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THE RATE OF CONVERGENCE TO EFFICIENCY IN THE BUYER'S BID DOUBLE AUCTION AS THE MARKET BECOMES LARGE

by

Mark A. Satterthwaite and Steven R. Williams

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Abstract

Consider a market with m sellers, each having a single item to sell, and m buyers, each desiring to buy at most one item. Each trader has a reservation value for the item independently drawn from the unit interval; sellers' values have distribution F_1 and buyers' values have distribution F_2. Sellers and buyers simultaneously submit offers and bids. These offers and bids determine a closed interval in which a market-clearing price may be selected. In the buyer's bid double auction (BBDA) the price selected is the upper endpoint of this interval. Trade then occurs at this price.

We consider Bayesian Nash equilibria in which all sellers use one strategy and all buyers use a second strategy. Each seller in the BBDA has a dominant strategy to set his offer equal to his reservation value. In response to these dominant strategies each buyer has an incentive to bid less than his reservation value. This strategic misrepresentation causes the BBDA to be _ex post_ inefficient. We show that the amount of misrepresentation by buyers must be small when the market is large. In fact, we prove that under all equilibrium responses of the buyers to the sellers' dominant strategies the difference between a buyer's bid and his reservation value is $O(1/m)$, regardless of the distributions F_1 and F_2. Thus, competitive pressures as the market grows quickly force buyers towards truthful revelation and the equilibrium towards the _ex post_ efficient, perfectly competitive allocation.

The Rate of Convergence to Efficiency in the Buyer's Bid Double Auction as the Market Becomes Large*

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1. Introduction

A goal of a market is to implement a Pareto efficient allocation of resources. Classical general equilibrium theory focuses on the existence of prices that implement an efficient allocation. Solving for such prices requires information about traders' preferences. This information is typically not possessed by any one individual or institution, for each trader typically has some private information about his own preferences. The market must somehow elicit the necessary private information if it is to implement an efficient allocation.

A major obstacle to accomplishing this is the incentive that traders may have to misrepresent their private information. In a market with prices this strategic behavior takes the form of distorting supply and demand in

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order to influence price. This behavior may cause ex post inefficiency, i.e., all potential gains from trade are not realized. Intuitively strategic behavior is only significant in small markets, for the ability to affect prices decreases as the number of traders in the market becomes large. In the limiting case of a market with a continuum of traders strategic behavior vanishes and traders willingly reveal their private information. Appropriate prices can then be calculated and efficiency results.

This paper develops the intuition that the number of traders is critical to the performance of a market by using Harsanyi's notion (1967-68) of a Bayesian game to model the impact of private information upon a simple market. Consider a market with m sellers, each having a single item to sell, and m buyers, each wanting to buy at most one item. Each trader has a reservation value for the item that is independently drawn from the unit interval; a seller's value is drawn from distribution F_1 and a buyer's value is drawn from distribution F_2. A trader privately knows his own reservation value. Each trader is risk neutral.

The items are reallocated according to the following rules. Sellers and buyers simultaneously submit offers and bids. These offers and bids determine a closed interval in which a market-clearing price can be selected. We choose as the price the upper endpoint of this interval. Trade then occurs at this price between those buyers whose bids are at least as great as it and sellers whose offers are strictly less than it. We call this procedure for allocating resources the buyer's bid double auction (BBDA) because in the one buyer-one seller case the buyer's bid determines the price whenever trade occurs.

We consider Bayesian Nash equilibria in which all sellers use one strategy and all buyers use a second strategy. Each seller in the BBDA has a dominant strategy to set his offer equal to his reservation value because he can not influence price when he trades. In response to these dominant strategies, each buyer has an incentive to bid less than his reservation value, which causes the BBDA to be *ex post* inefficient. We show, however, that the amount of misrepresentation by buyers must be small when the market is large. In fact, we prove that under all equilibrium responses of the buyers to truthful revelation by the sellers the difference between a buyer's bid and his reservation value is $O(1/m)$, regardless of the distribution of the reservation values. Thus, as the market grows large, competitive pressures quickly force buyers towards truthful revelation and the equilibrium outcome towards an *ex post* efficient, perfectly competitive allocation.

Two aspects of this result deserve emphasis. First, our rate of convergence is exact, not asymptotic. It allows comparison of the efficiency of equilibria of different sized markets, no matter how small or large. As our examples illustrate, the gain in efficiency is dramatic even as the number of traders varies over a small range, e.g., from $m = 2$ to $m = 16$. Second, our result applies to all Bayesian Nash equilibria in which the buyers symmetrically adopt the same response to the sellers' dominant strategy of truthful revelation. Large misrepresentations simply cannot be equilibrium behavior in a large market, regardless of which equilibrium is chosen.

Our result is analogous to a classic result in general equilibrium theory. Building on Debreu and Scarf's (1963) result on the convergence of the core to the Walrasian allocations, Debreu (1975) and Grodal (1975) showed that as a regular Arrow-Debreu economy is replicated, the maximum distance between a core allocation and its nearest Walrasian allocation is $O(1/m)$, where m is the number of replications.[1] Beyond the obvious fact that the same rate holds, both results show that in a large market all equilibrium outcomes are close to a Walrasian outcome. Some differences, however, between these results should be kept in mind. Our model confronts the inefficiency that private information and individual incentives causes, while the cooperative approach assumes that the outcome of trade is efficient. Additionally, our result concerns an explicit procedure for arriving at an allocation. The core results fail to explain how the core is reached. It is important to note, however, that the Arrow-Debreu framework is much richer than our elementary model.

The questions that give rise to this work have been articulated by Hayek (1945), Arrow (1959), and Hurwicz (1972) among others. Hayek emphasized the importance of modeling the impact of private information upon an economy: the resource allocation problem

> . . .is thus in no way solved if we can show that all the facts, if they were known to a single mind (as we hypothetically assume them to be given to the observing economist), would uniquely determine the solution; instead we must show how a solution is produced by the interactions of people each of whom possesses only partial knowledge (1945, p. 530).

Arrow (1959) criticized general equilibrium theory for failing to explain how Walrasian prices are formed. Hurwicz has been a pioneer in formally evaluating the informational and incentive feasibility of economic mechanisms:

[1] See Hildenbrand (1982) and Dierker (1982) for surveys of this topic.

> On the informational side, the question is whether the mechanism allows for the dispersion of information and limitations on the capacity of various units to process information. On the incentive side, there is the problem whether the rules prescribed by the mechanism are compatible with either individual or group incentives (1972, p. 298-9).

Because we model the BBDA as a Bayesian game, our result reflects the role that private information and individual incentives play within an explicit process of price formation. We do not, however, deal with limitations on rationality and information processing. Nevertheless, our result that all equilibrium strategies of the BBDA in a large market are close to truthful revelation suggests that cognitive limitations are unimportant in large markets.

An important precursor of our result is Roberts and Postlewaite's (1976) study of the noncooperative incentive that an agent within an Arrow-Debreu exchange economy has to act strategically. In their model each agent first reports an excess demand function, a competitive equilibrium is computed based on the reports, and finally goods are allocated according to the computed solution. They show that as a generic economy becomes large each agent's incentive to misreport his excess demand function vanishes. Their result, while related, is different from our result because it does not concern equilibrium behavior by the agents and it does not state a rate at which the incentive to misreport vanishes.

Chatterjee and Samuelson (1983) did important early work on the bilateral double auction using a Bayesian game model. Leininger, Linhart, and Radner (1986) and Satterthwaite and Williams (1987) extended the analysis of Chatterjee and Samuelson and demonstrated a great multiplicity of equilibria in the bilateral case. The mathematical techniques of this paper follow naturally from the techniques developed in Satterthwaite and

Williams (1987). Wilson (1985a) showed that double auctions achieve Holmstrom and Myerson's (1983) standard of interim incentive efficiency if the market is sufficiently large. This paper complements Wilson's result by showing that not only do large markets achieve interim incentive efficient performance, they converge at a specified, rapid rate to _ex post_ efficiency.

Myerson and Satterthwaite (1983) developed techniques for computing optimal trading mechanisms when reservation values are private on both sides of the market. For given distributions F_1 and F_2 the optimal mechanism maximizes the _ex ante_ expected gains from trade subject to the constraints of private information and strategic behavior. Gresik and Satterthwaite (1986, Th. 5) showed that if the _ex ante_ optimal mechanism is used, then the maximal gap between the reservation values of a buyer and a seller who are _ex post_ inefficiently excluded from trade is at most $O((\ell n\ m)^{1/2})/m)$. They conjectured that the tighter $O(1/m)$ rate of our result holds. Our convergence result improves upon their's in two ways. First it verifies their conjecture, for the order of the optimal mechanism's bound must be as small as the order of the BBDA's bound. Second, our result concerns a realistic trading procedure. The rules of the BBDA are stated in terms of the bids and offers; the Bayesian game framework is used not to define the BBDA, but to analyze the outcome of trade under this procedure when there is private information. By contrast, an optimal mechanism is defined in terms of the distributions F_1 and F_2; changing the distributions changes the optimal mechanism's rules for allocating the items. As Wilson (1985b) has emphasized, the rules of real-world trading mechanisms are independent of the underlying distributions.

Finally, McAfee and McMillan (1987) have surveyed the literature on one-sided and double auctions.[2] Their survey shows both the debt that our paper and other papers on double auctions owe to the literature on one-sided auctions and the distance that the double auction literature has to go before it reaches an equivalent level of sophistication. For example, our results are for the independent private values model only. Consideration of Milgrom and Weber's more general model (1982) of affiliated values has as yet proved intractable.

2. Notation, Model, and Preliminary Observations

Consider a market with m buyers ($m \geq 2$) and m sellers in which each seller wishes to sell an indivisible item and each buyer wishes to purchase at most one item.[3] Each seller has a reservation value independently drawn from the distribution F_1, and each buyer has a reservation value independently drawn from F_2. A trader's reservation value is his own private information. Each distribution F_i is a C^1 function whose density $f_i \equiv F_i'$ is positive at every point in $(0,1)$ and zero outside $[0,1]$. The distributions F_1 and F_2 are common knowledge among the traders. We use v_1 to denote a seller's reservation value and v_2 to denote a buyer's reservation value. A seller's utility is zero if he fails to sell his item

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[2] In a one-sided auction a seller with a known reservation value is attempting to maximize his revenue in selling an object(s) to a set of buyers whose reservation values are private. Thus the distinction between a one-sided auction and a double auction of the type we are studying is that in a double auction both buyers and sellers have private information while in a one-sided auction only the buyers have private information.

