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SUSCEPTIBILITY TO MANIPULATION

by

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The purpose of this paper is to analyze a certain kind of incentive and manipulation problem that occurs most prominently in positional voting. Positional voting is any election procedure equivalent to having the tally for N alternatives given by a voting vector $\underline{W}_N = (w_1,..,w_N)$, $w_j \geq w_{j+1}$, $w_1 > w_N = 0$, where w_j points are assigned to a voter's j^{TH} ranked alternative.[1] Already in the 1780's it was recognized that one of these systems, the Borda Count, (defined by $\underline{B}_N=(N-1,N-2,..,1,0)$ and to be denoted by BC), can be manipulated[2]. This weakness still is used as an argument against it. (See, for example, [9,15].) Of course, because of the Gibbard - Satterthwaite Theorem [3,13] we now know that all positional voting methods can be manipulated if $N \geq 3$.

If all systems can be manipulated, then a "second best" approach must be used. Namely, we should determine the positional voting system that is "least susceptible" to a successful manipulation of the relative ranking of some two alternatives. This involves determining which $\underline{W}_N$ gives a strategic vote the largest impact on the final tally. For N=3, the answer is $\underline{B}_3 = (2,1,0)$, the BC. Consequently, one might suspect, and it probably is true, that the BC is fairly suspectible to manipulation over all possible coalitions.

There are noteworthy examples where large coalitions, such as special interest blocs, try to manipulate an election. But quite often, manipulation attempts are due to the efforts of small coalitions and individuals. What happens here? To analyze this issue, we need to understand the frequency of opportunities where the alterred

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1. It is easy to see that a positive scalar multiple of $\underline{W}_N$ always yields the same election ranking, so identify all such multiples. Thus, the plurality method corresponds to (1,0,..0) or to any scalar multiple of it.

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2. Because $\underline{B}_3 = (2,1,0)$, a voter with the ranking $a_1 > a_2 > a_3$ wishing to influence the $\{a_1,a_2\}$ election outcome could strategically mark the ballot $a_1 > a_3 > a_2$ to give a_1 a two point, rather than a single point differential over a_2.

tally actually changes the outcome. In a sense that will be made precise, the surprising conclusion is that for N=3, the BC is the positional voting system that is least susceptible to manipulation! In other words, because $\underline{B}_3$ gives a strategic voter a stronger impact on the final tally, it is easy to concoct examples where the BC is the worst method; over all possible examples, the BC is the best. My analysis of manipulation involves a balance between "the frequency of opportunities to successfully manipulate the system" and "the impact of a strategically marked ballot on the tally". The best system optimizes a combination of these factors.

The BC is not the best choice for N>3; for N=4, the unique answer is (2,2,0,0) (or, equivalently, (1,1,0,0)), and for N=5 it is (2,2,1,0,0). It may appear that a pattern is beginning to develop where the answer for N=6 should be (2,2,2,0,0,0). There is a pattern, but it isn't the obvious one because for N=6, the best choice is, essentially, (6,6,5,1,0,0). (The precise answer requires some of the w_j's to be irrational numbers.) For larger values of N, the optimal choice of $\underline{W}_N$ is determined by the zeros of a set of algebraic equations. From these equations it appears that if N is sufficiently large, then the differences between the successive w_j's in $\underline{W}_N$ approaches a common value. This is, of course, a definition for the BC. My results also show that for $N \geq 3$, (1,0,0,..,0) (plurality voting) and (1,1,..,1,0) always share the dubious honor of being the most susceptible to binary manipulation.

In addition to positional voting, I analyze multiple voting systems. (A multiple systems is where each voter selects how to tally the ballot from among several specified methods.) Some of the better known multiple systems include cumulative voting, approval voting, and any positional voting procedure that tolerates "truncated" ballots. As one might suspect, the conclusion is that multiple systems are more susceptible to manipulation than any of the individual systems that define it. Moreover, the mathematical approach developed here extends to run-off elections, to other classes of social choice functions as well as to certain other allocation or decision problems that do not admit incentive structures.[3]

In two interesting papers Chamberlain [3] and Nitzan [8] consider a similar voting problem with related assumptions. On the surface, their conclusions appear to disagree with mine. I don't believe this is due to the differences in methodology.[4] We differ in conclusions because we address different issues and use different measures. This issue is briefly addressed at the end of Section 2.

A measure for manipulability should reflect that the goal is to encourage a sincere vote. "Telling the truth" must play a central role; we don't want a system that penalizes honesty. But, what system satisfies these conditions? To understand the motivation for my assumptions, I'll briefly review some of the incentive literature.

We know from the Gibbard - Satterthwaite theorem that there always are situations where someone can replace a sincere vote with a strategic one to get a personally better outcome. (This means, in game theoretic terminology, that telling the truth is not a dominant strategy.) Because it is impossible to attain our goal for all possible profiles, maybe we should restrict attention to certain "natural" ones. In incentive theory, one usually seeks those mechanisms where a person can't do better than telling the truth <u>when everyone else is sincere</u>. (So, sincere behavior is a Nash equilibrium.) This doesn't mean that such a system can't be manipulated, that there aren't strategies equally as good as being sincere, or that you can't try to manipulate the system; it only means that if everyone else is sincere, you cannot successfully manipulate the system to your advantage.

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3. This is by design; I view positional voting as an important prototype system for the incentive question posed here, so my approach has been influenced by this literature. For an informative survey of incentives, I suggest (Groves and Ledyard [5]).

