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IMMISERIZING GROWTH IN DIAMOND'S OVERLAPPING GENERATIONS MODEL: A GEOMETRICAL EXPOSITION

by

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Abstract

This paper demonstrates, using simple diagrams, that, in Diamond's (1965) overlapping generations model, capital saving technological progress could make all generations (including those which live during the transition period) worse off when the economy is dynamic inefficient. It is also shown that technological progress always makes some generations better off when the economy is dynamic efficient.

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1. Introduction

This paper demonstrates, using simple diagrams, that, in Diamond's (1965) overlapping generations model, capital saving technological progress could make all generations (including those which live during the transition period) worse off when the economy is dynamic inefficient. It is also shown that technological progress always makes some generations better off when the economy is dynamic efficient.[1]

No one should be surprised at the possibility of immiserizing growth in Diamond's model. As demonstrated in the noted papers by Bhagwati(1958) and Johnson(1967), an expansion of production frontier may reduce the welfare of the representative consumer in the presence of some distortions. Immiserizing growth can occur when the growth is sufficiently biased to increase the existing distortion. Allocations in Diamond's overlapping generations economy are not Pareto-optimal in the dynamic inefficiency region, because high labor income earned by the young consumers generates savings more than enough to maintain the golden rule capital stock. This suggests that capital saving technological progress, which raises the marginal productivity of labor and reduces the marginal productivity of capital, could be immiserizing. The old generation which has the capital stock when the technological progress occurs suffers from a capital loss. The young and all future generations will earn higher wages, which causes an increase in saving and therefore a decline in the interest rate. They could also suffer from technological progress, because a decline in the interest rate, which is a deterioration of their terms of trade since they are net savers in this economy, might be large enough to offset a primary gain of a rise in wages.

Some comments on closely related works are in order. Bhagwati (1958) and

Johnson (1967) developed their arguments in open economy settings. Their examples are incomplete in the sense that they took a terms-of-trade or an offer curve of the rest of the world as exogenous, instead of solving for general equilibrium. Aumann and Peleg (1974) and Dixit and Norman (1980), among others, constructed two country models and demonstrated that an increase in endowments in the home country can make home consumers worse off in a general equilibrium setting. These studies, however, also revealed that foreign consumers will be better off at the expense of home consumers: it is impossible for all consumers to be worse off in a closed economy in the absence of distortion. In Diamond's economy, allocations are not Pareto optimal when the steady state interest rate is lower than the growth rate of economy, i.e., when the economy is dynamic inefficient.[2] Therefore, Diamond's model is a natural framework within which to demonstrate the possibility of immiserizing growth.

Few works have investigated welfare implications of technological progress in an overlapping generations economy. Fried (1980), Galor (1986) and Matsuyama (1986) are the few exceptions. Fried and Matsuyama assume no population growth and therefore, discuss the dynamic efficient case only. They are mainly concerned with intergenerational distributions of the gains from technological progress. Galor examines the steady state welfare implications of Hicks-neutral technological change. Donsimoni and Polemarchakis (1985) investigated effects of an increase in labor endowment on the steady state welfare in Diamond's model. They show that, unless the economy is on the golden rule path, a labor endowment increase can reduce the steady state welfare.[3]

To the best of my knowledge, no works have demonstrated an immiserizing growth (i.e., an expansion of production frontier which reduces everybody's welfare) in a general equilibrium setting.

The paper is organized as follows. Section 2 expounds Diamond's model and introduces four loci, which play a crucial role in what follows. They are the iso-utility locus, the asset market equilibrium locus, the factor price frontier and the iso-savings locus. Section 3 provides a simple proof of the impossibility of immiserizing growth in an efficient case, using the first three loci. Section 4 turns to an inefficient case and demonstrates that technological progress could make all generations worse off. The proof is constructive in the sense that a factor price frontier which causes an immiserizing growth is drawn using the iso-savings locus.

2. Diamond's Model

The model is based on Diamond's classical contribution. Refer to Diamond (1965) for more details.