[3] We have excluded the bilateral case ($m = 1$) because its analysis is different from the $m \geq 2$ case. See Satterthwaite and Williams (1987).

and $p-v_1$ if he does sell and the market price is p. Similarly a buyer's utility is zero if he fails to buy and v_2-p if he does buy.

These are the common knowledge rules of the BBDA. Every trader simultaneously submits a bid/offer. These bids/offers are arrayed in increasing order $s_{(1)} \le s_{(2)} \le \ldots \le s_{(2m)}$ and the price p is set at $s_{(m+1)}$. Trade occurs among sellers whose offers are strictly less than p and buyers whose bids are greater than or equal to p. When ties occur, p may not be a market-clearing price. In order to explain exactly who trades under the BBDA we refer to Table 2.1.

Let s be the number of sellers whose offers exceed p, k be the number of sellers whose offers equal p, t be the number of buyers whose bids exceed p, and j be the number of buyers whose bids equal p. There are m-s-k offers and m-t-j bids less than p. Note that $s+k+t+j \ge m$ traders bid/offer at least as much as p, since $p = s_{(m+1)}$. Therefore

(2.01) $\quad t+j \ge m-s-k,$

which means the demand t+j at the price p is necessarily at least as large as the supply m-s-k.

Consider the case where a single bid/offer uniquely determines $s_{(m+1)}$, i.e., $j+k = 1$ and $t+s = m-1$. In (2.01) bring s+k to the left-hand side; the left-hand side then sums exactly to m and (2.01) holds with equality. In this case, supply exactly equals demand and every buyer whose bid is at least p purchases an item and every seller whose offer is less than p sells his item. Next consider the remaining case where at least two bid/offers equal $s_{(m+1)}$, i.e., $j+k \ge 2$ and demand t+j strictly exceeds supply m-s-k. The BBDA then prescribes that the supply of m-s-k items is allocated beginning with the buyer who bid the most and working down the list of buyers whose bids are at least p. If in this process a point is reached where two or more buyers submitted identical bids and the remaining supply of unassigned items is insufficient to serve them, then the available supply is rationed among these bidders using a lottery that assigns each an equal chance of receiving an item. This completes our definition of the BBDA.

Table 2.1. Determination of the market price.

	Sellers	Buyers
# bids/offers > $s_{(m+1)}$	s	t
# bids/offers = $s_{(m+1)}$	k	j
# bids/offers < $s_{(m+1)}$	m-s-k	m-t-j

We adopt the Bayesian game framework to analyze the outcome of trade. Within this framework a trader's reservation value is his <u>type</u> and his <u>strategy</u> is a function that specifies a bid/offer for each of his possible types. An <u>equilibrium</u> consists of a strategy for each trader such that, for each of his possible reservation values, the bid/offer his strategy specifies maximizes his expected utility given the other traders' strategies and the distributions of their reservation values.

We now identify some basic properties of equilibria in the BBDA. The most fundamental property is that a seller can not marginally influence the price p by altering his offer in any case where he sells his item. This follows from the BBDA's rule that a seller only sells if his offer is strictly less than the price $p \equiv s_{(m+1)}$. It follows that sellers have no incentive to act strategically, i.e., each seller's dominant strategy is to submit his reservation value as his offer.[4] Let $\tilde{S}$ denote this strategy:

$$(2.02) \qquad \tilde{S}(v_1) = v_1$$

for all $v_1 \in [0,1]$.

Theorem 2.1: In the BBDA, $\tilde{S}$ is a dominant strategy for each seller.

Proof: Select a strategy for each buyer and for all but one of the sellers, and let v_1 be the reservation value of the exceptional seller. This seller would be no worse off by submitting an offer of $b = v_1$ rather than $b' > v_1$ because: (i) if he sells the item with the offer b' at a price $p > b'$, then he also sells it with the offer of $b = v_1$ at the unchanged price p; and (ii) if he fails to sell the item with the offer $b' \geq p$, he can only gain if he instead offers $b = v_1$, for the price whenever he trades necessarily exceeds his offer. A similar analysis shows that the seller is no worse off with the offer of $b = v_1$ than an offer $b'' < v_1$. Q.E.D.

We assume throughout this paper that all sellers adopt the strategy $\tilde{S}$. We also assume that all buyers use the same strategy. We denote the common strategy of the buyers as $B: [0,1] \to \mathfrak{R}$ and we denote a set of strategies where each seller plays $\tilde{S}$ and each buyer plays B as $<\tilde{S},B>$.

In order to further establish the properties of equilibria $<\tilde{S},B>$ we need additional notation:

$\pi(v_2,b;B)$ = a buyer's expected utility when v_2 is his reservation value, b is his bid, and B is the common strategy of the other buyers;

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[4] A standoff equilibrium also exists in which all buyers bid zero, all sellers offer one, and no trade occurs.

$P(b;B)$ = the probability a buyer will trade when b is his bid and B is the common strategy of the other buyers;

$C(b;B)$ = the expected payment of a buyer when b is his bid and B is the common strategy of the other buyers.

Note that $\pi(v_2,b;B) = v_2P(b;B) - C(b;B)$ and that $P(\cdot;B)$ is a probability distribution on the interval $[0,1]$. Finally, $P(\cdot;B)$ is strictly increasing on this interval because (i) the density f_1 is positive on $(0,1)$ and (ii) each seller uses his dominant strategy $\tilde{S}$.

<u>Theorem 2.2</u>: Let $<\tilde{S},B>$ be an equilibrium in the BBDA. The function B must have the following properties: (i) $0 < B(v_2)$ for all $v_2 \in (0,1]$; (ii) $B(v_2) \leq v_2$ for all $v_2 \in [0,1)$; (iii) $B(v_2)$ is strictly increasing on $[0,1]$ and differentiable almost everywhere.

<u>Proof</u>: An important preliminary observation is this. Select a buyer. Suppose each seller uses $\tilde{S}$ and the other m-1 buyers use the strategy B, where no restriction is placed on B. For any $p \in (0,1)$, if the selected buyer bids p, then there is a positive probability that the price will be p and the selected buyer will receive an item at this price. This is true because, given any array of bids from the m-1 buyers using B, a positive probability always exists that the offers of the m sellers will fall such that exactly m of the bids/offers of these 2m-1 traders are strictly less than p, i.e., $p = s_{(m+1)}$.

This observation immediately implies (i) and (ii). If a buyer with reservation value $v_2 > 0$ bids $b'' \leq 0$, his expected utility is zero because no seller's offer will be less than b". Bidding $b' \in (0,v_2)$, however, provides him with a positive probability of a profitable trade. This proves (i). If a type v_2 buyer ($v_2 < 1$) bids $b > v_2$, then a positive probability exists that the price will be in $(v_2,b]$ and he will trade at a loss. Reducing his bid to $b = v_2$ eliminates these losses without eliminating any profitable trades. This proves (ii).[5]

We use an argument from Chatterjee and Samuelson (1983, Th. 1) to show that B must be nondecreasing. Let $v_2'' > v_2'$. Because <S,B> is an equilibrium, we have

$$\pi(v_2',B(v_2');B) - \pi(v_2',B(v_2'');B) \geq 0 \tag{2.03}$$

and

$$\pi(v_2'',B(v_2'');B) - \pi(v_2'',B(v_2');B) \geq 0. \tag{2.04}$$

Adding these inequalities, we obtain

$$\begin{aligned}\pi(v_2'',B(v_2'');B) &- \pi(v_2',B(v_2'');B) \\ &+ \pi(v_2',B(v_2');B) - \pi(v_2'',B(v_2');B) \geq 0.\end{aligned} \tag{2.05}$$

Recall that $\pi(v_2,b;B) = v_2P(b;B) - C(b;B)$. Therefore (2.05) becomes

$$(v_2'' - v_2')P(B(v_2'');B) + (v_2' - v_2'')P(B(v_2');B) \geq 0, \tag{2.06}$$

or equivalently

$$(v_2'' - v_2')[P(B(v_2'');B) - P(B(v_2');B)] \geq 0. \tag{2.07}$$

By assumption, $v_2'' > v_2'$; therefore, $P(B(v_2'');B) \geq P(B(v_2');B)$. Since $P(\cdot;B)$ is increasing, we conclude that $B(v_2'') \geq B(v_2')$.

We now show by contradiction that B cannot be constant over any interval with non-empty interior. Suppose that $B(v_2) = b'$ for all v_2 in such an interval I. The bounds that we have derived upon B imply that $0 < b' < 1$. Our argument rests upon the following point: the probability of trade $P(b;B)$ is discontinuous at $b = b'$. This is true because the following

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[5] We can not rule out extremely small bids (e.g., b = -1000) for a type zero buyer and extremely large bids for a type one buyer. These are probability zero cases that do not affect the expected utilities of other traders and therefore do no affect equilibrium calculations.

events occur simultaneously with positive probability: (i) each buyer's reservation value is in I and therefore all buyers bid b', (ii) at least one seller's offer is less than b', and (iii) at least one seller's offer is greater than b'.[6] Stipulations (i)-(iii) imply that the market price is b', the market fails to clear at this price, and the available units are allocated randomly among the buyers. Raising the selected buyer's bid from b' to $b'' > b'$ ensures that he receives an item with probability one in the stipulated situation, rather than with some probability less than one under the random allocation rule. Therefore an $\varepsilon > 0$ exists such that $P(b'';B) > P(b';B) + \varepsilon$ for all $b'' > b'$.

A buyer whose reservation value is in I has an incentive to raise his bid above b'. This is seen by computing his gain from increasing his bid from b' to b'':

$$(2.08) \qquad \begin{aligned} &\pi(v_2,b'';B) - \pi(v_2,b';B) \\ &= v_2(P(b'';B) - P(b';B)) + C(b';B) - C(b'';B) \\ &> v_2\varepsilon - (b'' - b'), \end{aligned}$$

where $(b''-b')$ is an upper bound on how much expected price can change when the buyer increases his bid to b''. For b'' sufficiently near b', (2.08) is positive, which completes the contradiction.

Finally the existence of B' almost everywhere follows from the monotonicity of B by a well-known theorem in analysis.[7] Q.E.D.

Two points should be emphasized about the monotonicity of the buyers' strategy in an equilibrium $<S,\tilde{B}>$. First, it implies that the probability of ties in the array of bids and offers is zero. Consequently we can ignore

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[6] Note that (ii) requires that $m \geq 2$.