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4. Their conclusions are based on computer experiments; mine are based on an analytic approach that permits precise statements. Incidently, Nitzan points out that "..an analytic derivation of the various .. measures seems to be a hopelessly complex task". In light of his comment, a contribution of this paper is the development of the mathematical structures that now permit such an analysis.

Positional voting methods form an important class of mechanisms where this incentive - manipulation problem isn't resolved even with the Nash approach. The next obvious step is to choose a (positional voting) system to minimize the likelihood that a voter can successfully manipulate the system when everyone else is sincere. This is the basis of this paper.

To determine which voting system minimizes the likelihood that honesty will be penalized, we need to measure how susceptible each $\underline{W}_N$ system is to being manipulated. Such a measure depends on the distribution of strategic voters, which, in turn, seems to be based on extra information. For instance, if I know there is a close contest between two candidiates, then I may try to manipulate the outcome to favor my preferred candidate; but if I have no added information, my unguided efforts could be counterproductive. Consequently, one approach toward strategic voting requires specifying who knows what, and who is saying what to whom.

An architect of a voting system for a "one time only" vote can use this extra information to select a method to encourage sincere voting. For example, suppose it is known that of the candidates {a,b,c}, a and b will closely contest the top position. Suppose the architect also knows either that nobody ranks c in top position, or that all voters with c as a top ranked candidate vote sincerely. It is easy to see that in this situation, *plurality voting*, (1,0,0), encourages sincere voting while (1,1,0) is the worst choice. On the other hand, if the architect knows that nobody ranks c in bottom place, then the answer is just the opposite: (1,0,0) is the worst system while (1,1,0) encourages a sincere vote. The theme of this example generalizes. In Section 2, Theorem 3 asserts that for *any choice* of a positional voting system, $\underline{W}_N$, there are distributions of the voters' preferences where $\underline{W}_N$ minimizes the susceptibility to manipulation. In other words, *with the appropriate assumptions, with the correctly constructed scenario, any system can be justified as being strategically the best*. Such added information isn't available when an *institutional voting system* is chosen. Such a system is used for all votes on all issues. It is used independent of changes in the organization. So, when choosing a $\underline{W}_N$ for an institution, we don't know which alternatives will be the target of a manipulation attempt, we don't know who is going to vote strategically, and we don't know how the preferences of the voters are split. Thus, a neutral approach is required. Therefore, my basic assumptions are: for any given set of three or more alternatives, it is equally likely for any pair of alternatives to be the target of a manipulation attempt, any profile or distribution of voters' types is equally likely, and it is equally likely that a strategic voter is of any particular type. In this sense, our results hold.

My emphasis is on how to minimize the likelihood that a pair of alternatives is manipulated. For other issues, such as the manipulation of a triplet, etc., the answers change. These questions are briefly addressed in the next section, and they can be analyzed with the mathematical techniques developed here.

2. Main Results.

The N alternatives $\{a_1,..,a_N\}$ define N! types of voters where each type corresponds to one of the N! strict rankings; that is, the voters are not indifferent between any two alternatives. Let p_j be the fraction of voters of the j^{TH} type, $j=1,..,N!$, and let $\underline{p}=(p_1,..,p_N)$. Each $p_j \geq 0$ and $..p_j=1$, so $\underline{p}$ is a point on the unit simplex, Si(N!), in the positive orthant of the N! dimensional Euclidean space. For a specified voting vector, $\underline{W}_N$, let $F(\underline{p};\underline{W}_N)$ be the obvious function that gives the tally of the ballot. More precisely, compute the vote tally for a_j (as determined by $\underline{p}$ and $\underline{W}_N$). Let this number be the j^{TH} component of an N-vector for the Euclidean space R^N, $j=1,..,N$. In this way, $F(\underline{p};\underline{W}_N)$ defines a vector in R_+^N; i.e.,

2.1 $$F(-;\underline{W}_N): Si(N!) \longrightarrow R_+^N,$$

where R_+^N is the positive orthant of R^N.

The election ranking admits a geometric interpretation. If the vector $F(\underline{p};\underline{W}_N)$ is closer to the axis defined by $\underline{e}_J$ than the one defined by $\underline{e}_K$ (this means that a_J received a higher vote than a_K), then the election ranking has a_J preferred to a_K. Consequently, if the coordinates of R^N are $(x_1,..,x_N)$, then the plane $x_J=x_K$ is the set of outcomes corresponding to indifference between a_J and a_K in the election outcome. Call this plane the "indifference surface between a_J and a_K."

If $\underline{p}$ represents the sincere rankings of the voters, the sincere outcome is $F(\underline{p};\underline{W}_N)$. If a voter, or a small group of voters, votes strategically, then the actual election tally is given by $F(\underline{p}';\underline{W}_N)$ where $\underline{p}' = \underline{p} + \underline{v}$ and where $\underline{v}$ represents the strategic change in voting. If $F(\underline{p};\underline{W}_N)$ and $F(\underline{p}+\underline{v};\underline{W}_N)$ are not on the same side of some indifference surface, then the manipulation, $\underline{v}$, affected the outcome. Whether this effect is successful or counterproductive depends on the sincere rankings of the manipulators. To start the analysis, I will formalize my earlier assumptions.

Assume that

1. any $\underline{p}$ in Si(N!) is equally likely;
2. each pair of alternatives is equally likely to be the target of an attempted manipulation.

