightarrow \tau_{t+1}(F)$ be upper semicontinuous in order to show that y_t^* exists.

Now define $\Psi_t: \ell_{t-1}(F) \rightarrow \tau_t(F)$ in the following way:

$$\Psi_t(\ell_{t-1}(x)) := \{y \in F / \ell_{t-1}(y) = \ell_{t-1}(x), \text{ such that}$$

y_t satisfies $y_{it} \in E_{it}(y_{-it})$ and $\tau_{t+1}(y) \in \Psi_{t+1}(\ell_t(y))\}$.

Hence, the correspondence Ψ_t is a refinement of the correspondence Ψ_{t+1} in the sense that the paths in the graph of Ψ_t are contained in the graph of Ψ_{t+1}. The refinement ensures that in period t none of the players have any incentive to deviate. We will want to show that the correspondence

Ψ_t: $\ell_{t-1}(F) \to \tau_t(F)$ is upper semicontinuous. For every $i \in N$ and $t \to 1$, we can define the set

$$B_{it} := \{y \in F / u^i(y) \geq b_{it}(\ell_{t-1}(y), y_{-it})\}.$$

We now define Ψ_t: $\ell_{t-1}(F) \to \tau_t(F)$ by its graph as follows:

$$gr\Psi_t = [gr\Psi_{t+1}] \cap [\cap_{i=1}^n B_{it}].$$

We will show that Ψ_t: $\ell_{t-1}(F) \to \tau_t(F)$ is nonempty valued, closed valued and upper semicontinuous.

Lemma 4.6: The $gr\Psi_t$ is closed.

Proof: Let x be a limit point of B_{it}. From the continuity of the function b_{it}: $grA_{-it} \to \mathbb{R}$, it follows that for any $\{x^k\}_{k=1}^{\infty}$ in B_{it} such that $x^k \to x$, we have $b_{it}(\ell_{t-1}(x^k), x^k_{-it})$ converging to $b_{it}(\ell_{t-1}(x), x^k_{-it})$. Since $u^i(x^k) \geq b_{it}(\ell_{t-1}(x^k), x^k_{-it})$ for all k, we have $u^i(x) \geq b_{it}(\ell_{t-1}(x), x_{-it})$. Therefore, $x \in B_{it}$. Hence, B_{it} is a closed subset of F. Since $gr\Psi_{t+1}$ is also a closed subset of F, we have that

$$gr\Psi_t = [gr\Psi_{t+1}] \cap [\cap_{i=1}^n B_{it}]$$

is a closed subset of F. []

Lemma 4.7: The correspondence $\psi_t: \ell_{t-1}(F) \rightarrow \tau_t(F)$ is nonempty valued, closed valued and upper semicontinuous.

Proof: From Lemma 4.5, we know that for every history $\ell_{t-1}(x) \in \ell_{t-1}(F)$ there exists a $y_t^* \in (\ell_{t-1}(F))$ such that

$$y_{it}^* \in E_{it}(y_{-it}^*)$$

for all $i \in N$.

That is, there exists $y \in gr\psi_{t+1}$ such that $\tau_{t-1}(y) \in \psi_{t+1}(\ell_{t-1}(x), y_{-it}, y_{it})$ and $u^i(y) \geq b_{it}(\ell_{t-1}(x), y_{-it})$ for all $i \in N$. Therefore, $\tau_t: \ell_{t-1}(F) \rightarrow \tau_t(F)$ is nonempty valued. From Lemma 4.6, its graph is a closed subset of F. Therefore, it is upper semicontinuous and closed valued. This completes the proof. []

What we have shown from Lemma 4.1 to Lemma 4.7 is that if we start with an upper semicontinuous and closed valued ψ_{t+1}, then it is possible to define a nonempty valued, closed valued and upper semicontinuous correspondence ψ_t such that the histories prescribed by the correspondence are such that the players have no incentive to deviate in period t knowing that after that they will be punished with histories prescribed by the correspondence ψ_{t+1}, or be rewarded by some history prescribed by the correspondence ψ_{t+1}. Hence, ψ_t can be thought of as selecting those histories that are supported by equilibrium strategies of the subgames generated by histories up to period $t - 1$, by promises of punishments or rewards by histories prescribed by ψ_{t+1}. We can again define a nonempty valued, closed valued upper semicontinuous correspondence ψ_{t-1} by prescribing that plays follow paths dictated by the correspondence ψ_t. We can do this in the same way as in defining ψ_t from ψ_{t+1}.