[7] See Royden (1968, p. 96).

ties and the randomized allocations that they necessitate. Second, the argument in Theorem 2.2 can be applied to double auctions besides the BBDA to show that when $m \geq 2$ an equilibrium common strategy of either side of the market must be increasing over all intervals in which the probability of trade is positive. Equilibrium strategies in the bilateral case may not be increasing; Leininger, Linhart, and Radner (1986), for instance, have derived step function equilibria in the bilateral split-the-difference double auction. Such equilibria, however, do not exist in this double auction when $m \geq 2$.

3. The First Order Approach

In this section we consider a buyer's first order condition for maximizing his expected utility conditional on his reservation value v_2, the use of a common strategy B by the other m-1 buyers, and the use of S by each seller. If <S,B> is an equilibrium, then this conditional expected utility is maximized at $B(v_2)$. We interpret the first order condition as a differential equation that must be satisfied almost everywhere by any function B that defines an equilibrium <S,B>. Conversely, we show that any increasing function B defines an equilibrium <S,B> if (i) B satisfies the differential equation, (ii) B respects the bounds $0 < B(v_2) < v_2$ for all $v_2 \in (0,1]$, and (iii) the distribution F_1 of each seller's reservation value satisfies a monotonicity condition.

The first order condition is formally derived in the Appendix. Here we state the condition and describe it intuitively. In order to state it we must define three probabilities:

K_m = the probability that bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-1 buyers using strategy B and m-1 sellers using $\tilde{S}$.

L_m = the probability that bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-2 buyers using strategy B and m sellers using $\tilde{S}$.

M_m = the probability that the bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in a sample of m-1 buyers using strategy B and m sellers using $\tilde{S}$.

Recall that $s_{(k)}$ is the kth order statistic in the specified sample of bids and offers (i.e., the bid/offer that ranks k<u>th</u> from the bottom).

Suppose a type v_2 buyer considers raising his bid by Δb above the value b, which may or may not equal the value $B(v_2)$. His incremental expected utility is

$$\left[mf_1(b)K_m\Delta b + (m - 1)\frac{f_2(\bar{v}_2)}{B'(\bar{v}_2)} L_m\Delta b\right](v_2 - b) - M_m\Delta b \tag{3.01}$$

where $\bar{v}_2 = B^{-1}(b)$. The buyer has two considerations in raising his bid. First, it may increase his probability of obtaining an item and, second, it may increase by Δb the price he pays for an item that he would have received at price b. These two considerations correspond respectively to the two terms in (3.01), which we now explain in detail.

The term in brackets represents the probability that raising his bid by Δb causes the selected buyer to go from not receiving an item to receiving an item. If initially he does not receive an item, then some buyer or seller's bid/offer above b determines the price p. If raising his bid is to benefit the buyer, then p must be in $(b,b+\Delta b)$, i.e., p must be just above b so that he can jump over it and become one of the buyers who purchases an item.

Select a seller in addition to the selected buyer. The probability that this seller's offer falls in the interval $(b,b+\Delta b)$ is $f_1(b)\Delta b$. Conditional on it falling in the interval and on the selected buyer bidding b, the probability that this offer determines the market price is K_m. Note that this probability is calculated on a sample of the remaining m-1 bids and m-1 offers because the selected buyer's bid and the selected seller's offer are fixed. Any of the m sellers could have been selected, so the probability that by increasing his bid the selected buyer jumps over a price-determining seller's offer is $mf_1(b)K_m\Delta b$. A similar argument shows that $(m-1)f_2(\bar{v}_2)L_m\Delta b/B'(\bar{v}_2)$ is the probability that the selected buyer jumps over a price-determining buyer's offer as he increases his bid. The density of a buyer's bids at b is $f_2(\bar{v}_2)/B'(\bar{v}_2)$, not $f_2(\bar{v}_2)$, because the buyer's distribution of bids is different from the distribution of his reservation values. Finally, the selected buyer's expected gain from increasing his bid and potentially going from being a nonrecipient of an item to being a recipient is the term in brackets times the gain when this happens. This gain is v_2-p or, in the limit as $\Delta b \to 0$, v_2-b.

On the other side of the ledger is $M_m\Delta b$. If the buyer is the trader whose bid determines the price, then raising his bid Δb increases the price that he pays for the item by Δb. The expected cost of raising his bid is therefore Δb times the probability M_m that he is in fact the price-determining trader.

From (3.01) we obtain the formula for the marginal expected utility of a type v_2 buyer whose bid is b:

$$\frac{d\pi(v_2,b;B)}{db} = \left[mf_1(b)K_m + (m-1)\frac{f_2(\bar{v}_2)}{B'(\bar{v}_2)}L_m\right](v_2-b) - M_m. \tag{3.02}$$

If $\langle \tilde{S}, B\rangle$ is an equilibrium, then B satisfies the first order condition $d\pi(v_2,B(v_2);B)/db = 0$ at all reservation values v_2 where B' exists.

To obtain a differential equation in the strategy B we must define the probabilities K_m, L_m, and M_m so that their values are functions only of the point (v_2,b):

$$(3.03)\quad K_m(v_2,b) = \Sigma_{i=0}^{m-1}\binom{m-1}{i}^2 F_1(b)^{m-1-i}(1-F_1(b))^i F_2(v_2)^i(1-F_2(v_2))^{m-1-i},$$

$$(3.04)\quad L_m(v_2,b) = \Sigma_{i=1}^{m-1}\binom{m}{i}\binom{m-2}{i-1}F_1(b)^{m-i}(1-F_1(b))^i F_2(v_2)^{i-1}(1-F_2(v_2))^{m-i-1},$$

$$(3.05)\quad M_m(v_2,b) = \Sigma_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}F_1(b)^{m-i}(1-F_1(b))^i F_2(v_2)^i(1-F_2(v_2))^{m-1-i}.$$

The probabilities K_m, L_m, and M_m in (3.01-02) are obtained by evaluating expressions (3.03-05) at $v_2 = B^{-1}(b)$.

That $K_m(B^{-1}(b),v_2)$ is the probability that the bid b lies between $s_{(m-1)}$ and $s_{(m)}$ in a sample of m-1 buyers using strategy B and m-1 sellers using strategy $\tilde{S}$ can be seen as follows. The statement that b lies between $s_{(m-1)}$ and $s_{(m)}$ means that m-1 bids/offers are below b and that the remaining m-1 bids/offers in the sample are above b. We sum the probabilities of all possible events in which exactly m-1 bids/offers are less than b. A total of m-1 bids/offers less than b may be obtained by i bids and m-1-i offers less than b. For a particular selection of i buyers and m-1-i sellers, the probability that only their bids/offers are less than b is $F_1(b)^{m-1-i}(1-F_1(b))^i \times F_2(v_2)^i(1-F_2(v_2))^{m-1-i}$ where $v_2 = B^{-1}(b)$. $F_1(b)$ is the probability that a particular seller (using strategy $\tilde{S}$) offers less than b, and $F_2(v_2) = F_2(B^{-1}(b))$ is the probability that a particular buyer (using strategy B) bids less than b. The term $\binom{m-1}{i}^2 = \binom{m-1}{i}\binom{m-1}{m-1-i}$ is the number of ways of simultaneously choosing i buyers from m-1 buyers and m-1-i sellers from m-1 sellers. Similar arguments show that L_m and M_m are given by (3.04) and (3.05).[8]

A differential equation in the strategy B is obtained by setting (3.02) equal to zero and regarding K_m, L_m, and M_m as functions of v_2 and b. Suppose $\langle \tilde{S}, B \rangle$ is an equilibrium. Because B is necessarily increasing we can invert B and regard a buyer's reservation value v_2 as a function of his bid b, i.e., $v_2 = v_2(b) \equiv B^{-1}(b)$ and $\dot{v}_2 \equiv dv_2(b)/db = 1/B'(v_2)$. Substituting $\dot{v}_2$ into the differential equation and solving gives

$$(3.06) \qquad \dot{v}_2 = \frac{M_m(v_2,b) - mf_1(b)K_m(v_2,b)(v_2-b)}{(m-1)f_2(b)L_m(v_2,b)(v_2-b)}$$

$$(3.07) \qquad \dot{b} = 1$$

where the tautology $\dot{b} \equiv db/db = 1$ has been added. Written in this form, the differential equation defines a vector field $(\dot{v}_2, \dot{b})$.

If $\langle \tilde{S}, B \rangle$ is an equilibrium, then (3.06-07) hold at every point $(v_2, B(v_2))$ at which $B'(v_2)$ exists. To establish a converse, we assume that the distribution F_1 of a seller's reservation value satisfies the following monotonicity property:

$$(3.08) \qquad c(v_1) \equiv v_1 + F_1(v_1)/f_1(v_1) \text{ is increasing for } v_1 \in [0,1].$$

Given (3.08), if a solution curve to (3.06-07) defines an increasing function $b = B(v_2)$, then $\langle \tilde{S}, B \rangle$ is an equilibrium in the BBDA.

<u>Theorem 3.1</u>: If $\langle \tilde{S}, B \rangle$ is an equilibrium in the BBDA, then $B(v_2) = b$ and $\dot{v}_2 = 1/B'(v_2)$ satisfy (3.06-07) at every $v_2 \in [0,1]$ at which $B'(v_2)$ exists. Conversely, suppose (3.08) holds and B is a C^1 function on $[0,1]$ such that (i) $B'(v_2) > 0$ and $0 < B(v_2) < v_2$ for all $v_2 \in (0,1]$

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[8] See David (1981, ch. 2) for a discussion of this type of probability calculation.

and (ii) $B(v_2) = b$ and $\dot{v}_2 = 1/B'(v_2)$ satisfy (3.06-07) at every $v_2 \in (0,1]$. Then $\langle S,\tilde{B}\rangle$ is an equilibrium of the BBDA.

The proof is in the Appendix.

4. The Geometry of Solutions

Theorem 2.2 states that if $\langle S,\tilde{B}\rangle$ is a equilibrium, then $0 \leq B(v_2) \leq v_2 \leq 1$. The graph of an equilibrium strategy B therefore lies within the triangle $0 \leq b \leq v_2 \leq 1$ (see Figure 4.1). Following an approach developed in Satterthwaite and Williams (1987), we describe the vector field (3.06-07) on this triangle in order to gain insight into the equilibria of the BBDA.