To determine the "success" of an attempt to manipulate, I follow the lead of the Nash equilibrium by analyzing the situation where a strategic voter, (or a small group of strategic voters) tries to reverse the relative ranking of a particular pair while all other voters vote sincerely. Assume without loss of generality that the strategic voter(s) attempt to influence the relative election rankings of a_1 and a_2 in the direction $a_1 \rangle a_2$. Obviously, this means that the voters have the relative ranking $a_1 \rangle a_2$.

To motivate the definition of $\underline{v}$, let N=3 and let the voter types be labelled as:

Type	Ranking	Type	Ranking
1	$a_1 > a_2 > a_3$	6	$a_2 > a_1 > a_3$
2	$a_1 > a_3 > a_2$	5	$a_2 > a_3 > a_1$
3	$a_3 > a_1 > a_2$	4	$a_3 > a_2 > a_1$

A strategic voter is of type 1,2, or 3. The most successful way this voter can manipulate the system is to mark the ballot as a type 2 voter, and a type 2 voter votes sincerely. Any other choice either isn't strategically maximal, or it is counterproductive. But, when a type 1 voter pretends to be type 2, the profile changes. For instance, suppose for 20 voters that the sincere profile is (6/20,3/20,2/20,2/20,3/20,4/20). With the stategic vote, the new profile is (5/20,4/20,2/20,2/20,3/20,4/20), so the change due to the attempted manipulation is 1/20 times (-1,1,0,0,0,0). Likewise, a change of a type 3 voter to a type 2 voter is given by a multiple (1/m) of (0,1,-1,0,0,0) where m is the total number of voters. Following our theme of neutrality, assume that:

3. it is equally likely for a manipulating voter to be of any type; such a voter assumes a strategy (voter type) to maximize the effect of the manipulation. If there are several maximal strategies, the voter selects the one most consistent with the voter's actual type. (Consistency is understood to imply that a maximal number of the relative rankings of pairs is preserved.)

The second part of this assumption has meaning only when $N \geq 4$.

Example. If a manipulating voter is of the type $a_1 > a_2 > a_3 > a_4$, then a maximal strategy would be to assume either the type $a_1 > a_3 > a_4 > a_2$ or $a_1 > a_4 > a_3 > a_2$. Of these two strategies, the first one is more consistent with the voter's true type.

The first part of Assumption 3 means that for N=3, it is equally likely for the manipulating voter to be of type 1 or type 2. Therefore, the <u>expected</u> <u>manipulation</u> is a scalar multiple of (-1/2,1,-1/2,0,0,0,). In general, $\underline{v}$, the <u>expected</u> <u>manipulation</u> <u>vector</u> (EMV), is determined in this manner by Assumptions 1-3. The scalar multiple of the EMV depends on whether there is one manipulating voter or a small group of manipulating voters, and on the total number of voters. For instance, with 20 voters, the multiple is q/20 where q is the number of manipulating voters. Everything that follows holds if the magnitude of the EMV, $|\underline{v}|$, is sufficiently small. Consequently, these results hold for a single voter in a sufficiently large population (say, N>20), or for a small coalition within a large group.

Definition. Let m>2 and an EMV $\underline{v}$ be given. Define $\mu(\underline{W}_N, m)$, the m voter measure of (binary) susceptibility of $\underline{W}_N$, to be the number of $\underline{p}$'s in Si(N!) with a common denominator m such that $F(\underline{p}+\underline{v}, \underline{W}_N)$ has the (relative) ranking $a_1 > a_2$ while $F(\underline{p}, \underline{W}_N)$ has the ranking $a_2 \geq a_1$.

This measure indicates, for m voters, what fraction of all possible profiles admit a situation where the relative ranking of a_1 and a_2 could be alterred. Clearly, a smaller measure of susceptibility means there are fewer opportunities to successfully manipulate the outcome, so an optimal system, $\underline{W}_N$, minimizes this measure. Unfortunately, each m admits a continuum of optimal choices of $\underline{W}_N$'s, and this continuum of answers changes with the value of m. But, this is what we should expect; if two choices of $\underline{W}_N$ are essentially the same, then the election outcomes always agree if there are only a few voters. For instance, if m<10, then an election tallied with (1,0,0) always agrees with the outcome when tallied with (1,e,0), $e \leq 0.1$. As a result, the "optimal choice for m" includes several vectors $\underline{W}_N$ where the election outcomes always are the same for m voter profiles. Distinctions in the election outcomes for these systems do arise when the number of voters increases; this means that some $\underline{W}_N$'s no longer are optimal.

This dependence on the number of voters, m, can be removed by keeping only those $\underline{W}_N$ that are optimal for all sufficiently large values of m. This motivates the next definition.

Definition. A voting system $\underline{W}_N'$ is susceptibility efficient if

2.2 $\quad \mu(\underline{W}_N', m) \leq \mu(\underline{W}_N, m)$

for all choices of $\underline{W}_N$ and for all sufficiently large values of m. The voting system $\underline{W}_N'$ is susceptibility inefficient if the inequality 2.2 is reversed.

A straightforward argument proves that $\underline{W}_N$ is susceptibility efficient if $\underline{W}^N$ is the minimum point for the next measure.