(a) Consumers: A consumer lives for two periods, but works only during the first. He supplies one unit of labor inelastically. Thus the labor force at time t, L_t equals the number of young consumers at this date. It is assumed that the population grows at rate $n>0$, i.e., $L_t=L_0(1+n)^t$. A consumer born at period t solves the following maximization problem:

$$\text{Maximize } U(c^y{}_t, c^o{}_{t+1}) \quad \text{subject to} \quad c^y{}_t+c^o{}_{t+1}/(1+r_{t+1})=w_t,$$

where $c^y{}_t$ and $c^o{}_{t+1}$ are his consumptions when young and old. His wage income is w_t, and r_{t+1} is the rate of return of holding one unit of capital from t to t+1. The utility function $U(c^y,c^o)$ satisfies the standard neoclassical assumptions which guarantee a unique interior solution to the problem. It is also assumed that consumptions when young and old are normal goods. Let $S(w_t,r_{t+1})$ denote the optimal savings $w_t - c^y_t$, given w_t and r_{t+1}. The normality assumption implies that $0<S_1<1$.[4] Let $H^S(w,r)$ denote the slope of an <u>iso-savings locus</u> $S(w,r)=\bar{S}$ at (r,w) in the (r,w) plane. Then,

(1) $$H^S(w,r)=-S_2(w,r)/S_1(w,r).$$

Let $V(w_t,r_{t+1})$ denote the indirect utility function, i.e., $V(w_t,r_{t+1})=U(w_t-S(w_t,r_{t+1}),(1+r_{t+1})S(w_t,r_{t+1}))$. An increase in wages increases the utility $(V_1=U_1>0)$. An increase in the interest rate also increases utility $(V_2=U_2S>0)$, because the consumer is a net saver in this economy. This implies that the <u>iso-utility locus</u> $V(w,r)=\bar{V}$ is downward sloping in the (r,w) plane. Let $H^U(w,r)$ denote the slope of an iso-utility locus at (r,w). Using the envelope theorem,

(2) $$H^U(w,r)=-S(w,r)/(1+r)<0.$$

(b) <u>Production</u>: The economy produces one final good, using labor L_t and the capital stock K_t. The technology employed satisfies constant returns to scale: $F(K_t,L_t)=L_tf(k_t)$, where k_t is the capital stock per worker. It is assumed that $f(k)$ has standard neoclassical properties and that production using this technology is competitive. Then, $r_t=f'(k_t)$ and $w_t=f(k_t)-k_tf'(k_t)$. It is useful to describe the production technology in its dual form. The cost of producing one unit of output is given by the cost function $C(w_t,r_t)$, which is independent of output produced due to the constant returns to scale assumption. Then, the <u>factor price frontier</u> (FPF) of this economy is $C(w,r)=1$, which is downward sloping and convex towards the origin in the (r,w) plane. Let $H^F(w,r)$ denote the slope of FPF at (r,w). Then,

(3) $$H^F(w,r)=-k=-f'^{-1}(r)<0.$$

Later, technological progress is introduced by a shift of FPF.

(c) Asset Market Equilibrium: Given r_{t+1}, firms invest so as to equalize the marginal productivity of capital and interest rate: $k_{t+1} = f'^{-1}(r_{t+1})$. Only the young have an incentive to hold assets in this economy. Therefore, asset market equilibrium locus (AME) is given by $(1+n)f'^{-1}(r_{t+1}) = S(w_t, r_{t+1})$. Let $H^A(w,r)$ denote the slope of this locus at (r,w). Then,

(4) $$H^A(w,r)=[(1+n)-f''(f'^{-1}(r))S_2(w,r)]/[f''(f'^{-1}(r))S_1(w,r)].$$

(d) Dynamics: Capital at date zero (k_o) is given by history and the dynamics of the economy are described by:

(5a) $$(1+n)f'^{-1}(r_{t+1})=(1+n)k_{t+1}=S(w_t,r_{t+1}), \qquad \text{(AME)}$$

(5b) $$C(w_t,r_t)=1, \qquad \text{(FPF)}$$

and

(5c) $$r_o=f'(k_o).$$

I also assume that Diamond's stability assumption that AME (5a) and FPF (5b) have a unique intersection $P^*=(r^*,w^*)$ and that:

(6) $$\left| H^A(w^*,r^*) \right| > -H^F(w^*,r^*) > 0.$$

The dynamic path of the economy, given by equations (5a-c) converges to $P^*=(r^*,w^*)$. This path is efficient if $r^*>n$ and inefficient if $r^*<n$.[5] The steady state capital stock is $k^*=f'^{-1}(r^*)$.

I first show the following lemma, which plays a crucial role in what follows.

Lemma:

$$H^F(w^*,r^*) \lesseqgtr H^U(w^*,r^*) \quad \text{as} \quad r^* \gtreqless n. \tag{7}$$

Proof: In the steady state, (5a) holds. Therefore, by (2) and (4), $H^U(w^*,r^*)=-S(w^*,r^*)/(1+r^*)=-[(1+n)/(1+r^*)]k^*=[(1+n)/(1+r^*)]H^F(w^*,r^*) \gtreqless H^F(w^*,r^*)<0$ as $r^* \gtreqless n$. Q.E.D.