Formula (3.06) defines $\dot{v}_2$ as a real number at every point on the triangle except along the edges XY where $b = 0$ and XZ where $v_2 = b$. At points X and Z, $\dot{v}_2$ is indeterminate; between X and Y it is negative infinity and between X and Z it is positive infinity. To obtain well-defined values for the vector field $(\dot{v}_2,\dot{b})$ everywhere except X and Z we consider the field's normalization $\bar{v} = (\dot{v}_2,\dot{b})/|(\dot{v}_2,\dot{b})|$. This normalization does not affect the solution curves. Note that $\bar{v}$ is nonsingular at every point on the triangle except X and Z.

Inspection of the field along the three edges and at the vertices allows us to identify three sets where solution curves enter the triangle and one set where they leave the triangle. A solution curve enters at each point where the field points inward. Multiple solutions may enter through X where $\dot{v}_2$ is indeterminate. The field $\bar{v}$ equals (1,0) and therefore points into the triangle along the edge XZ. It also points inward along the edge YZ at points where $F_1(b) > f_1(b)(1-b)$. A solution curve exits at any point where the field points outward. This occurs only on YZ (perhaps including vertex Z) at points where $F_1(b) < f_1(b)(1-b)$. Figure 4.2 shows three solution curves for the case where F_1 and F_2 are the uniform distribution and $m = 2$. Curve ρ_1 enters from the edge XZ, ρ_2 enters from the vertex X, and ρ_3 enters from the lower half of the edge YZ. All exit along the upper half of edge YZ.

Curve ρ_2 meets the conditions of Theorem 3.1 and therefore defines an equilibrium $\langle\tilde{S},B\rangle$. Curve ρ_1 may be a segment of an equilibrium strategy B, but it is unclear how to complete its definition for reservation values that lie to the left of the point on XZ where it enters the triangle.[9] Finally curve ρ_3 does not determine an equilibrium because it does not define the buyer's bid b as an increasing function of v_2, i.e., $\dot{v}_2$ is negative along some segments of ρ_3.

The failure of ρ_3 to determine an equilibrium illustrates an extremely important property of $\dot{v}_2$. Inside the triangle an open region necessarily exists where $\dot{v}_2$ is negative; formally we define this region as

$$\Gamma_m(F_1,F_2) \equiv \{(v_2,b):\ \dot{v}_2 < 0\}, \tag{4.01}$$

where the dependence in (3.06) of $\dot{v}_2$ on (v_2,b), F_1, F_2, and m is suppressed. We use γ_m to label the upper edge of Γ_m. Note Γ_m always contains the edge XY and some portion of the edge YZ. For the case in which F_1 and F_2 are uniform and $m = 8$, Figure 4.1 shows Γ_m as the region below the curve γ_m connecting X and W. The set Γ_m is important because the graph of any function B that defines an equilibrium $\langle\tilde{S},B\rangle$ must lie outside Γ_m at every point where B is differentiable. In the next section we show that as m

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[9] Extending B's graph down along the edge XZ towards X does not define an equilibrium. At each point on this extension B′ exists and yet (3.05) is not satisfied. This violates Theorem 3.1. It may be possible to extend ρ_1 by jumping to some lower solution curve.

increases, Γ_m grows and forces all equilibrium strategies towards edge XZ, which corresponds to truthful revelation. This is the fundamental insight that underlies our convergence result.

The set Γ_m can be interpreted in terms of marginal expected utility. Choose a point (v_2,b) in Γ_m and suppose an equilibrium $<\tilde{S},B>$ did exist such that $B(v_2) = b$ and B is differentiable at v_2. Theorem 2.2 states that $B'(v_2) \geq 0$. Select a buyer. If the other traders use their equilibrium strategies, formula (3.02) implies that the selected buyer's marginal expected utility is necessarily positive at (v_2,b) because, by the definition of Γ_m, a negative number would be needed in place of $B'(v_2)$ in (3.02) in order to make his marginal expected utility zero. The selected buyer therefore has an incentive to raise his bid above $B(v_2) = b$, which contradicts the assumption that $<\tilde{S},B>$ is an equilibrium.

In the remainder of this section we discuss the relationship between the shape of Γ_m and the equilibria of the BBDA. To this end we introduce property λ: a subset A of the triangle $0 \leq b \leq v_2 \leq 1$ satisfies <u>property</u> λ if a continuous function $\lambda(v_2) = b$ exists such that $A = \{(v_2,b): b < \lambda(v_2)\}$, i.e., A consists of all points within the triangle below the graph of λ. Thus Γ_m satisfies property λ if γ_m is the graph of a single-valued function $\lambda(v_2) = b$. In Figure 4.1 γ_m is such a graph and therefore Γ_m satisfies property λ. Figure 4.3 illustrates a region Γ_m that fails to satisfy property λ.

Property λ is significant for two reasons. If Γ_m satisfies property λ, then the graph of any equilibrium strategy B must lie completely outside Γ_m. This can be seen by referring to Figure 4.3 and using Theorem 2.2's results that an equilibrium B is increasing and almost everywhere differentiable.

Suppose the graph of an equilibrium strategy B contains point C in Γ_m. When property λ holds there exists a rectangle within Γ_m whose bottom edge is on XY and whose upper right corner is C. Since the strategy B is increasing its graph must pass through this rectangle. Because B is differentiable almost everywhere, it must be differentiable at some point in the rectangle where $\dot{v}_2$ is necessarily negative. This contradicts the requirement that B is increasing. Note that when property λ fails to hold the graph of an equilibrium B could conceivably pass through Γ_m at an isolated point. This is illustrated in Figure 4.3 by the graph consisting of the curve from X to D, the point E, and the curve from F to G.

The second point concerning property λ is its role in a theorem on the existence of equilibria.

<u>Theorem 4.1</u>: For given F_1, F_2, and m, if $\Gamma_m(F_1,F_2)$ satisfies property λ, then a C^1 function B exists that defines an equilibrium <S,B> for the BBDA.

The proof is in the Appendix.

The weakness of this theorem is that it does not specify conditions on F_1, F_2, and m under which Γ_m satisfies condition λ. Direct calculation, however, suggests that property λ has some generality. Figure 4.4 graphs γ_m, the upper boundary of Γ_m, for values of m equal to one, eight, and sixteen when F_1 and F_2 are the uniform distribution. Visual inspection indicates that property λ holds in these cases.

Further evidence on existence is provided by the following example of a linear equilibrium when reservation values are uniformly distributed. The strategy

$$B(v_2) = \frac{m}{m+1} v_2 \tag{4.02}$$

defines an equilibrium $\langle \tilde{S},B \rangle$ in this case for the market with m sellers and m buyers. Proof is by direct substitution in (3.06). Curve ρ_2 in Figure 4.2 depicts this solution for m = 2. Substitution of m = 1 in (4.02) defines the equilibrium that Williams (1987) computed for the bilateral BBDA in the uniform case. The example is nice because closed form solutions to (3.06-07) are in general difficult or impossible to obtain.

5. Convergence of All Equilibria to Truthful Revelation

The complement of Γ_m in the triangle $0 \le b \le v_2 \le 1$ contains the edge XZ where the buyer's bid b equals his reservation value v_2. In this section we show that as m increases the vertical distance between the boundary γ_m and the edge XZ is $O(1/m)$. The graph of an equilibrium strategy B must lie between γ_m and XZ at almost all values of v_2. This permits us to show that in equilibrium the difference between a buyer's reservation value and his bid is $O(1/m)$, no matter what his reservation value and no matter which equilibrium $\langle \tilde{S},B \rangle$ is chosen.

Rearrangement of (3.06) gives us an inequality defining the region in which $\dot{v}_2$ is nonnegative:

(5.01) $\dot{v}_2 \ge 0$ if and only if $v_2 - b \le \dfrac{1}{f_1(b)} \times N_m(v_2,b)$

where N_m is defined as the ratio

(5.02) $$N_m(v_2,b) \equiv \frac{M_m(v_2,b)}{mK_m(v_2,b)}.$$

The left-hand side of the second inequality in (5.01) is the amount by which the buyer's bid misrepresents his reservation value. Only the right-hand side depends on the number of traders. We therefore focus on the behavior of N_m as m increases. Two theorems, whose proofs are in the Appendix, describe this behavior.

Theorem 5.1: For each pair of numbers $0 < b \leq v_2 < 1$ and all $m \geq 1$, the ratio $N_m(v_2,b)$ is strictly decreasing in m.

The functions K_m, L_m, M_m, and hence N_m are well-defined in the m = 1 case, which permits us to state Theorem 5.1 using m = 1. The statement of Theorem 5.2 uses the notation

$$(5.03) \qquad z(v_2,b) = \frac{F_2(v_2)(1-F_1(b))}{F_1(b)(1-F_2(v_2))}.$$

Theorem 5.2: If $m \geq 2$ and (v_2,b) satisfies $0 < b \leq v_2 < 1$, then

$$(5.04) \qquad N_m(v_2,b) < \frac{2F_1(b)}{m} \max \left[1, z(v_2,b) \right].$$

These theorems have the following interpretation. Consider m′ < m″. If, for m′, $\dot{v}_2$ is negative at some point (v_2,b), then Theorem 5.1 implies that it is also negative for m″. The region Γ_m therefore grows monotonically in m, i.e., for m′ < m″, $\Gamma_{m'} \subset \Gamma_{m''}$. Theorem 5.2 describes the rate at which these regions grow.

The main result of the paper follows from substituting the inequalities of Theorem 5.2 into (5.01).

Theorem 5.3: Consider the BBDA when sellers' reservation values are drawn from F_1 and seller's reservation values are drawn from F_2. A continuous function $\kappa(v_2;F_1F_2)$ of v_2 exists such that, for any $m \geq 2$ and any equilibrium <S,B> in a market of size m,

$$(5.05) \qquad v_2 - B(v_2) \le \frac{\kappa(v_2;F_1,F_2)}{m}$$

at every v_2 in the open interval (0,1).