Definition. Let the EMV $\underline{v}$ be such that $|\underline{v}|$ is sufficiently small. Define $\mu(\underline{W}_N)$, the measure of (binary) susceptibility of $\underline{W}_N$, to be the volume of $\{\underline{p}$'s in Si(N!)$\mid F(\underline{p}+\underline{v}, \underline{W}_N)$ gives the (relative) ranking $a_1 > a_2$ while $F(\underline{p}, \underline{W}_N)$ corresponds to the ranking $a_2 \geq a_1\}$.

Theorem 1. For all choices of $N \geq 3$, the systems $(1,0,0,..,0)$ and $(1,1,..,1,0)$ are susceptibility inefficient and $\mu((1,0,..0)) = \mu((1,1,..,1,0))$.

For $N=3$, the unique susceptibility efficient system (USES) is the BC. For $N=4,5$, the USES are, respectively, $(2,2,0,0)$ and $(2,2,1,0,0)$. Also, the following ratios hold:

2.3 $\mu((1,0,0))/\mu(\underline{B}_3) = 1.027$
$\mu((1,0,0,0))/\mu((1,1,0,0) = 4.6064$
$\mu((2,1,1,0))/\mu((1,1,0,0) = 3.358$
$\mu(\underline{B}_4)/\mu((1,1,0,0)) = 2.014$
$\mu((1,0,0,0,0))/\mu((2,2,1,0,0) = 2254.003$
$\mu(\underline{B}_5)/\mu((2,2,1,0,0)) = 13.914$

Theorem 1 can be extended to all values of N by using Eq. 3.23; this equation gives the measure of susceptibility for any $\underline{W}_N$.

The ratios in Eq. 2.3 compare the vulnerability of the different systems. I found it surprising that the ratios are so close to unity if N=3. This suggest that manipulability may not be a significant distinguishing factor among positional voting systems for N=3, so other criteria about the various systems may dictate the final choice of the voting system. On the other hand, our results for N=4,5 serve as a strong argument against using (1,0,..,0) or (1,..,1,0). Here, the BC doesn't fare too badly, so it might be adopted should it satisfy other criteria; e.g., see [10,11,14].

The fact that $\mu((1,0,..,0)) = \mu((1,1,..,1,0))$ reflects an important symmetry admitted by voting systems; a symmetry that plays a key role in the proofs. To describe the symmetry, R will denote an involution, or a "reversing mapping" in 3 different settings. First, let R be a map that reverses the ranking of a profile. For instance, for N=3, if $\underline{p}$ is of type 1, then $R(\underline{p})$ is of type 4; if $\underline{p}$ is of type 2, then $R(\underline{p})$ is of type 5; etc. Next, let R be the mapping that reverses a ranking. For instance, $R(a_2 \succ a_1 \succ a_3) = a_3 \succ a_1 \succ a_2$. With these related definitions of R, it is obvious that

2.4 $F(R(\underline{p}),\underline{W}_N) = R(F(\underline{p},\underline{W}_N))$.

The next use of R is to reverse a voting vector. This takes some explanation. Let $\underline{E}_N = (1,1,..,1)$. Such a vector isn't a voting vector because it doesn't distinquish among the candidates. However, if $a>0$, then an election tallied with

$\underline{W}_N$ and with $a\underline{W}_N + b\underline{E}_N$ always agree, so we can identify all such vectors. Define $R(\underline{W}_N)$ to be the class of voting vectors $-a\underline{W}'_N + b\underline{E}_N$ where $\underline{W}'_N$ is the vector $(w_N, w_{N-1}, .., w_1)$. For example, $R((1,0,0)) = (1,1,0)$ because $(1,1,0) = -(1,0,0)' + (1,1,1) = -(0,0,1) + (1,1,1)$. These three involution operators are related.

Proposition. For $N \geq 3$, $R(F(\underline{p}, \underline{W}_N)) = F(R(\underline{p}), R(\underline{W}_N))$.

Proof. The ranking given by $R(F(\underline{p}, \underline{W}_N))$ always agrees with the ranking given by $b\underline{E}_N - F(\underline{p}, \underline{W}_N) = b\underline{E}_N + F(\underline{p}, -\underline{W}_N)$ (even though $-\underline{W}_N$ isn't a voting vector, this can be computed) $= F(R(\underline{p}), R(\underline{W}_N))$.

Corollary. For any $N \geq 3$, $\mu(\underline{W}_N) = \mu(R(\underline{W}_N))$.

This corollary generalizes the assertion that $\mu((1,0,..,0) = \mu((1,..,1,0)$.

Theorem 1 concerns the social welfare ranking of some two alternatives, but they need not be the two top ranked ones. What happens if we restrict attention only to those profiles of voters where a_1 and a_2 are contesting the top position? Because of symmetry considerations, the answer remains the same.

Theorem 2. The systems that least susceptible to a binary manipulation of the top two ranked alternatives are $\underline{B}_3$ for N=3, (1,1,0,0) for N=4, and (2,2,1,0,0) for N=5.