Figures Ia and Ib show positions of three loci passing through the steady state $P^*=(r^*,w^*)$. Curve UU' represents the iso-utility locus $V(w,r)=V(w^*,r^*)$, curve AA' the asset market equilibrium locus and curve FF' the factor price frontier. The stability condition (6) implies that AA' is (in absolute terms) steeper than FF'. Lemma implies that UU' is less steep than FF' when the economy is efficient and steeper than FF' when the economy is inefficient.[6] Therefore, a decline in wages with a rise in capital incomes along the FPF reduces the steady state utility in the efficient case and increase it in the inefficient case. For example, suppose that, in addition to capital stock, there exists assets that bring a real rent, such as land, paintings or jewels, and assume that rents per capita are constant. Then, as shown in Tirole (1985), the economy is efficient. (Otherwise, the present value of assets with rents would be infinite.) Higher rents per capita crowd out more capital stock, shifting AA' to the right in Figure Ia. A new steady state is given by a point like Q. The economy is farther away from the golden rule and steady state utility is lower.

On the other hand, if the economy is inefficient, steady state utility increases by crowding out the capital stock. For example, suppose that the government issues short term bonds, keeping the debt-labor ratio, b, constant. The government also spends $g_t=(n-r_t)b$ per capita each period for

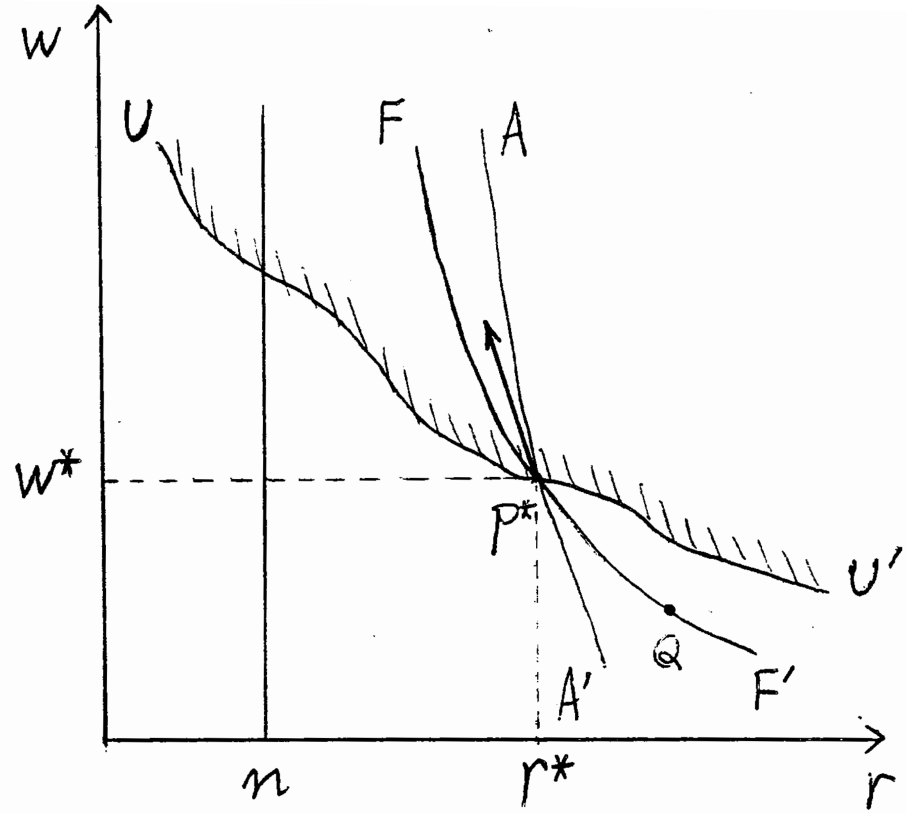

Figure Ia : Efficient Case

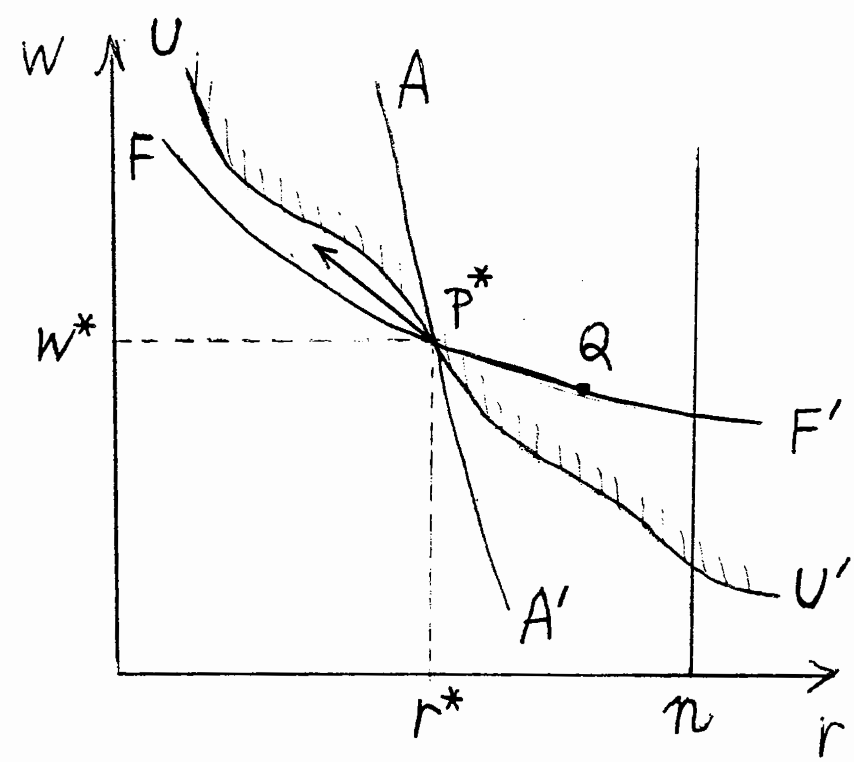

Figure Ib: Inefficient Case

purposes which do not affect utility, such as foreign aid. A higher debt-labor ratio shifts AA' to the right in Figure Ib. A new steady state is given by a point like Q. The government debt crowds out the capital stock and the economy moves closer to the golden rule and therefore, steady state utility is higher.

In the following sections, I will return to the model with no rents and no government bonds and discuss the possibility of immiserizing growth caused by technological progress.

3. Efficient Case: Impossibility of Immiserizing Growth

Using Figure Ia, I will prove that, when the economy is efficient, technological progress always makes some generations better off. To do so, it is assumed that new technology $G(K,L)=Lg(k)$ is also constant returns to scale and satisfies the standard neoclassical assumptions. Then technological progress can be expressed as an upward shift of FF'.