Proof. We first show that B satisfies (5.05) at all reservation values $v_2 \in (0,1)$ where $B'(v_2)$ exists. Fix v_2 and let $\bar{b}$ denote $B(v_2)$. From (5.01) and Theorem 5.2 we have

$$(5.06) \qquad v_2 - \bar{b} \le \frac{N_m(v_2,\bar{b})}{f_1(\bar{b})} < \frac{2}{m}\,\frac{F_1(\bar{b})}{f_1(\bar{b})}\max\Big[\,1,\ z(v_2,\bar{b})\,\Big].$$

A finite bound on $v_2 - \bar{b}$ that does not involve $\bar{b}$ is obtained by maximizing the right-hand side of (5.06) over a closed interval that contains $\bar{b}$. The bid $\bar{b}$ is bounded above by v_2 and below by zero. The right-hand side, however, may be infinite at $\bar{b} = 0$. This complication is sidestepped by bounding $\bar{b}$ away from zero. The region Γ_2 is an open set that contains the triangle's lower edge XY. Theorem 5.1 implies that the point $(v_2,\bar{b})$ lies above the region Γ_2. Choose a continuous function μ on (0,1) such that the graph of μ lies within Γ_2 and μ is greater than zero. The bid $\bar{b}$ therefore satisfies $\mu(v_2) \le \bar{b} \le v_2$. Define

$$(5.07) \qquad \kappa(v_2) \equiv \max_{\mu(v_2)\le b\le v_2} \frac{2F_1(b)}{f_1(b)}\max\Big[\,1,\ z(v_2,b)\,\Big].$$

For convenience, we suppress the dependence of κ on F_1 and F_2. Note that κ is continuous in v_2 because μ is continuous. With this definition of κ, (5.05) holds at all points where B' exists.

We now show that (5.05) also holds at reservation values in (0,1) where B' does not exist. Consider the set D_m of reservation values v_2 and bids b that violate (5.05):

(5.08) $$D_m \equiv \{(v_2,b):\ 0 < b \le v_2 < 1 \text{ and } v_2 - \kappa(v_2)/m > b\}.$$

The set D_m is open because κ is continuous. It also satisfies property λ because its upper boundary is the graph of the function $v_2 - \kappa(v_2)/m$. Suppose, contrary to the theorem, that some $(v_2,B(v_2)) \equiv (v_2,\bar{b})$ is an element of D_m. We now repeat an argument from Section 4. A rectangle within D_m exists whose base is on the edge XY and whose upper right corner is $(v_2,\bar{b})$. Because B is increasing, the graph of B must pass through the rectangle. Somewhere on this segment of the graph B' must exist, which contradicts the above result that (5.05) holds wherever B is differentiable. Q.E.D.

As an illustration, we follow the proof of Theorem 5.3 to compute the function κ when each trader's reservation values are uniformly distributed. The boundary γ_2 of Γ_2 is complicated. By Theorem 5.1 the boundary γ_1 of Γ_1 lies within Γ_2. It is easy to compute the function μ that describes this boundary γ_1. Formula (3.06) for $\dot{v}_2$ implies that

(5.09) $$\gamma_1 = \{(v_2,b):\ v_2 = b + F_1(b)/f_1(b)\}.$$

In the uniform case, γ_1 is the graph of the function $b = \mu(v_2) = v_2/2$. We now compute κ using (5.07). From (5.03), $z(v_2,b) = v_2(1-b)/(1-v_2)b$. Note that $z(v_2,b) \ge 1$ for $v_2/2 \le b \le v_2$. Formula (5.07) therefore simplifies to

(5.10) $$\kappa(v_2;F_1,F_2) = \max_{v_2/2\le b\le v_2} \frac{2bv_2(1-b)}{(1-v_2)b} = \frac{v_2(2-v_2)}{(1-v_2)},$$

which means that in the uniform case, the difference between a buyer's reservation value v_2 and his bid is less than or equal to $v_2(2-v_2)/(1-v_2)m$.

It is important to emphasize that the bound on $v_2 - B(v_2)$ in Theorem 5.3 is loose because it is based on the set Γ_m. A solution curve ρ passing through a point (v_2, b) outside Γ_m may not define an equilibrium $\langle \tilde{S}, B \rangle$, for ρ may pass through Γ_m somewhere else (e.g., curve ρ_3 in Figure 4.2). The graph of an equilibrium strategy B therefore lies outside the set consisting of all solution curves that pass through Γ_m, which is a strictly larger set than Γ_m. On the other hand, we believe that the rate of convergence $O(1/m)$ is sharp because numerical computation of equilibria for various values of m and the example (4.02) suggest that equilibria converge to truthful revelation at the same rate as γ_m.

6. Additional Comments

1. A simple partial equilibrium calculation provides insight into our convergence result. It reveals that the driving force behind the $O(1/m)$ rate is the relative rates at which the likelihood of obtaining an item by increasing one's bid and the likelihood of simply driving up price go to zero as the number of traders increases. Consider a market with 2m traders in which $F \equiv F_1 \equiv F_2$ (with density f) and m is large. Select a buyer with reservation value v_2 and suppose he believes that in addition to the sellers all other buyers will truthfully report their reservation values because the market is large. In the sample of bids and offers from the 2m-1 other traders, let g be the density of the critical bid/offer $s_{(m)}$ that the selected buyer must beat with his bid b in order to receive an item. As before, let M_m be the probability that the bid b lies between $s_{(m)}$ and $s_{(m+1)}$ in this sample and thus determines the market price. Adapting (3.02) to this simplified situation, the buyer chooses his bid to satisfy

(6.01) $\quad (v_2-b)g(b) = M_m$.

When the buyer considers raising his bid, the left-hand side is his marginal expected gain from increasing his likelihood of receiving an item and the right-hand side is his marginal expected cost from driving up the price. Formulas in David (1981, p. 9) give

(6.02) $\quad M_m = \binom{2m-1}{m} F(b)^m (1-F(b))^{m-1}$

(6.03) $\quad g(b) = (2m-1)f(b) \binom{2m-2}{m} F(b)^{m-1} (1-F(b))^{m-1}$.

Substitution into (6.01) implies

(6.04) $\quad v_2 - b = \dfrac{F(b)}{mf(b)}$,

which is the same rate that we obtained in Theorem 5.3.

2. The results of this paper are true for the more general case in which the number of buyers may differ from the number of sellers. Our proof of this is not included here because it would substantially complicate the paper. We do, however, outline the proof and discuss some implications. Assume that there are n buyers and q sellers where $n,q \geq 2$. Our characterization of the traders' equilibrium strategies is the same: $\tilde{S}$ is the dominant strategy of each seller, and an equilibrium, common response B of the n buyers satisfies the necessary conditions of both Theorems 2.2 and 3.1 (with appropriate changes in (3.06)). The first order condition for buyers and the tautology $\dot{b} = 1$ again define a vector field on the triangle $0 \leq b \leq v_2 \leq 1$. A region Γ_{nq} exists in which $\dot{v}_2$ is negative. The key step in our proof is a demonstration that Γ_{nq} is increasing in both n and q, i.e., if $n' \leq n$ and $q' \leq q$, then $\Gamma_{n'q'} \subset \Gamma_{nq}$. In particular, if $m = \min(n,q)$, then $\Gamma_{mm} \subset \Gamma_{nq}$. Note that Γ_{mm} is identical to Γ_m as defined in (4.01). Theorem 5.3 uses the set Γ_m to bound the amount of misrepresentation in the

market with m buyers and m sellers. Because $\Gamma_m \equiv \Gamma_{mm} \subset \Gamma_{nq}$, this bound also applies to the market with n buyers and q sellers.

This result has two implications. First, the amount of misrepresentation converges to zero in any sequence of markets in which both the number of buyers and the number of sellers increases without bound. Second, Theorem 5.3's $O(1/m)$ result generalizes to any sequence of markets in which the ratio of buyers to sellers is fixed as the market grows. Specifically, fix n_0 and q_0, let $m_0 = \min(n_0, q_0)$, and let τ index a sequence of markets with τn_0 buyers and τq_0 sellers. The amount of misrepresentation $v_2 - B(v_2)$ in this market is $O(1/\tau m_0) = O(1/\tau)$.

It is interesting to apply these results to a sequence of markets in which the number of buyers, the number of sellers, and the ratio of the number of sellers to the number of buyers all converge to infinity. As the sequence progresses the increasing ratio of sellers to buyers implies that the expected return to each buyer of a given type increases, for it becomes increasingly likely that there will be sellers available who are willing to trade at "low" prices. At the same time each buyer has a diminishing incentive to misrepresent his reservation value. As explained above, this is true because the marginal expected change in the market price that he causes by decreasing his bid Δb converges to zero more rapidly than does the probability that he will be excluded from trade because he changes his bid. A buyer therefore benefits from the increasing ratio of sellers to buyers, despite his diminishing "market power", or ability to influence the market price.

Another interesting observation is that the strategy (4.02) defines an equilibrium $\langle S, \tilde{B} \rangle$ when there are m buyers and F_1 and F_2 are uniform, regardless of the number of sellers. Our results for markets with unequal numbers of buyers and sellers only apply when both sides of the market grow large. This example suggests, however, that in the BBDA it is competition among the buyers alone that drives the market to efficiency.

3. A basic insight of the literature in social choice theory on strategy-proofness is that strategic behavior is only avoidable in mechanisms where individuals can not affect each other's allocations.[10] See, for example, Satterthwaite and Sonnenschein (1981). In the BBDA traders affect each other's allocations by affecting the expected price. The ability to affect price vanishes rapidly as the market grows. The social choice results therefore suggest that strategic behavior should vanish as the market grows large. Our result shows that this in fact happens.

4. For small markets the BBDA and other double auctions are <u>ex post</u> inefficient, i.e., when the market closes potential gains from trade may be "left on the table." Equilibrium strategies are fully revealing of traders' reservation values; consequently when the market closes the traders know if further gains from trade are possible. Cramton (1984) has criticized one-shot double auctions on this point. Specifically, he argues that the use of a one-shot double auction implicitly assumes that traders precommit not to reopen the market even when it is common knowledge that further gains from trade exist. Such precommitment may be difficult or impossible to maintain. Our results suggest that this criticism of one-shot double auctions lacks force in large markets because the expected value of the unrealized gains from trade rapidly vanishes as the market grows.

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[10] The one important exception is the family of Groves mechanisms.