Theorems 1 and 2 are based on certain neutrality assumptions motivated by the requirements of an institutional mechanism. These conclusions need not hold for a "one-shot" decision process where we have additional information about the organization. This extra information concerning the profiles changes the definition of the EMV, $\underline{v}$. For instance, suppose it is <u>not</u> equally likely for a strategic voter to be of type 3 or of type 1. Instead, suppose with probability c a strategic voter is of the first type and with probablity 1-c the voter is of the third type. Then the EMV, $\underline{v}_c$, is a multiple of $(-c,1,-(1-c);0,0,0)$ rather than $(-1/2,1,-1/2;0,0,0)$. The choice of the optimal voting method now requires finding the $\underline{W}_N$ that minimizes the volume of $\{\underline{p}$'s in $Si(N!) \mid F(\underline{p}+\underline{v}_c, \underline{W}_N)$ gives the (relative) ranking $a_1 > a_2$

while $F(\underline{p},\underline{W}_N)$ corresponds to the ranking $a_2 \geq a_1$}. The answer depends upon the choice of c. In what follows, let c denote the <u>organization's</u> <u>manipulation</u> <u>characteristic</u>.

Theorem 3. Let $\underline{W}_3$ be given. There exists a value for an organization's manipulation characteristic, c, so that $\underline{W}_3$ is the USES.

This statement extends to all values of N as well as to multiple systems such as approval or cumulative voting.

Definition [12]. A <u>simple</u> voting system for N alternatives is where all ballots are tallied with a specified $\underline{W}_N$. A <u>multiple</u> <u>voting</u> <u>system</u> for N alternatives is where
1) there is a specified set of at least two voting vectors, $\{\underline{W}^j{}_N\}$, where the difference between any two of them is not a scalar multiple of (1,1,..,1), and
2) each voter selects any one of the voting vectors to tally his ballot.

Examples. The set of vectors for <u>cumulative</u> <u>voting</u>, as used in Illinois [15,12], is {(3,0,..0), (2,1,0,..,0), (3/2,3/2,0,..,0), (1,1,1,0,..,0)}

The set of vectors for <u>approval</u> <u>voting</u> is {(1,0,...0), (1,1,0,..,0), ...,(1,..,1,0)}.

The simple system defined by $\underline{W}_N$ admits a <u>truncated</u> <u>ballot</u> if alternative voting vectors are admitted to tally ballots where not all candidates are listed and if $\underline{W}_N$ and the alternative vectors define a multiple system. For instance, if $\underline{B}_N$ is used to tally the ballots, each voter needs to rank of all N candidates. To tally a ballot where only k of the candidates are ranked, suppose $\underline{B}_k$ is used. (Alternatively, the vector $\underline{B}_{k+1}$ or (N-1,..,N-(k+1),0,..,)) could be used.) This defines a multiple system.

Our next theorem asserts that multiple systems are more susceptible to manipulation than any of the component simple systems that define it. This conclusion is reasonable; after all, each simple system provides certain strategies and opportunites to manipulate the system. We should suspect that a multiple system makes available all of these opportunities and strategies. Thus, because there are added opportunities to successfully change the outcome of an election, a multiple system should be more susceptible to manipulation. The actual argument is more complicated, but this intuition serves us well.

Theorem 4. Let $N \geq 3$. If $\underline{W}_N$ is one of the voting vectors from a specified multiple voting system, then the multiple system is more susceptible to binary manipulation than the system given by $\underline{W}_N$.

Corollary. For any $N \geq 3$, both cumulative and approval voting are more susceptible to binary manipulation than plurality voting. Therefore, both cumulative and approval voting are more susceptible to binary manipulation than any simple voting system. A system that admits truncated ballots is more susceptible to binary manipulation than the original system.

Niemi [7] contends that when certain assumptions are relaxed, approval voting begs to be manipulated. We're using different arguments and a different approach, but Theorem 4 and the corollary show that this is true not only for approval voting, but also for <u>any</u> multiple system admitting multiples of (1,0,..0) or (1,..,1,0). Consequently, this is true for cumulative voting, cardinal voting, and a simple system that admits truncated ballots. On the surface, these statements appear to contradict the many results supporting approval voting. They don't because the conclusions supporting approval voting are based on particular organizational assumptions such as dicotomous preferences; therefore, the differences in conclusions are partially explained by Theorem 3. If fact, it follows from Theorem 1 that approval voting has to be more susceptible to manipulation than many other multiple systems.

There is an interesting difference between the manipulation of a multiple and a simple system. In a simple system, a strategic voter has to misrepresent his rankings of the alternatives. In a multiple system, a strategic voter can do this, or he can remain true to his ranking while strategically choosing one of the other admitted tallying methods. In other words, multiple methods not only provide additional strategies to manipulate the system, but these added strategies are even sanctioned! (For an analysis of truncated ballots that captures some of this spirit, see Brams and Fishburn [2]. For an analysis of the "completely indeterminate" properties of multiple systems, see Saari and Van Newenhizen [12].)

My emphasis has been on manipulating the relative ranking of a pair of candidates. There are other interesting kinds of manipulation that I haven't considered. For instance, a voter may wish to manipulate the outcomes of N-1 of the N candiates, or to manipulate the outcome of some subset. A more interesting problem is to determine which $\underline{W}_N$ minimizes the probability of a voter successfully manipulating the outcome of <u>any</u> subset of candidates. While I haven't completely analyzed this question, the symmetry considerations that are crop up in Section 3 and elsewhere [10] lead me to suspect that the optimal answer for this more general problem either is $\underline{B}_N$, the BC, or it approaches this vector for larger values of N.