Proposition 1: Suppose that the economy is initially in the steady state $P^*=(r^*,w^*)$ where $r^*>n$. Then any small technological progress always makes some generations better off.[7]

Proof: Technological progress changes consumption by the old generation from $[1+f'(k^*)](1+n)k^*$ to $[1+g'(k^*)](1+n)k^*$, where $(1+n)k^*$ is the original steady state capital stock per old consumer. To make the old generation strictly worse off, $f'(k^*)$ must be greater than $g'(k^*)$. This implies that, in Figure Ia, AA' must shift leftward in the neighborhood of P*. Since technological progress shifts FF' upward, the intersection of the two loci moves in the direction indicated by the arrow. The new steady state must lie above UU'.

Since the indirect utility function is independent of technology, this proves that any technological progress which make the old generation worse off must increase steady state welfare. Q.E.D.

The logic underlying this result is quite straightforward. When the economy is efficient, the young consumer does not save enough to maintain the golden rule capital stock. Any technological progress, which makes the old consumer worse off increases wages and therefore, savings. It also reduces the golden rule capital stock. The technological progress brings the economy closer to the golden rule, and thereby improves steady state welfare.

4. Inefficient Case: Immiserizing Growth

A casual look at Figure Ib suggests that, in an inefficient case, technological progress which makes the old generation worse off (an upward shift of FF' and a leftward shift of AA') could also reduce the steady state welfare, as indicated by the arrow. However, Figure Ib is not appropriate as a tool to demonstrate immiserizing growth, because a shift of AA' is not independent of a shift of FF'.

In this section, I will demonstrate the possibility of immiserizing growth in an inefficient case. To do so, the iso-savings locus through $P^*=(r^*,w^*)$, the SS' curve, is introduced. From equations (1) and (2), the slope of UU' at P^*, $H^U(w^*,r^*)$, is numerically smaller than that of SS' at P^*, $H^S(w^*,r^*)$, when:

(8) $S_1(w^*,r^*)>e(w^*,r^*)=(1+r^*)S_2(w^*,r^*)/S(w^*,r^*)=d\log S/d\log(1+r)$.

holds. This inequality means that the marginal propensity to save is greater than the gross interest rate elasticity of savings at $P^*=(r^*,w^*)$. On the other hand, if $S_1(w*,r*)<e(w*,r*)$, the slope of UU' at P*, $H^U(w*,r*)$, is numerically greater than that of SS' at P*, $H^S(w*,r*)$.