Hence, we can define a sequence of such correspondences. We will use this kind of construction to show the existence of subgame perfect equilibrium points.

Let ψ_t^k: $\ell_{t-1}(F) \to \tau_t(F)$ be defined as follows. $\psi_{k+\nu}^k(\ell_{k+\nu-1}(x)) := \tau_{k+\nu}(F)$. That is, for any $t > k$, the correspondence assigns the whole set of feasible paths to histories. This is upper semicontinuous and closed valued. For $t \leq k$,

$$[gr\psi_t^k] = [gr\psi_{t+1}^k] \cap [\cap_{i=1}^n B_{it}].$$

Hence, for any $t \leq k$, the correspondences are defined recursively by using the construction laid out in Lemmas 4.1 to 4.7.

Now define the correspondence

$$\psi_t^0\colon \ell_{t-1}(F) \to \tau_t(F)$$

by

$$\psi_t^0(\ell_{t-1}(x)) = \cap_{k=1}^{\infty} \psi_t^k(\ell_{t-1}(x)).$$

<u>Lemma 4.8</u>: The sequence of correspondences $\{\psi_t^0\}_{t=1}^{\infty}$ are all nonempty valued, closed valued and upper semicontinuous, and satisfy the condition that

$$gr[\psi_t^0] = [gr\psi_{t+1}^0] \cap [\cap_{i=1}^n B_{it}]$$

Proof: Since each correspondence ψ_t^k is nonempty-valued and closed-valued, by the finite intersection property of compact sets $\psi_t^0(\ell_{t-1}(x)) = \cap_{k=1}^{\infty} \psi_t^k(\ell_{t-1}(x))$ is nonempty for all histories $\ell_{t-1}(x)$. Hence, the correspondence is nonempty valued. Now,

$$
\begin{aligned}
gr[\psi_t^0] &= gr[\cap_{k=1}^{\infty}\psi_t^k] \\
&= \cap_{k=1}^{\infty} [gr\psi_t^k] \\
&= \cap_{k=1}^{\infty} \{[gr\psi_{t+1}^k] \cap [\cap_{i=1}^{n} B_{it}]\} \\
&= [\cap_{k=1}^{\infty} gr\psi_{t+1}^k] \cap [\cap_{i=1}^{n} B_{it}] \\
&= gr\ [\cap_{k=1}^{\infty} \psi_{t+1}^k] \cap [\cap_{i=1}^{n} B_{it}] \\
&= [gr\ \psi_{t+1}^0] \cap [\cap_{i=1}^{n} B_{it}].
\end{aligned}
$$

Since this graph is closed, the proof is complete. []

We now claim that the set ψ_1^0 is the set of histories that will be generated by perfect equilibrium points of the infinite horizon games. We now state the main result of this section.

Theorem 4.9: Infinite horizon games with simultaneous moves (IHGSM) satisfying Assumptions 1-5 always have subgame perfect equilibrium points. The set ψ_1^0 is the set of histories that can be generated by the perfect equilibrium paths of an IHGSM.

Proof: We will prove the results by showing that for every $y \in \psi_1^0$, there is a strategy combination $h \in H$ whose equilibrium path is y and which is subgame perfect. Let

$$W_{it}(\ell_{t-1}(x)) := \{\tau_t(y)/y \in \text{graph } \psi_t^0,$$
$$\ell_{t-1}(y) = \ell_{t-1}(x) \text{ and } u^i(y) = w_{i(t-1)}(\ell_{t-1}(x))\}.$$

Hence, $W_{it}(\ell_{t-1}(x))$ is the collection of paths in the subgame generated by $\ell_{t-1}(x)$ which give player i the worst payoff for paths along $\psi_t^0(\ell_{t-1}(x))$. By Lemma 4.8 this is well defined for every feasible history $\ell_{t-1}(x)$ up to time t.

For each t and i, choose $z_t^i(\ell_{t-1}(x)) \in W_{it}(\ell_{t-1}(x))$ which is possible if we invoke the axiom of choice.