Symmetry considerations also provide insight into why my conclusions appear to differ from those in [3,8]. For instance, with my measure and with N=3, I find that the BC is the best, but Chamberlain [3] finds that when the BC is compared with the plurality vote, it isn't the best. This isn't overly surprising because we use different measures, and so we should expect different conclusions. But much more is involved. An important variable seems to be the number and kinds of paradoxes a voting system admits. Common sense suggests that the more paradoxes, or unexpected outcomes there are, the more opportunites a stategic voter has to direct the final outcome. This suggests we should determine which systems admit the fewest paradoxes. As shown in [10,11], the unique answer is the BC, and this is compatible with our conclusions.

There is a way to understand why the different measures can give radically different conclusions. A profile is identified with an election outcome, and, conversely, each election outcome defines a set of profiles. But, how are these profiles arranged? The geometry of these regions is developed in [10] and partly described in Section 3. Rather than giving a technical description, I will outline the basic conclusions by using an analogy with the geometry of rectangles. Of all the rectangles with the same perimeter, the square has the maximum volume, while the degenerate rectangle has the minimum volume of zero. Of all rectangles with the same volume, the square has the minimum perimeter while the degenerate rectangle has the largest. The "minimum configuration" changes depending on whether the measure to be minimized is volume or perimeter length. In both cases, the maximum and minimum configuration is either the rectangle with the most symmetry - the square, or the rectangle with the least symmetry, - the degenerate one.

A similar situation occurs with the geometry of the distribution of profiles. Should a measure of manipulation allow <u>any</u> coalition to manipulate the results, the emphasis is on the "volume" of the distribution of profiles. This is related to some of the measures used in [3]. By analogy with the rectangles, we might expect the more symmetrical regions to fare poorly with this measure, while the least symmetrical fare better. This happens. On the other hand, by considering small groups of manipulating voters, as I do, the emphasis is on how many profiles are near the boundary between election rankings. So, my measure involves the "surface area" or "the perimeter" of the region. Here we might expect the more symmetrical regions to fare better, and they do.

The symmetry of the regions of profiles is determined by the symmetry properties of the voting vector $\underline{W}_N$. Because only pairs of alternatives are analyzed, this symmetry is partly captured by the operator(s) R. For example, the BC has "symmetrical regions" because $\underline{B}_3$ is the only vector satisfying $R(\underline{W}_3)=\underline{W}_3$. In general, for each N, the optimal $\underline{W}_N$ satisfies this symmetry condition. On the other hand, if $\underline{W}(u) = (1,u,0)$, then the "least symmetrical" choice is given by the extreme values for u; these values define the voting vectors (1,0,0) and (1,1,0). This again supports our conclusion.

3. Proofs

The number of subscripts needed for the proofs grows rapidly with the value of N, and this proliferation of notation tends to obscure the basic ideas. Therefore, I first prove the theorems for N=3 before giving the general proof.

Assume that the election for $\{a_1,a_2,a_3\}$ is tallied with $\underline{W}_3=(w_1,w_2,w_3)$. Using the notation introduced in Section 2, the tally for a_1 is $p_1w_1 + p_2w_1 + p_3w_2 + p_4w_3 + p_5w_3 + p_6w_2$ while that for a_2 is $p_1w_2 + p_2w_3 + p_3w_3 + p_4w_2 + p_5w_1 + p_6w_1$. Thus, the set of points, $\underline{p}$, leading to a tie vote between a_1 and a_2 is

3.1 $\langle \underline{E}_6, \underline{p} \rangle = 1$,

3.2 $\langle \underline{N}, \underline{p} \rangle = 0$,

where $\underline{N} = (w_1-w_2, w_1-w_3, w_2-w_3, w_3-w_2, w_3-w_1, w_2-w_1)$, $\underline{E}_6 = (1,..,1)$, and $\langle -,- \rangle$ is the usual Euclidean scalar product.

The election rankings remain the same whether a profile is tallied with $a\underline{W}_3+b\underline{E}_3$, $a>0$, or with $\underline{W}_3$. So, choose a and b so that the new vector is $(1,u_1,-1)$ where $-1 \le u_1 \le 1$. With this notation, the BC corresponds to $u_1=0$, the plurality vote corresponds to $u_1=-1$, and $(1,1,0)$ corresponds to $u_1=1$. Also,

3.3 $\underline{N} = (1-u_1, 2, u_1+1, -(1+u_1), -2, -(1-u_1))$.

It follows from the symmetry proposition that the answers are the same for $(1,u_1,-1)$ and $(1,-u_1,-1)$, so we can and do assume that u_1 is in the interval $[0,1]$.

The goal is to find the volume of the $\underline{p}$'s that are close enough to the hyperplane defined by Eqs. 3.1 and 3.2 so that $\underline{p}+\underline{v}$ will cross the hyperplane. Here, $\underline{v}$ is the appropriately small positive multiple of $(-1/2,1,-1/2,0,0,0)$. I'm only interested in the ratios of maeasures, so I will supress this and all other common multiples. If a precise formula for a measure is needed, these multiples need to be restored.) This is the volume defined by the hyperplane and the length of the component of $\underline{v}$ that is orthogonal to the hyperplane. So, for any choice of u_1, the measure is given by

3.4a $\mu(\underline{W}_3) = \mu(u_1)$ is a common scalar multiple of $\langle \underline{N}/|\underline{N}|, \underline{v} \rangle$(surface vol. of the hyperplane defined by Eqs. 3.1, 3.2) where $|\underline{N}|$ is the length of $\underline{N}$.

It is a simple computation to show that

3.4b $\langle \underline{N}/|\underline{N}|, \underline{v} \rangle = 3/2(3+u_1^2)^{1/2}$.

