

DISCUSSION PAPER NO. 74

AN ADDENDUM: PURE INTEREST AND  
THE GENERAL CASE OF TIME STATE  
PREFERENCE FOR MONETARY INCOME

by

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## PREFACE

Due to some extent of interest already generated in the question, this paper is being issued in its rough "draft" form, which needs be read in conjunction with the Discussion Paper No. 73 to which it is an addendum. Meanwhile, a self-contained version of it, giving due attention to all the "nitty-gritty" details is also being prepared. This way, the interested reader does not have to wait too long to see the implications in the general case.

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0. INTRODUCTION

In the paper "Proof that the Existence of Pure Interest Rate Fixes the Admissible Functional Forms of Cardinal Utility for Monetary Income," I showed that, if we assume that an individual discounts monetary incomes at some rate which is his own rate of pure interest, then the following is a theorem of the requirement of consistency of the preference behavior: For any lottery, the time adjusted value of its certainty equivalent must always equal the certainty equivalent of the corresponding time adjusted lottery. (The time adjusted lottery corresponding to any given lottery is constructed by adjusting for time each of the possible outcomes of the lottery individually, using the individual's own rate of pure interest, and leaving all else the same.) Also, I used this "Fundamental Consistency Condition" to derive the admissible functional forms of the cardinal utility for monetary income for

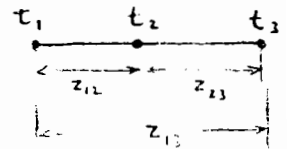
the case when the cardinal utility is assumed to be one and the same for incomes at any time slice whatever.

In this "Addendum," I study the implications of the Fundamental Consistency Condition for the general case when the last mentioned assumption is dropped, i.e., when the individual's current preferences are dependent upon the "time state" so that, for each  $t$  ( $t \in T$ ), the cardinal utility functions  $u_t: M \rightarrow \mathbb{R}$  for monetary incomes  $m \in M$  at time  $t$  are not required to be necessarily one and the same.

ADDENDUM: THE GENERAL CASE OF TIME STATE PREFERENCE AND PURE INTEREST

Drop Assumption II (see 2.5), i.e., consider the general case when, for each  $t \in T$ , the utility functions  $u_t: M \rightarrow \mathbb{R}$  for monetary income at time  $t$  are not required to be necessarily one and the same.

Denote  $z_{12}$  = Adjustment factor for  $(t_1, t_2)$   
 $z_{23}$  = " " "  $(t_2, t_3)$   
 $z_{13}$  = " " "  $(t_1, t_3)$



where  $z_{12}, z_{23}, z_{13} \in \Theta$  and  $t_1, t_2, t_3 \in T$ .

Also, denote  $u_1, u_2, u_3$  to be the utility functions for monetary incomes at times  $t_1, t_2, t_3$  respectively.

By the Fundamental Consistency Theorem (see 2.3), for an arbitrary lottery  $\ell = \{((m_1, t_1) : p_1), \dots, ((m_n, t_1) : p_n)\}$  with  $m_1, \dots, m_n \in M$ , we have

$$u_1(c) = \sum_{i=1}^n p_i u_1(m_i) \Leftrightarrow u_2(z_{12} \cdot c) = \sum_{i=1}^n p_i u_2(z_{12} \cdot m_i) \dots\dots(1)$$

Now, rewrite  $u_2(z_{12} \cdot m) \equiv g_{12}(m) \quad (m \in M) \dots\dots\dots(2)$

Then, (1) can be rewritten as:

$$u_1(c) = \sum_{i=1}^n p_i u_1(m_i) \Leftrightarrow g_{12}(c) = \sum_{i=1}^n p_i g_{12}(m_i) \dots\dots\dots(3)$$

Now, recall the fact that each  $u_t \quad (t \in T)$  is determined upto an

affine transformation: (3) says that  $u_1$  and  $g_{12}$  are the same upto an affine transformation. Hence,

$$g_{12}(m) = \Upsilon(z_{12}) + \varphi(z_{12}) \cdot u_1(m) \quad (m \in M) \quad \dots\dots\dots(4)$$

where  $\Upsilon(z_{12})$  and  $\varphi(z_{12})$  are constants which depend upon the value of  $z_{12}$ .

Clearly, it does not matter whether the lottery  $\ell$  is adjusted first to the time  $t_2$  and then to the time  $t_3$  or is adjusted directly to the time  $t_3$ . If we consider the adjustment to take place in two steps, then the following is obtained for all  $m \in M$ :

$$u_3(z_{23}(z_{12}, m)) = g_{13}(m) = \Upsilon(z_{23}) + \varphi(z_{23}) \cdot u_2(z_{12}, m) \quad \dots\dots\dots(5)$$

$$= \Upsilon(z_{23}) + \varphi(z_{23}) \cdot [\Upsilon(z_{12}) + \varphi(z_{12}) \cdot u_1(m)] \quad \dots\dots\dots(6)$$

$$= \Upsilon(z_{23}) + \varphi(z_{23}) \cdot \Upsilon(z_{12}) + \varphi(z_{23}) \cdot \varphi(z_{12}) \cdot u_1(m) \quad \dots\dots\dots(7)$$

Making the adjustment in a single step, and noting that  $z_{12} \cdot z_{23} = z_{13}$ ,

$$g_{13}(m) = \Upsilon(z_{13}) + \varphi(z_{13}) \cdot u_1(m) \quad \dots\dots\dots(8)$$

Finally, comparing (7) and (8), we have the following equations

holding simultaneously:

$$\varphi(z_{12} \cdot z_{23}) = \varphi(z_{13}) = \varphi(z_{23}) \cdot \varphi(z_{12}) \quad \dots\dots\dots(9)$$

$$\Upsilon(z_{12} \cdot z_{23}) = \Upsilon(z_{13}) = \Upsilon(z_{23}) + \varphi(z_{23}) \cdot \Upsilon(z_{12}) \quad \dots\dots\dots(10)$$

In the above, (9) is the classical Cauchy's functional equation which is the same as Equation (20) of the main paper (see p. 21); and (10) is the same functional equation as Equation (15) of the main paper (see p. 20); this is also reducible to a classical Cauchy's equation [Equation (18) of the main paper (see p. 21)]. There is one difference: we have not established what values should be assigned to  $\gamma$  and  $\varphi$  at some characteristic points; and, also,  $\gamma$  and  $\varphi$  don't have to be strictly increasing monotonic.

Without going into the nitty-gritty details, it now transpires that (9) and (10) are simultaneously satisfied with any of the following solutions only:

$$\gamma \equiv k \quad \text{and} \quad \varphi(x) = |x|^c \cdot \text{Sign } x \quad \dots\dots\dots(11)$$

$$\varphi \equiv k \quad \text{and} \quad \gamma(x) = c \cdot \log |x| \cdot \text{Sign } x \quad \dots\dots\dots(12)$$

$$\varphi \equiv k \quad \text{and} \quad \gamma \equiv \kappa \quad \dots\dots\dots(13)$$

(where  $k$  and  $\kappa$  are constants)

Of these, the first two cases were already covered in the main paper. So, this leaves the general solution, when all the  $u_t$ 's ( $t \in T$ ) are not necessarily the same, to be of the form:

$$u_2(z_{12} \cdot m) = g_{12}(m) = k + \kappa u_1(m) \quad (m \in M) \quad \dots\dots\dots(14).$$

Thus, the utility functions are obtained by stretching the m axis by the discount factor (or the time adjustment factor).