Now, define $h \in H$ as follows. Let $y \in \psi_1^0$ be a feasible history. Then for any time t and history $\ell_{t-1}(x)$ up to time t - 1, we define

$$h_i(\ell_{t-1}(x)) := \begin{cases} y_{it} \text{ if } \ell_{t-1}(x) = \ell_{t-1}(y) \text{ for all } i \in N. \\ [z^j_{(k-1)}(\ell_{k-2}(x))]_{it} \text{ if } j \in N \text{ is the largest integer} \\ \text{such that } x_{j(k-1)} \neq h_{j(k-1)}(\ell_{k-2}(x)), \text{ where } k \text{ is the} \\ \text{smallest integer } 1 \leq k \leq t - 1 \text{ such that} \\ k \leq s \leq t - 1 \text{ satisfying } \ell_s(x) \in \text{graph } h_s. \end{cases}$$

Hence, if there are no deviations the strategy is to play along the path $y \in \psi_1^0$. If there is a deviation then the strategy is to punish one of the deviators by playing a path in the appropriate set.

We will now show that the strategy n-tuple $h \in H$ defined above is a perfect equilibrium point of the infinite horizon game.

Let $\ell_{t-1}(x)$ be any history up to time t - 1. The strategy $h \in H$ is such that on the subgame generated by $\ell_{t-1}(x)$, the subgame strategy $h|_{\ell_{t-1}(x)}$ has a path that is in $\psi_t^0(\ell_{t-1}(x))$. From the definition of $\Psi_t^0(\ell_{t-1}(x))$ any $y \in \mathrm{gr}\psi_t^0$ such that $\ell_{t-1}(y) = \ell_{t-1}(x)$ is such that if player $i \in N$ deviates from $h|_{\ell_{t-1}(x)}$ in period t by playing x_{it}, then his payoff is

$$w_{it}(\ell_{t-1}(x),y_{-it},x_{it}) \leq b_{it}(\ell_{t-1}(x),y_{-it}) \leq u^i(y).$$

Since this holds for every subgame and every player $i \in N$ we have shown that $h \in H$ as defined is a subgame perfect equilibrium of the infinite horizon game.

Since we defined $h \in H$ for an arbitrary $y \in \psi_1^0$, we have completed the proof. []

5. Some Special Classes of Infinite Horizon Games

Infinite horizon games with perfect information as discussed in Harris (1985) essentially rule out repeated games, an important class of infinite horizon games which have been extensively studied in game theory and in game theoretic applications to economics. Since the class of infinite horizon games that we have studied includes repeated games as a special case, it is of interest to see how Theorem 4.9 applies to this class of games. Let

$$G := \{X^j, u^j\}_{j \in N}$$

be a one-shot game with X^j being finite for each player. Let $G_\mu := \{\mu(X^j),\ u^j_\mu\}_{j \in N}$ be the corresponding one-shot game in which $\mu(X^j)$ is be a one-shot game with X^j being finite for each player. Let $G_\mu := \{\mu(X^j),\ u^j_\mu\}_{j\in N}$ be the corresponding one-shot game in which $\mu(X^j)$ is the mixed strategy space of player $j \in N$ and u^j_μ is the expected payoff of player $j \in N$. Let

$$X(\mu) := \otimes^n_{i=1} \mu(X^j)$$

denote the outcomes of the one-shot game G_μ. Then $X(\mu)$ is a compact convex set.

Define $G^\infty := \{H_j, u^j_\infty\}_{j\in N}$

as the supergame obtained by repeating G_μ an infinite number of time so that $h^j \in H_j$ is a sequence of functions $\{h^j_t\}^\infty_{t=1}$ such that

$$h^j_t\colon [X(\mu)]^{t-1} \to \mu(X^j)$$

is a function from the histories up to time $t - 1$ into the strategy set of player $j \in N$. The payoff function in the supergame G^∞ is defined as:

$$u^j_\infty(x) := \Sigma^\infty_{t=1} \delta^t u^j_\mu(x_t)$$

where $x_t \in X(\mu)$ and $x = \{x_t\}^\infty_{t=1}$ is an outcome path of G^∞.

<u>Theorem 5.1</u>: G^∞ has a subgame perfect equilibrium point.

Since $X(\mu)$ is a compact convex set in the usual topology, and the actions spaces $\mu(X^j)$ are invariant over time, Assumption 5 is also satisfied.

Also, since $u^j_\mu: X(\mu) \to \mathbb{R}$ is the expected payoff function of player $j \in N$, it is concave over $X(\mu)$. Since the set of feasible histories is $X := \otimes^\infty_{t=1} [X(\mu)]$, it follows that $u^j_\infty: X^\infty \to \mathbb{R}$ is concave over histories. This shows that Assumption 4 holds.