(When more complicated incentive or susceptibillity issues are investigated, such as assuming that the number of strategic voters varies with types and with the number of each type, this same formulation holds. The main difference is that Eq. 3.4b now becomes a function of $\underline{p}$. This means that the formulation, Eq 3.4a, can be identified with a higher dimensional "flux" or "fluid flow" problem where the flow is defined by Eq. 3.4b.)

The hypersurface in Eq. 3.4a is four dimensional surface in six dimensional space. To find the surface volume, we use two different parametric representations to reduce the problem to an integration problem over a region in a four dimensional space. The first change of variables is obtained by solving Eq. 3.1 for p_5.

3.5 $p_1=y_1,\ p_2=y_2,\ p_3=y_3,\ p_4=y_4,\ p_6=y_5,\ p_5=1-\sum y_j.$

The common integrating factor, $6^{1/2}$, for this change of variables is independent of u_1, and so we supress it. The new domain is

3.6 $y_j\geq 0$, $j=1,..,5$, and $\langle \underline{E}_5,\underline{y}\rangle\leq 1$.

The equation $\langle \underline{N},\underline{p}\rangle=0$ becomes

3.7 $\langle \underline{N}',\underline{y}\rangle = 2,$

where $\underline{N}'=(3-u_1,4,u_1+3,1-u_1,1+u_1)$.

The last parametric representation is obtained by solving Eq. 3.7 for y_2, this is the other variable with a coefficient independent of u_1.

3.8 $y_1=x_1$, $y_3=x_2$, $y_4=x_3$, $y_5=x_4$, and y_2 is obtained from Eq. 3.7 and the x_j's.

For this change of variables, the integrating factor is $\{4^2+(3-u_1)^2+(3+u_1)^2+(1-u_1)^2+(1+u_1)^2\}^{1/2}$, which is a scalar multiple of

3.9 $\{9 + u_1^2\}^{1/2}$.

Using this change of variables with $\langle \underline{E}_5,\underline{y}\rangle\leq 1$ and with the constraint $y_2\geq 0$, the geometry of the domain is given by

3.10 $\langle \underline{N}_j,\underline{x}\rangle \leq 2$, $j=1,2$, and $x_j\geq 0$

where $\underline{N}_1=(1+u_1,1-u_1,3+u_1,3-u_1)$ and $\underline{N}_2=(3-u_1,u_1+3,1-u_1,1+u_1)$.

There is a symmetry between $\underline{N}_1$ and $\underline{N}_2$; each is a permutation of the other.

(This results from the $a_1 > a_2$ and $a_2 > a_1$ symmetry.) But their symmetry relationship is even stronger. If we allow u_1 to range over its natural domain of [-1,1], rather than [0,1], then the pairs (x_1,x_4) and (x_2,x_3) exchange roles and coefficients as u_1 changes sign. (This can be used to prove the proposition.)

The variables $\underline{x}$ only define the domain, so the volume is determined by elementary geometric arguments. Let $\underline{e}_j$ be the unit vector with unity in the j^{TH} component. Then the four vertices defined by the bounding hyperplane $\langle \underline{N}_1, \underline{x} \rangle = 2$ are $2\underline{e}_1/(1+u_1)$, $2\underline{e}_2/(1-u_1)$, $2\underline{e}_3/(3+u_1)$, $2\underline{e}_4/(3-u_1)$, and the region defined by $\langle \underline{N}_1, \underline{x} \rangle \leq 2$ is defined by these four points and $\underline{0}$. Correspondingly, the region defined by $\langle \underline{N}_2, \underline{x} \rangle \leq 2$ is given by the five points $\underline{0}$, $2\underline{e}_1/(3-u_1)$, $2\underline{e}_2/(3+u_1)$, $2\underline{e}_3/(1-u_1)$, $2\underline{e}_4/(1+u_1)$. The feasible region, or the domain, is given by the intersection of these two regions. By symmetry (of $\underline{N}_1$ and $\underline{N}_2$) this defines two congruent regions where one of them is defined by the five points $\underline{0}$, $2\underline{e}_1/(3-u_1)$, $2\underline{e}_2/(3+u_1)$, (1/2,0,0,1/2), and (0,1/2,1/2,0). Thus, the total volume is twice the volume of each of these congruent regions. The volume of this object is a fixed scalar multiple of $1/(3-u_1)(3+u_1) = 1/(9-u_1^2)$.

From Eqs. 3.9, 3.4 and the above, it follows that

3.11 $$\mu(u_1) = \{SF/(9-u_1^2)\}\{(9+u_1^2)/(3+u_1^2)\}^{1/2}$$

where SF is the "scalar factor" determined by the suppressed common constants. Obviously, this function has its minimum at $u_1=0$ - the Borda Count - and its maximum at $|u_1|=1$ - at (1,0,0) and (1,1,0). This proves the first part of Theorem 1 for N=3. The second part is obtained by computing $\mu(0)/\mu(1) = \mu(0)/\mu(-1)$.

To prove Theorem 2 for N=3, Eq. 3.4a is integrated over the region corresponding to $a_1=a_2>a_3$. Because of the symmetry, the surface volume corresponding to $a_1=a_2>a_3$ equals that corresponding to $a_3>a_1=a_2$. Therefore, the value of the new integral is 1/2 the volume computed in Eq. 3.1. This means the measure of susceptibility is half that for the general ranking. Consequently, the optimal answer of the Borda Count and the worst choices of

(1,0,0), (1,1,0) remain.