5. The BBDA is one example of the sealed bid k-double auction. The more general formulation of the k-double auction is to set price as $(1-k)s_{(m)} + ks_{(m+1)}$ where k is a fixed parameter in the interval [0,1]. The BBDA is the k-double auction in which k = 1. All the results of this paper have exact parallels for the seller's offer double auction in which k = 0. Our analysis of these two extreme cases is greatly facilitated because in each case traders on one side of the market truthfully reveal their reservation values. The analysis becomes more difficult when k is in the open interval (0,1) because then both sides of the market can affect price; as a consequence, both sides act strategically. As of this writing we have been unable to obtain the O(1/m) convergence result for this more general case. We conjecture, however, that it is true for two reasons. First, in the general case, regions analogous to Γ_m exist; as in the BBDA, large distortions of reservation values by either the buyers or sellers can be ruled out. The key insight in our analysis of the BBDA thus applies in the more general case. Second, numerical computation of equilibria in the general case supports the conjecture that all differentiable equilibria converge to truthful revelation as O(1/m).

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Appendix

Proof of Theorem 3.1. To prove the necessary part of the theorem, it is sufficient here to derive formula (3.02) for $d\pi/db$. The result in the theorem concerning (3.06-07) then follows from the discussion in the text. We derive the marginal expected utility at bid b of a type v_2 buyer who is bidding against m sellers, each using strategy $\tilde{S}$, and m-1 buyers, each using an increasing function B as their strategy. Let $x = s_{(m)}$ and $y = s_{(m+1)}$ where $s_{(k)}$ denotes the kth largest bid/offer in the array of 2m-1 bids/offers received from the other traders, and let e(x,y) denote the joint density of x and y. Note that e(x,y) = 0 whenever x > y. Table A.1 catalogues the three distinct utility consequences of the bid b. For example, if b should be greater than y, then the selected buyer receives an item at price y and has utility v_2-y.

Table A.1
Possible Outcomes of a Bid b

Case No.	Case Definition	*Ex post* Utility
I	$b < x < y$	0
II	$x < b < y$	$v_2 - b$
III	$x < y < b$	$v_2 - y$

Note: Ties are a probability zero event because all traders use increasing strategies.

The expected utility of bidding b is

(A.01) $$\pi(v_2,b;B) = \int_b^1\int_0^b (v_2-b)e(x,y)dxdy + \int_0^b\int_0^y (v_2-y)e(x,y)dxdy$$

where the first integral is the expected gain from the case II outcomes and the second integral is the expected gain from the case III outcomes. Differentiating with respect to b, we obtain

(A.02) $$\frac{d\pi}{db} = -\int_0^b (v_2-b)e(x,b)dx + \int_b^1 (v_2-b)e(b,y)dy - \int_b^1\int_0^b e(x,y)dxdy + \int_0^b (v_2-b)e(x,b)dx.$$

The first and fourth terms cancel, (v_2-b) factors out of the second term, and the remaining integrals have straightforward probability interpretations:

(A.03) $$d\pi/db = (v_2-b)g(b) - Pr(x<b<y)$$

where g(b) is the density of the order statistic x evaluated at b. This density can be shown to equal the term in brackets in (3.02) using the standard technique in David (1981, p. 9). Similarly, $Pr(x<b<y) = M_m(B^{-1}(b),b)$. This gives us (3.02) and completes our discussion of the theorem's necessary part.

Sufficiency of the first order approach is proven as follows. Given a function B that meets the theorem's requirements, we must show that $\pi(v_2,b;B)$ is maximized at $b = B(v_2)$. Arguments in the proof of Theorem 2.2 show that we can restrict attention to $b \in (0,v_2]$. Two cases must be considered: $b \in (0,B(1)]$ and $b \in (B(1), v_2]$. Consideration of the first case is facilitated by defining

(A.04) $$J_m(v_2,b;B) \equiv mf_1(b)K_m(v_2,b) + (m-1)f_2(b)L_m(v_2,b)/B'(v_2).$$

Formula (3.02) then becomes

(A.05) $$d\pi(v_2,b;B)/db = J_m(B^{-1}(b),b;B)(v_2-b) - M_m(B^{-1}(b),b)$$

and the differential equation (3.06) is equivalent to

(A.06) $\quad J_m(v_2,B(v_2);B)(v_2-B(v_2)) - M_m(v_2,B(v_2)) = 0.$

Formula (A.05) can be rewritten as

(A.07) $$d\pi(v_2,b;B)/db = J_m(B^{-1}(b),b,B)(v_2-B^{-1}(b)) + J_m(B^{-1}(b),b;B)(B^{-1}(b)-b) - M_m(B^{-1}(b),b).$$

If we evaluate the differential equation (A.06) at $v_2 = B^{-1}(b)$, we obtain the last line in (A.07). We therefore have $d\pi/db$ equal to the top line. Note that (i) $J_m(B^{-1}(b),b;B)$ is positive for all $0 < b < B(1)$, (ii) $d\pi(v_2,b;B)/db$ is zero at $b = B(v_2)$, and (iii) the function B^{-1} is increasing since B is increasing. The marginal expected utility $d\pi(v_2,b;B)/db$ therefore changes from positive to negative at $b = B(v_2)$, which establishes that $\pi(v_2,b;B)$ is maximized on $(0,B(1)]$ at $b = B(v_2)$.

Consider now the remaining case $b \in (B(1),v_2]$. While the marginal expected utility $d\pi(v_2,b;B)/db$ is discontinuous at $b = B(1)$, the expected utility $\pi(v_2,b;B)$ is continuous in b on [0,1] because B is a C^1 function. It is therefore sufficient to prove that $d\pi(v_2,b;B)/db$ is negative over $(B(1),v_2]$. For a bid b in this interval (A.03) is

(A.08) $$\begin{aligned}\frac{d\pi(v_2,b;B)}{db} &= (v_2-b)mf_1(b)K_m(1,b) - M_m(1,b)\\ &= (v_2-b)mf_1(b)[1-F_1(b)]^{m-1} - mF_1(b)[1-F_1(b)]^{m-1}\\ &= \frac{m}{f_1(b)}[1-F_1(b)]^{m-1}\left[v_2 - b - \frac{F_1(b)}{f_1(b)}\right].\end{aligned}$$

Consider the last line of (A.08). The monotonicity property (3.08) implies that the expression in brackets is decreasing in b and it is also increasing in v_2. Consequently if some line of (A.08) is negative at $v_2 = 1$ and $b = B(1)$, then each line is negative for any v_2 over the entire interval $(B(1),v_2]$.

We show that the first line is negative at $v_2 = 1$ and $b = B(1)$ by considering the solution B at that point. By hypothesis $\dot{v}_2$ is positive at all points $(v_2, B(v_2))$. The numerator of the right-hand side of (3.06) determines the sign of $\dot{v}_2$; at $(1, B(1))$ this numerator is $-[1-B(1)]mf_1[B(1)]K_m[1,B(1)] + M_m[1,B(1)] > 0$. The negative of this expression is the first line of (A.08) evaluated at $v_2 = 1$ and $b = B(1)$. Q.E.D.

Proof of Theorem 4.1. Theorem 3.1 implies that it is sufficient to find an increasing function B on [0,1] such that (i) $0 < B(v_2) < v_2$ for $v_2 \in (0,1]$ and (ii) the graph of B is a solution curve to the vector field $\bar{v}$ that (3.06-07) defines. Refer to Figure 4.1. Property λ guarantees the existence of a function λ that defines the upper boundary of Γ_m. We first prove the existence of a solution curve that enters the triangle through the point X, proceeds through the interior of the triangle, and exits through the edge YZ.

W is the point $(1, \lambda(1))$. Solution curves exit the triangle only through the interval WZ. Solution curves may enter the triangle at X, and do enter through the points on the open interval XZ and through points along the open interval YW. Let E_1 denote the points in WZ that lie on a solution curve that enters through XZ, and let E_2 denote the points in WZ that lie on a solution curve that enters through YW. Because $\bar{v}$ is nonsingular and the intervals XZ and YW are open, both E_1 and E_2 are nonempty and open; note also that E_1 and E_2 are disjoint. Since the interval WZ is connected, the set of points in WZ that lie on a solution curve that enters through X must be nonempty, i.e., there must be a solution curve to the vector field $\bar{v}$ that enters through X, proceeds through the interior of the triangle, and exits through WZ. Note in particular that any such curve must exit at a point where $\dot{v}_2$ is positive because the nonempty set E_2 lies below this exit point.

We now show that a solution curve that enters through X necessarily lies within the region where $\dot{v}_2$ is positive. Note this key point: any solution curve that intersects the graph of λ must have entered the triangle along the open interval YW, for as one traces backwards from the graph of λ along the solution curve, b decreases while v_2 increases. A solution curve that enters through X cannot intersect the graph of λ, and it exits at a point above the graph of λ. It therefore lies above the closure of Γ_m. Q.E.D.

In proving Theorems 5.1 and 5.2 we use the following formula for $N_m/F_1(b)$:

$$\text{(A.09)} \qquad \frac{N_m(v_2,b)}{F_1(b)} = \frac{\Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i} z^i}{\Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i}(m-i) z^i}.$$

The right-hand side has been derived from (5.02) by (i) factoring out F_1 from $M_m(v_2,b)$ and canceling, (ii) dividing the numerator and denominator by $[F_1(b)(1-F_2(v_2))]^{m-1}$, and (iii) substituting

$$\text{(A.10)} \qquad m\binom{m-1}{i}^2 = \binom{m-1}{i}\binom{m}{i}(m-i)$$

into the denominator.

<u>Proof of Theorem 5.1.</u> It is sufficient to prove that N_m/F_1 is strictly decreasing in m. Substitute j for i as the index of the terms in the formula for N_{m+1}/F_1 that is given by (A.09). Next, compute the numerator of $(N_m - N_{m+1})/F_1$:

$$\text{(A.11)} \qquad \left[\Sigma_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i} z^i\right]\left[\Sigma_{j=0}^{m}\binom{m}{j}\binom{m+1}{j}(m+1-j) z^j\right]$$
$$- \left[\Sigma_{i=0}^{m-1}\binom{m-1}{i}\binom{m}{i}(m-i) z^i\right]\left[\Sigma_{j=0}^{m}\binom{m}{j}\binom{m+1}{j} z^j\right].$$

The proof will be completed by showing that all of the coefficients of this polynomial are nonnegative, and some are strictly positive.