Proposition 2: Suppose that the economy is initially in the steady state $P*=(r*,w*)$ where $r*<n$. Then, there exists a technological progress which makes all generations worse off.

Proof:

Part 1: First, I will show a proof for the case when inequality (8) holds. The proof is constructive. It is divided into three steps.

Step 1: the construction of a FPF. (See Figure II[8])

Choose a point $P**=(r**,w**)$ which lies below UU', above FF' and SS' and to the left of $P*=(r*,w*)$. (This is possible since UU' is steeper than FF' and SS' by equations (7) and (8).) Let $k**=(1+n)^{-1}S(w**,r**)$. Since P** lies above SS', $S(w**,r**)>S(w*,r*)$. Therefore, $k**=(1+n)^{-1}S(w**,r**)> (1+n)^{-1}S(w*,r*)=k*$. Draw a line through P**, l_1l_1', whose slope is $-k**$. Draw a line through P**, l_2l_2', whose slope is $-k*$. Next, choose a point $P_o=(r_o,w_o)$ in the interior of the area surrounded by UU', l_1l_1', l_2l_2', FF' and SS'. By choosing P_o close enough to P**, it is ensured that line P_oP** (not drawn in Figure II) does not intersect with UU'. Draw a line through P_o, l_3l_3', whose slope is $-k*$. Finally, draw a downward sloping convex curve, which is tangent to l_1l_1' at P** and to l_3l_3' at P_o and lies above FF' everywhere. Denote this convex curve ff'. By construction, ff' lies below UU' in the interval $[r**,r_o]$.

Step 2: the construction of an AME locus. (See Figure III).

Now, let us draw the AME locus, when the FPF is given by ff'. Call this locus aa'. First, if the interest rate is equal to r**, firms' demand for