Since all Assumptions 1–5 of Theorem 4.9 hold, the theorem applies and the proof is complete. []

While it is comforting to know that the theorem applies to a fairly wide class of infinite horizon games, repeated games are only a special case of infinite horizon games in which the action spaces do not change over time or with histories.

There are infinite horizon games in economics which are not repeated games. We will study a class of such games. A version of this class of games appears in Fudenberg and Levine (1983).

We assume there are n firms. Firms $j \in N$ has an initial capital stock given by k^j_0. At any time t, firm j's feasible investment strategies are described by $[0, \bar{I}^j_t]$, where $\bar{I}^j_t$ denotes the maximum investment that can occur.

The actual investment that a firm can make is a function of its capital stock in the previous period. Hence, its feasible investment possibilities at time t, given the capital stock K^j_{t-1} at time t is given by $[0, i^j_t(K^j_{t-1})] \subseteq [0, \bar{I}^j_t]$. The function $i^j_t: [0, \bar{K}^j_{t-1}] \to [0, \bar{I}^j_t]$ is assumed to be monotone increasing and concave on $[0, \bar{K}^j_{t-1}]$, where $\bar{K}^j_{t-1}$ is the maximum capital stock that the firm can accumulate up to time t - 1.

The amount $i_t^j(K_{t-1}^j)$ is the maximum that firm j can invest when K_{t-1}^j is the capital stock in time t - 1.

We assume that given the capital stocks K_t^j of the firms in time t, there is an equilibrium in the product market with associated net revenues (i.e., total revenues minus operating costs). These net revenues, for instance, could be the result of a Nash equilibrium in quantities given the short run costs at time t. We are, therefore, implicitly assuming that the choice of quantities at time t has no effect on the game later. We will call this the <u>Fudenberg-Tirole-Spence assumption</u>.

Each firm's instantaneous profit in time period t is net revenues minus the investment in that period, and we will denote it by

$$\pi_t^j\colon\ ((n,t) \to \mathbb{R}$$

where

$$I(n,t)\ :=\ \otimes_{j=1}^{n}\ [0,\bar{I}_t^j].$$

The payoff function $\pi_t^j(I_t^1,\ldots,I_t^n)\ :=\ v_t^j(K_t^1,\ldots,K_t^n)\ -\ I_t^j$, where $v_t^j(K_t^1,\ldots,K_t^n)$ are the net revenues of the firm in period t.

We will assume that $v_t^j(\bullet)$ is concave. Then, it is not difficult to check that $\pi_t^j(\bullet)$ is concave. The payoff function of the firm over the infinite horizon is then given y

$$\pi_\infty^j(I)\ :=\ \Sigma_{t=1}^{\infty}\ \delta_j^t\pi_t^j(I_t)$$

where I denotes an entire history of investment by the firms and I_t is the investment made in period t. $\delta_j \in (0,1)$ is the discount rate of firm j.

$$\pi^j_\infty(I) := \Sigma^\infty_{t=1} \delta^t_j \pi^j_t(I_t)$$

where I denotes an entire history of investment by the firms and I_t is the investment made in period t. $\delta_j \in (0,1)$ is the discount rate of firm j.

Since $\pi^j_t(\cdot)$ is concave for all $t \geq 1$, we have $\pi^j_\infty(\cdot)$ is concave over the histories of investments.

Theorem 5.2: The infinite horizon game of strategic investment has subgame perfect equilibria.

Proof: Assumptions 1-3 of Theorem 4.9 is clearly satisfied.

Assumption 4 holds for the payoff functions. Since

$K^j_{t-1} = \Sigma^{t-1}_{\nu=1} I^j_\nu + K^j_0$, and the investment function $i^j_t: [0,\bar{K}^j_{t-1}] \to [0,\bar{I}^j_t]$ is assumed to be monotone increasing and concave on $[0,\bar{K}^j_{t-1}]$, Assumption 5 holds.

Since all the assumptions of the theorem hold, it follows that by Theorem 4.9 we have the result. []

6. Conclusion

We have given a solution to the problem posed in Harris (1985). However, in the course of the proof we used the fact that the player set N is finite when we used Kakutani's fixed point theorem. Also, the strategy spaces were assumed to be finite dimensional. It is suggested here that both assumptions could possibly be relaxed if, instead of Kakutani's fixed point result, we use Glickberg's theorem which is the infinite dimensional counterpart of Kakutani's fixed point result.

The other point that one would like to make is that though we have applied the result to two classes of infinite horizon games, it is not difficult to see that many other interesting classes of infinite horizon games could fit into our framework.

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