For $0 \le k \le 2m-1$, the coefficient of z^k is

$$\text{(A.12)} \qquad \Sigma_{\substack{i+j=k \\ 0\le i\le m-1 \\ 0\le j\le m}} \binom{m-1}{i}\binom{m}{i}\binom{m}{j}\binom{m+1}{j}(i+1-j).$$

We now pair terms in this expression with the following formula: the i=u, j=v term is paired with the i=v-1, j=u+1 term. Some terms may be left out by this pairing; there is no term to pair with the i=k, j=0 term (if such a term exists for the given value of k), and a term of the form i=u, j=u+1 is paired with itself. It is easy to see from (A.12), however, that a term with j=0 is positive, and a term with i+1=j is zero. Except for these special cases, the formula pairs each term in (A.12) with a different term. Note that this pairing is well-defined, i.e., if i',j' is assigned to i",j" by the formula, then i",j" is assigned to i',j'.

We now rewrite the sum of i=u, j=v term and the i=v-1, j=u+1 term. Factoring out the i=u, j=v term, we have

$$\text{(A.13)} \qquad \binom{m-1}{u}\binom{m}{u}\binom{m}{v}\binom{m+1}{v}(u+1-v) + \binom{m-1}{v-1}\binom{m}{v-1}\binom{m}{u+1}\binom{m+1}{u+1}(v-u-1)$$

$$= \binom{m-1}{u}\binom{m}{u}\binom{m}{v}\binom{m+1}{v}(u+1-v)\left\{1- \left[\frac{v}{u+1}\right]^2\right\}.$$

Note that the signs of the last two terms of the product on the second line of (A.13) are the same. The expression (A.13) is therefore positive except when u+1=v, which is a case that was discussed above. Q.E.D.

<u>Proof of Theorem 5.2</u>. The inequality (5.04 is equivalent to the following pair of inequalities: (i) if $z(v_2,b) \le 1$, then $N_m(v_2,b)/F_1(b) <$

$2/m$, and (ii) if $z(v_2,b) \geq 1$, then $N_m(v_2,b)/F_1(b) < 2z(v_2,b)/m$. We begin by proving the first inequality. Using (A.09), it is sufficient to show that

$$\text{(A.14)} \qquad \Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i}\left[1-\frac{2(m-i)}{m}\right]z^i$$

is negative for $0 < z \leq 1$. Multiplying through by m, we obtain

$$\text{(A.15)} \qquad \Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i}(2i-m)z^i$$

Note that the coefficient of z^i is positive if $i > m/2$, zero if $i = m/2$, and negative if $i < m/2$. Excluding the $i=m/2$ term (if it is present) and the $i=0$ term (which is clearly negative), we now pair the remaining terms with the following formula: for $1 \leq u < m/2$, the $i=u$ term is paired with the $i=m-u$ term. The sum of the $i=u$ and $i=m-u$ terms reduces as follows:

$$\text{(A.16)} \qquad \binom{m-1}{u}\binom{m}{u}(2u-m)z^u + \binom{m-1}{m-u}\binom{m}{m-u}(m-2u)z^{m-u}$$

$$= \binom{m-1}{u}\binom{m}{u}(2u-m)z^u\left[1 - \frac{u}{m-u}z^{m-2u}\right].$$

Since $u < m/2$, and $z \leq 1$, it is true that (i) $(2u-m) < 0$, (ii) $u/(m-u) < 1$, and (iii) $z^{m-2u} \leq 1$. The second line of (A.16) is therefore negative, and it follows that (A.15) is also negative. This completes the proof of the first inequality.

We now turn to the second inequality. Again using (A.09), it is sufficient to show that

$$\text{(A.17)} \qquad \Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i}z^i - \frac{2}{m}\Sigma_{i=0}^{m-1} \binom{m-1}{i}\binom{m}{i}(m-i)z^{i+1}$$

is negative when $z \geq 1$. After reindexing the right-hand summation by replacing i with i-1 and then multiplying (A.17) by m, we obtain

$$(A.18)\qquad m - 2mz^m + \Sigma_{i=1}^{m-1}\left[m\binom{m-1}{i}\binom{m}{i} - 2(m-i+1)\binom{m-1}{i-1}\binom{m}{i-1}\right]z^i$$

Since $z \geq 1$, $m-2mz^m$ is negative. It is thus sufficient to focus on the remaining summation.

By factoring, this summation can be rewritten as

$$(A.19)\qquad \Sigma_{i=1}^{m-1}\binom{m-1}{i}\binom{m}{i}\left[m - \frac{2i^2}{m-i}\right]z^i.$$

The coefficient of z^i is negative when $i > m/2$, zero when $i=m/2$, and positive when $i < m/2$. Excluding the $i=m/2$ term (if it exists), we pair terms as in the proof of the theorem's first part: for $1 \leq u < m/2$, the $i=u$ term is paired with the $i=m-u$ term. The sum of these terms is

$$(A.20)\qquad \binom{m-1}{u}\binom{m}{u}\left[m-\frac{2u^2}{m-u}\right]z^u + \binom{m-1}{m-u}\binom{m}{m-u}\left[m - \frac{2(m-u)^2}{u}\right]z^{m-u}.$$

The proof is completed by showing that the sum (A.20) is negative. Since $z \geq 1$ and the $i=m-u$ term is negative, it is sufficient to show that

$$(A.21)\qquad \binom{m-1}{u}\left[m-\frac{2u^2}{m-u}\right] + \binom{m-1}{m-u}\left[m-\frac{2(m-u)^2}{u}\right]$$

is negative. By factoring out $\binom{m-1}{u}/(m-u)$, this reduces to

$$(A.22)\qquad \binom{m-1}{u}\left[m(m-u) - 2u^2 + mu - 2(m-u)^2\right]/(m-u).$$

The expression in brackets equals $-(m-2u)^2$, which shows that (A.20) is negative. Q.E.D.

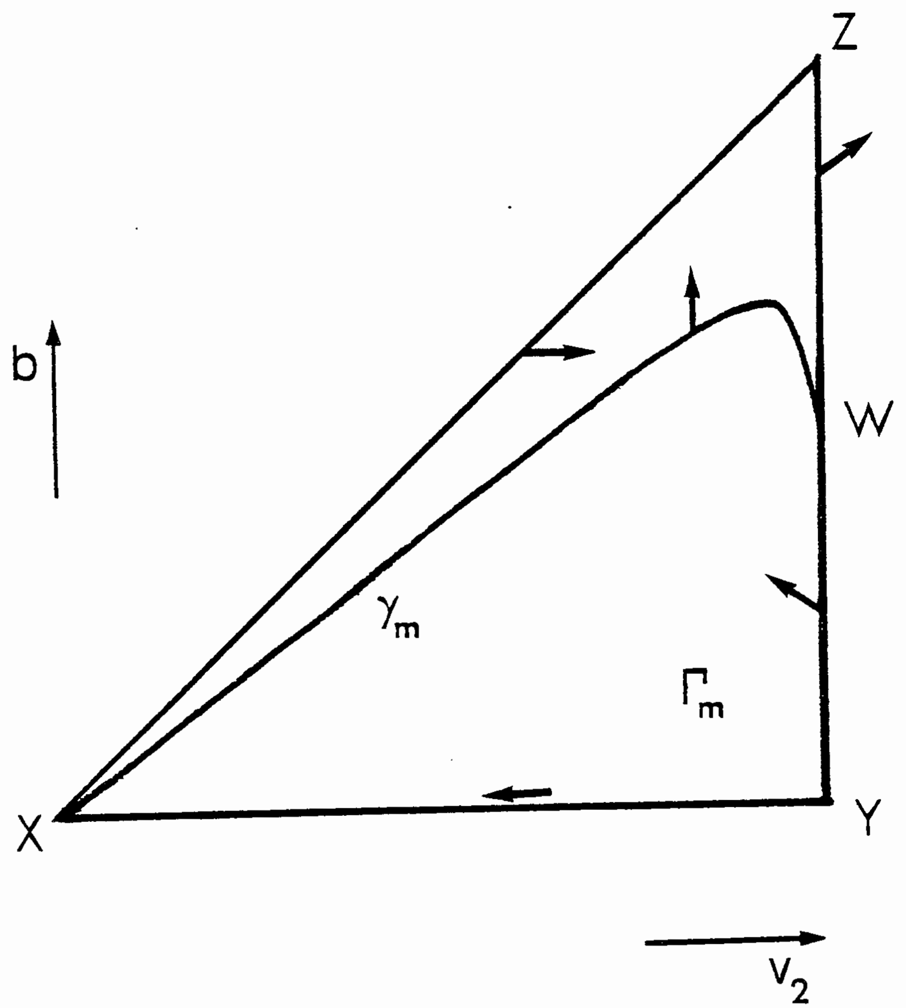

Figure 4.1. If $\langle S,\tilde{B}\rangle$ is an equilibrium, then the graph of B lies in the triangle XYZ defined by the inequalities $0 \le b \le v_2 \le 1$. The arrows show the direction of the vector field $(\dot{v}_2,\dot{b})$ on the edges and at a point on γ_m.