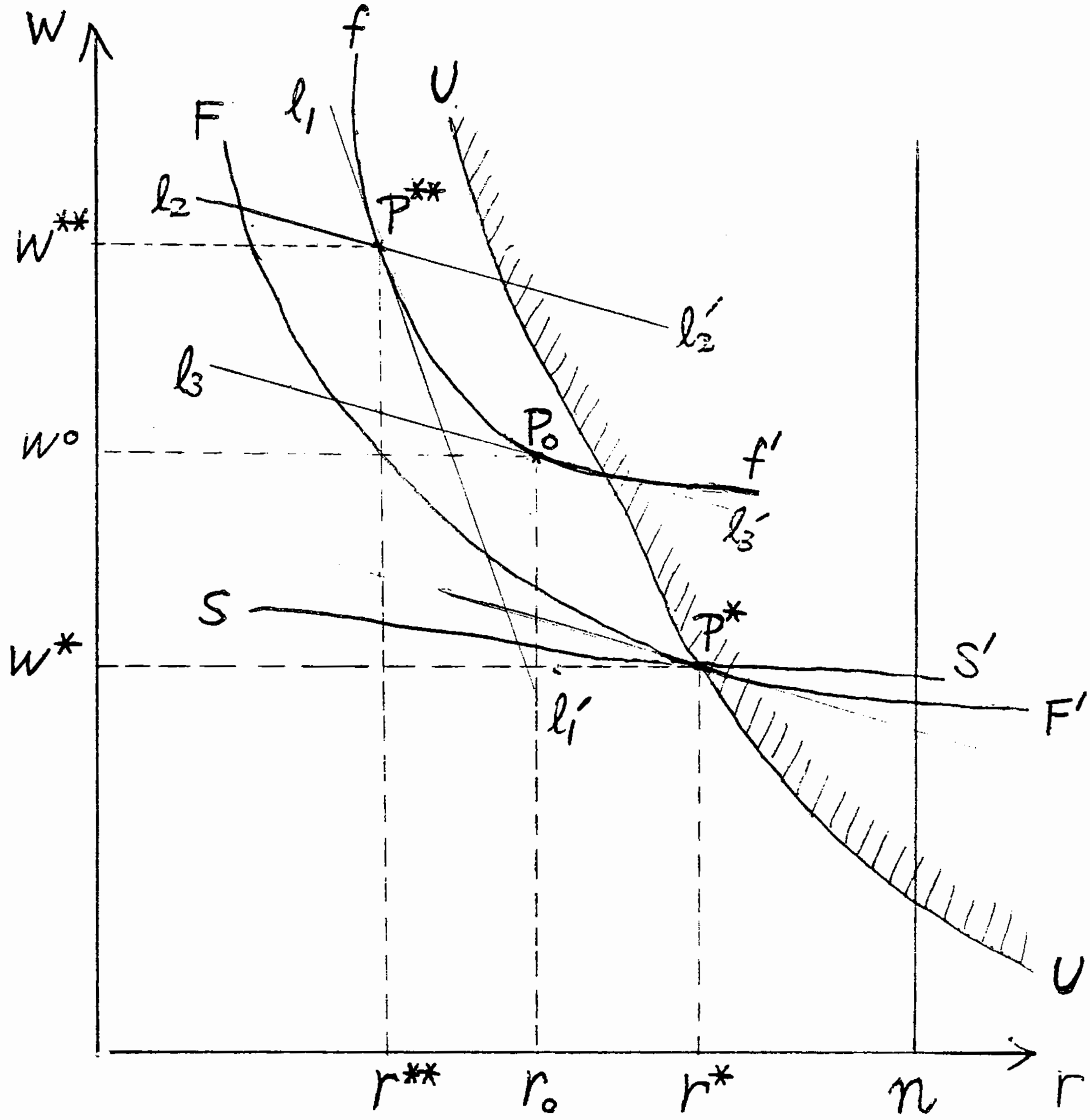

Figure II

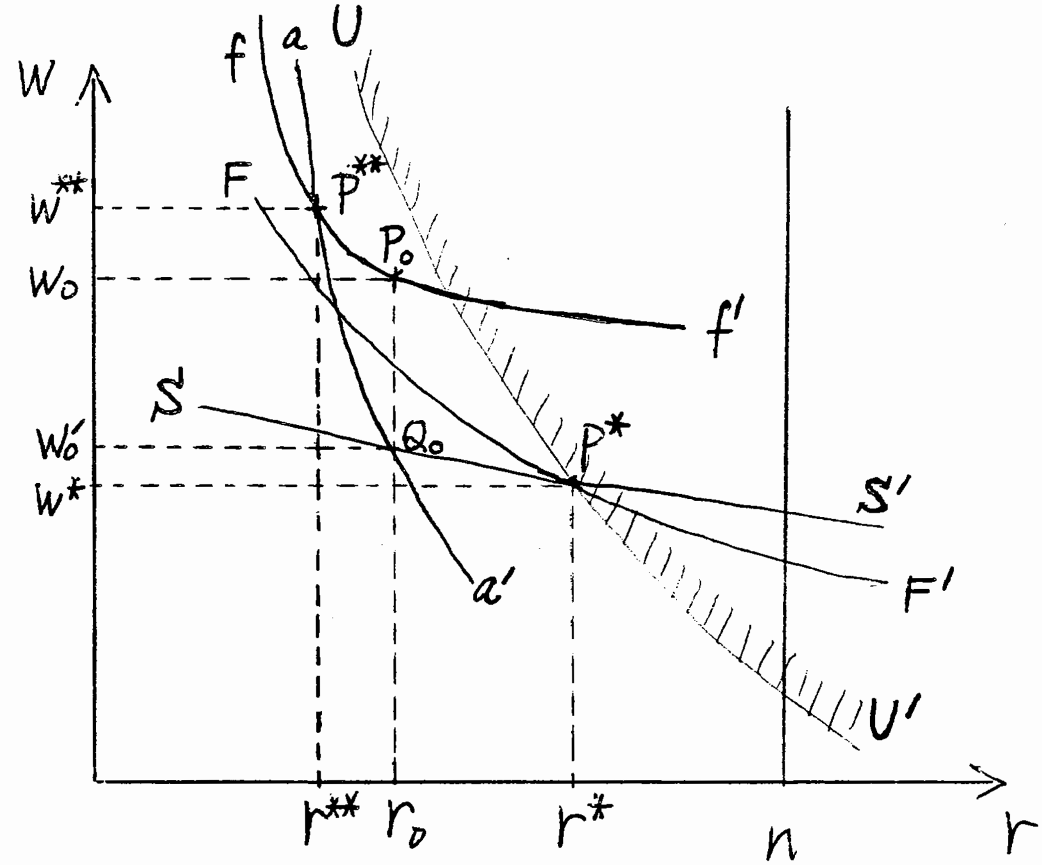

Figure III

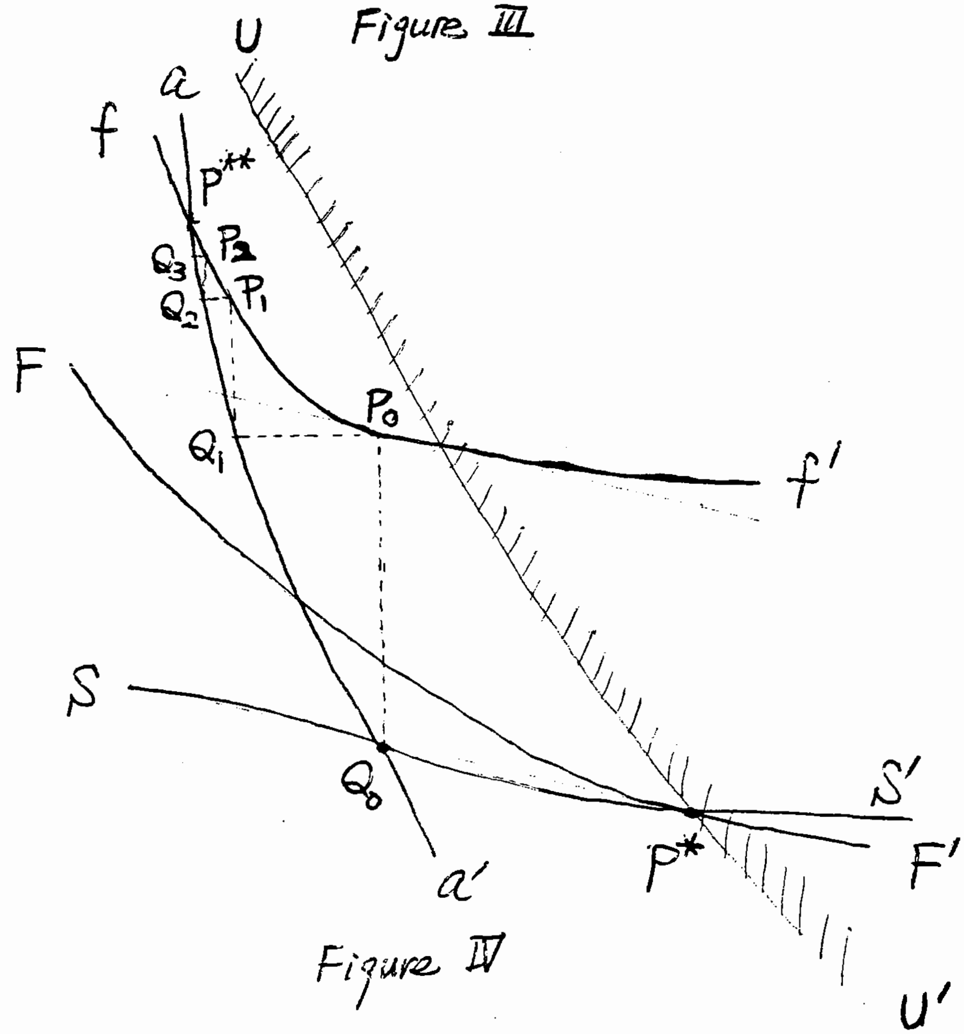

Figure IV

capital per worker is equal to k** by construction of ff' and equation (3). Since (1+n)k**=S(w**,r**), the asset market clears at P**=(r**,w**). Second, denote the intersection of SS' and $r=r_o$ by $Q_o=(w_o',r_o)$. Then, $S(w_o',r_o)=S(w^*,r^*)$. If the interest rate is r_o, the demand for capital per worker is k* by construction of ff'. Therefore, $(1+n)k^*=S(w^*,r^*)=S(w_o',r_o)$ implies that the asset market clears at Q_o. Hence aa' can be drawn by connecting P** and Q_o. Note that aa' lies below ff' in the interval (r^{**},r_o) and therefore, below UU' in the interval $[r^{**},r_o]$. (If aa' intersects with ff' in (r^{**},r_o), then choose the intersection closest to Q_o and relabel it as P**.)

Step 3: proof that technological progress which shifts the FPF from FF' to ff' is an immiserizing growth. (See Figure IV)

Suppose that the economy is initially at P*. The capital stock per worker is k*. Suppose that technological progress shifts the FPF from FF' to ff'. Then factor prices jump to the point on ff' where the slope is -k*, i.e., P_o by construction of ff' (see equation (5c)). The interest rate drops from r* to r_o. Therefore, the old consumer is made worse off. The dynamics of the economy is depicted by sequence of P_o, Q_1, P_1, Q_2, P_2, Q_3,,,,,P*. (see equations (5a) and (5b)). The utility of the i-th generation after the technological progress is given by Q_i. By construction of ff' and aa', Q_i lies below UU'. This proves that technological progress can make all generations worse off, when inequality (8) holds.