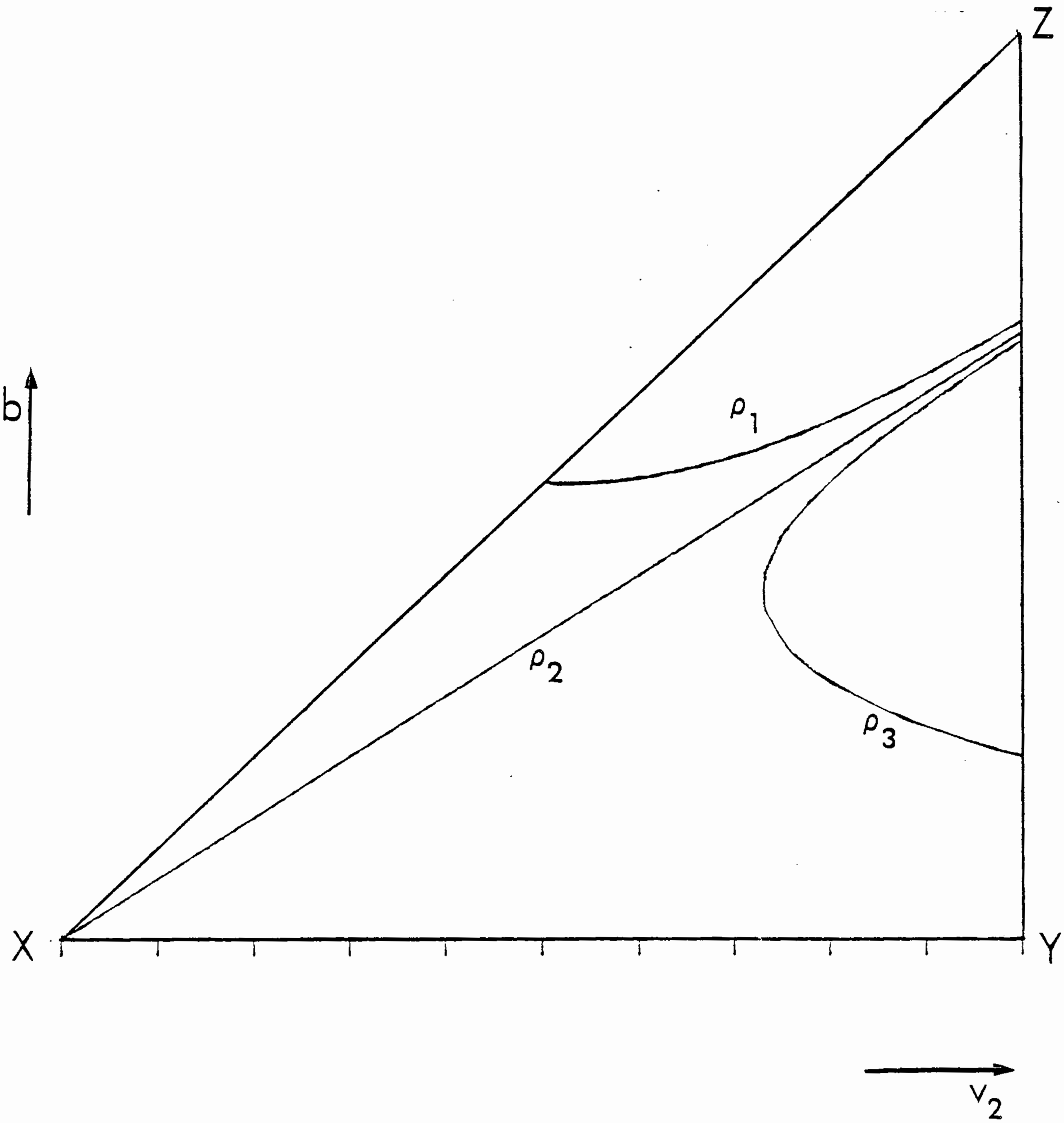

Figure 4.2. The curves ρ_1, ρ_2, and ρ_3 are solutions to the differential equation (3.06-07) when m = 2 and reservation values are distributed uniformly. Only ρ_2 defines an equilibrium.

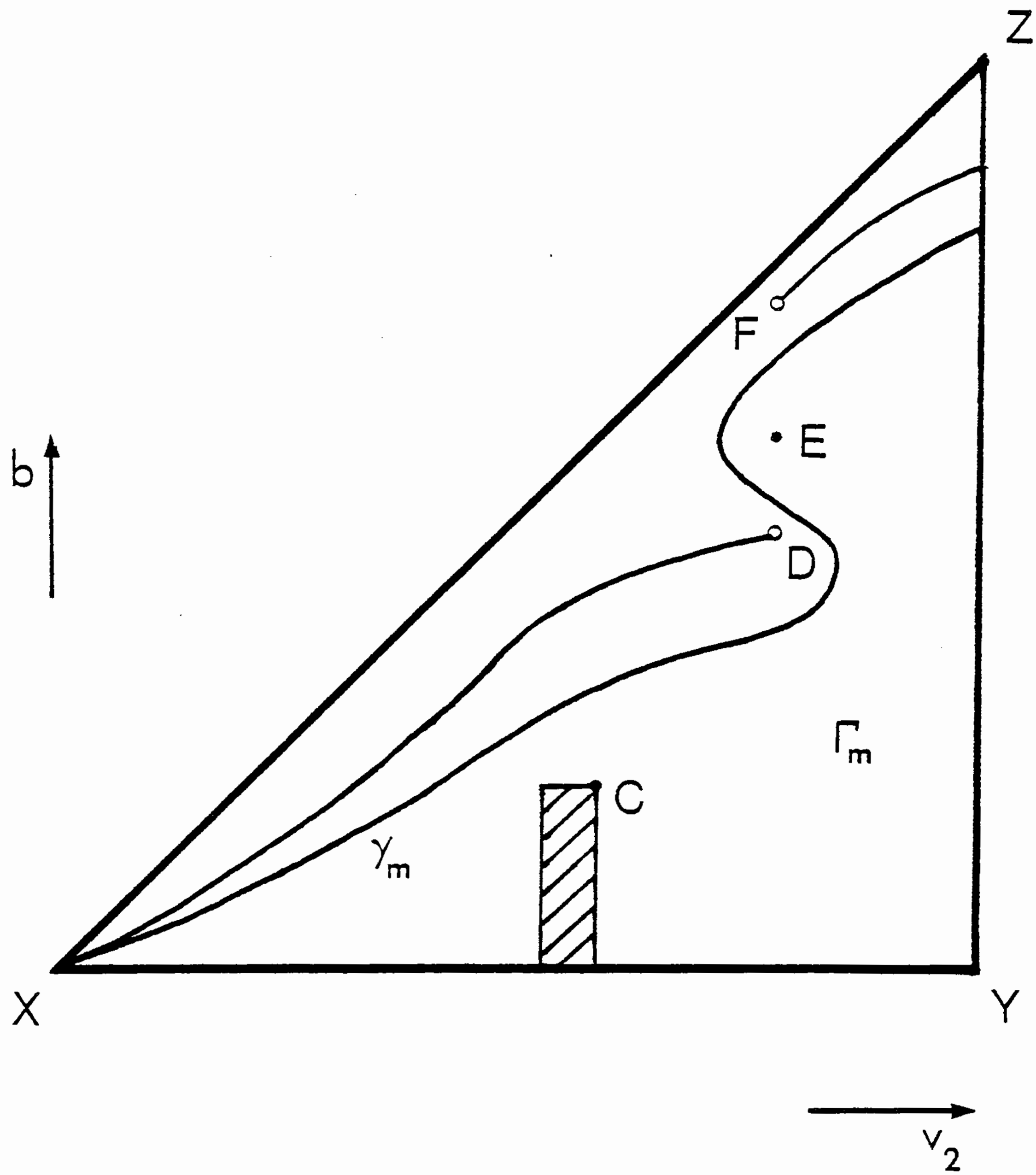

Figure 4.3. The boundary γ_m does not define b as a function of v_2; therefore property λ is not satisfied. This may permit the existence of an equilibrium strategy whose graph enters Γ_m at isolated points such as E.

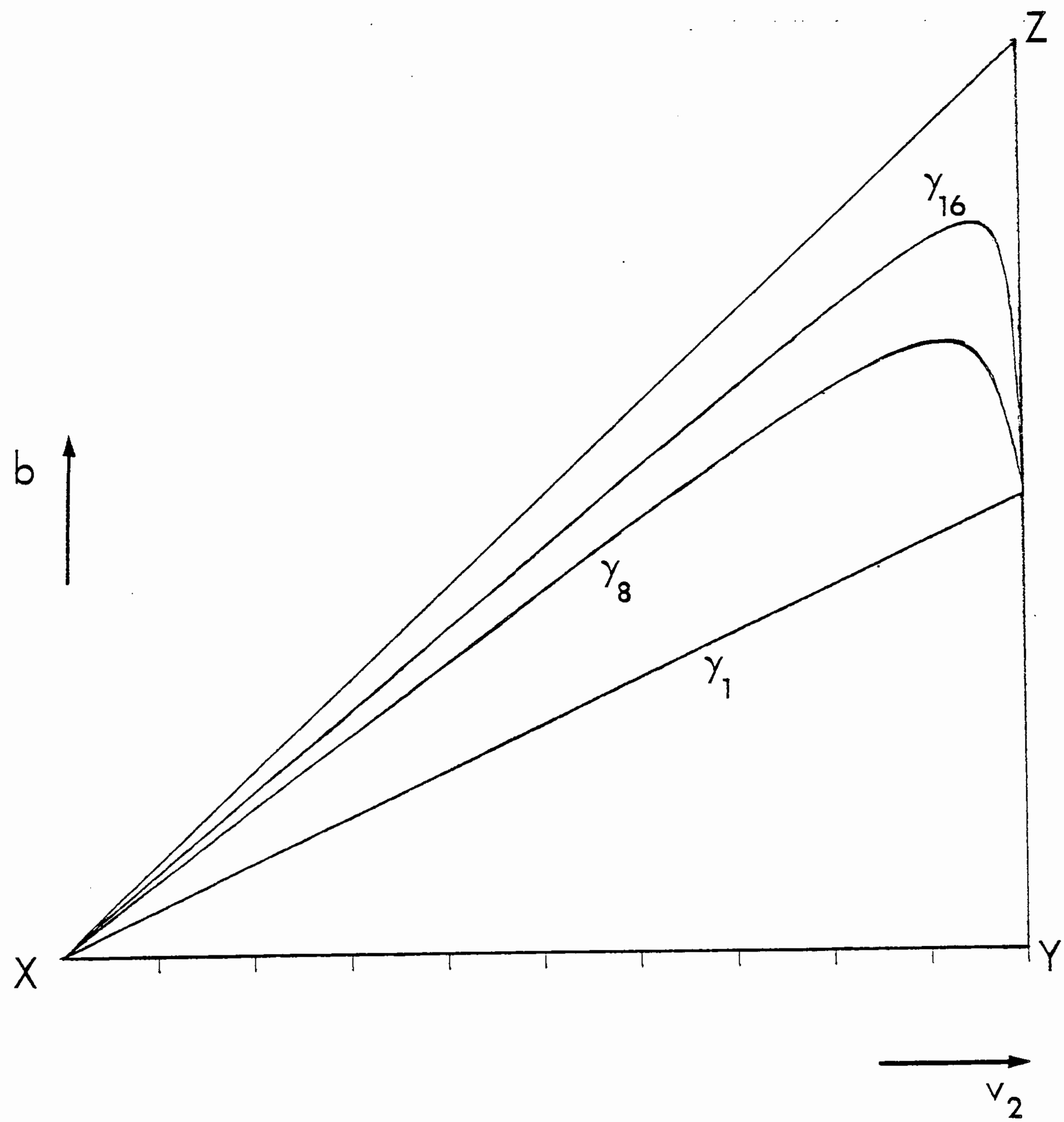

Figure 4.4. The boundaries γ_1, γ_8, and γ_{16} are shown for the uniform case. The graph of any equilibrium strategy B in a market with 2m traders must lie above γ_m almost everywhere. The edge XZ corresponds to the strategy of truthful revelation.