Part 2: Next, I will give a sketch of a proof for the case when inequality (8) does not hold without figures, since the basic technique is the same as in Part 1.

Choose a point P**=(r**,w**) which lies below UU', above FF' and left to SS'. Let $k^{**}=(1+n)^{-1}S(w^{**},r^{**})$. Since P** lies left to SS', S(w**,r**)<

$S(w^*,r^*)$. Therefore, $k^{**}=(1+n)^{-1}S(w^{**},r^{**})<(1+n)^{-1}S(w^*,r^*)=k^*$. Choose a point $P_o=(r_o,w_o)$ such that P_o lies northwest to P^{**}, the slope of line segment P_oP^{**} is numerically greater than $-k^*$, smaller than $-k^{**}$, and that the triangle formed by points P_o, P^{**} and (r^{**},w_o) lies below UU' and SS'. Finally, draw a downward sloping convex curve, which passes through P_o with slope equal to $-k^*$ and passes through P^{**} with the slope equal to $-k^{**}$ and lies above FF' everywhere. Then, one can show that technological progress which shifts the FPF from FF' to this convex curve is an immiserizing growth, by following the same steps as in Step 2 and Step 3 of Part 1.

Combining Part 1 and Part 2, this completes the proof of proposition 2.

References:

Aumann, R. J. and B. Peleg "A Note on Gale's Example." Journal of Mathematical Economics 1 (August 1974):209-211.

Balasko, Y. and K. Shell. "The Overlapping-Generations Model I: the Case of Pure Exchange Without Money." Journal of Economic Theory 23 (June 1980): 281-306.

Bhagwati, Jagdish N. "Immiserizing Growth: A Geometrical Note." Review of Economic Studies 25 (June 1958):201-205.

Diamond, Peter A. "National Debt in a Neoclassical Growth Model.: American Economic Review 55 (December 1965):1126-1150.

Dixit, A. and V. Norman, Theory of International Trade Cambridge, Cambridge University Press, 1980.

Donsimoni, M. P. and H. M. Polemarchakis. "Intertemporal Equilibrium and Disadvantageous Growth." First Boston Working Paper Series FB-85-35, Graduate School of Business, Columbia University, 1985.

Fried, Joel. "The Intergenerational Distribution of the Gains from Technical Change and from International Trade." Canadian Journal of Economics 13 (February 1980):65-81.

Galor, O. "The Long-Run Implications of a Hicks-Neutral Technological Progress." Working Paper #86-20, Brown University, 1986.

Johnson, Harry G. "The Possibility of Income Losses from Increased Efficiency or Factor Accumulation in the Presence of Tariffs." Economic Journal 77 (March 1967):151-154.

Matsuyama, Kiminori. "Welfare Analysis of Technological Progress in a Two Sector Overlapping Generations Model." mimeo. Harvard University. 1986.

Shell, Karl. "Notes on the Economics of Infinity." Journal of Political Economy 79 (October 1971):1002-1011.

Tirole, Jean. "Asset Bubbles and Overlapping Generations." Econometrica 53 (September 1985):1071-1100.

Footnotes:

1. Technological progress may reduce the <u>steady state</u> welfare even in a dynamic efficient case: see Fried (1980).

2. The non-optimality of overlapping generations model is not due to the "missing trading links" as sometimes suggested. It is due to the "double infinity" of the model: see Shell (1971).

3. Donsimoni and Polemarchakis call their result "immiserizing growth" or "disadvantageous growth". This is very misleading. By restricting their analysis to the steady state, they ignore the effect on the generations which live during the transition period. What happens in their model is a transfer of welfare gains from future generations to current generations. In fact, one can easily verify that an increase in labor endowment always increase the welfare of the current old generation, which has the title to the capital stock when the change occurs.

4. S_1 and S_2 denote the derivatives of S with respect to its first and second arguments. All derivatives which appear in this paper assumed to exist and be finite.

5. I do not discuss the case $r^*=n$. The analysis of this non-generic case needs to be algebraic, utilizing the Balasko and Shell (1980) criterion.

6. In Figures Ia and Ib, AA' is downward sloping, but this need not be the case. AA' could be less steep than UU' in Figure Ib.

7. A large technological progress could make all generations worse off, when it is large enough to drive the economy into the inefficiency region, or when it is large enough to produce a multiple of the steady states.

8. In Figures II, III and IV, SS' is drawn so that $H^F(r^*,w^*)<H^S(r^*,w^*)<0$, but this need not be the case. However, the proof can be applied to other cases and need not be altered, as long as $H^U(r^*,w^*) < H^S(r^*,w^*)$, i.e., when inequality (8) holds.