To prove Theorem 3, the only difference is that $\underline{v}$ is replaced with $\underline{v}_c$ = $(-c,1,-(1-c),0,0,0)$, so $\langle \underline{N},\underline{v}_c\rangle = 2 + c(1-u_1) + (1-c)(1+u_1)$. Therefore, $\mu_c(u_1)$ becomes $\{SF\langle \underline{N},\underline{v}_c\rangle/(9-u_1^2)\}\{(9+u_1^2)/(3+u_1^2)\}^{1/2}$. The minimum point for c=1 is the boundary value $u_1=1$, for c=1/2 it is $u_1=0$, and for c=0 it is the other boundary point $u_1=-1$. Furthermore, the minimum point, u_1, is a continuous function of c. Therefore, it follows from the intermediate value theorem that any value of u_1 is the optimal choice for some choice of c in [0,1]. (The extension of this result for larger values of N requires introducing additional variables to describe the strategic behavior and/or the numbers of each type of voter. For multiple voting systems, we need still more variables to describe how a voter chooses among the voting vectors.)

For $N\geq 3$ alternatives $\{a_1,..,a_N\}$, normalize the voting vector so that $\underline{W}_N=(1,u_1,u_2,..,u_{N-2},-1)$ where $u_j\geq u_{j+1}$, $1\geq u_1$, and $u_{N-2}\geq -1$. The first condition on the $\underline{p}$'s corresponds to Eq. 3.1 and it asserts that the domain is the simplex.

3.12 $\langle \underline{E}_N ,\underline{p}\rangle = 1.$

Eq. 3.2 is determined by by the difference between the tallies for a_1 and a_2. So, if p_j corresponds to the j^{TH} type of voter, j=1,..,N!, then the coefficient for p_j is determined by how this voter ranks a_1 and a_2. The coefficient equals the difference between the weights assigned to a_1 and to a_2. (See the definition of $\underline{N}$ in terms of the w_j's.) The following vector lists all of these possibilities where a_1 is ranked above a_2. The first series is where a_1 is top ranked, the second is where a_1 is second ranked, etc.

3.13 $\underline{M} = (1-u_1,1-u_2,..1-(-1),u_1-u_2,..,u_{N-2}-(-1))$

If a_1 and a_2 are interchanged in a ranking, then the coefficient changes sign. Therefore, $-\underline{M}$ captures all of the possibilities for $a_2>a_1$. For each fixed ranking of a_1 and a_2, there are (N-2)! rankings of the remaining (N-2) alternatives. This means that with an appropriate labeling of the types of voters, Eq 3.2 is replaced with

3.14 $\langle \underline{N},\underline{p}\rangle = 0$ where $\underline{N}=(\underline{M},..,\underline{M},-\underline{M},..,-\underline{M})$.

(Both $\underline{M}$ and $-\underline{M}$ are repeated $(N-2)!$ times.)

To determine $\underline{v}$, note that there are $N(N-1)/2$ ways to rank the a[illegible]ves a_1 and a_2. For each ranking, there are $(N-2)!$ ways to rank the remaining alternatives. According to the consistency assumption (3), a strategic voter selects a voter type where the relative rankings of $\{a_3,..,a_N\}$ is sincere, but where a_1 is ranked first and a_2 is ranked last. This means the expected change in $\underline{p}$ due to a strategic voter is a scalar multiple of $\underline{v}=(-1,..,-1,(N(N-1)/2)-1,-1,..,-1)$ where the one positive value is in the same component as the value 2 in $\underline{M}$. By the neutrality assumption, the EMV is a scalar multiple of $\underline{V}=(\underline{v},..,\underline{v},\underline{0},..,\underline{0})$.

The generalization of Eq. 3.4b is now immediate.

3.15 $\langle \underline{N}/|\underline{N}|,\underline{V}\rangle$ is a scalar multiple of
$$\{(N-2)(N-1)-\sum_j u_j(N-2j-1)\}/\{4+2[N-2+\sum_j u_j^2]+(\sum_{j<k}(u_j-u_k)^2\}^{1/2}$$

Eqs. 3.12, 3.14 define a $N!-2$ dimensional surface in a $N!$ dimensional space. To find the value of Eq 3.4a, two parametric representations are used to reduce the problem to an integration over a domain in a $N!-2$ dimensional space. The first parametric representation is much the same as Eq. 3.5 where we solve for a p_j that has coefficient -2. (Thus, a_2 is top ranked and a_1 is bottom ranked.) The condition that $p_j \geq 0$ defines the extension of Eq. 3.6.

3.16 $\langle \underline{E}_{N-1},\underline{y}\rangle \leq 1$.

The integrating factor from this change of variables depends only on the number of alternatives, so it is supressed.

In $\langle \underline{N},\underline{p}\rangle=0$, $-2p_j = -2(1-\langle \underline{E}_{N-1},\underline{y}\rangle)$, so the generalization of Eq. 3.7 becomes

3.17 $\langle \underline{N}',\underline{y}\rangle = 2$ where $\underline{N}' = 2\underline{E}_{N-1}+\underline{N}''$ and where $\underline{N}''$ is obtained from $\underline{N}$ by dropping the coordinate direction corresponding to p_j.

The last parametric representation is determined from Eq. 3.17. In the same way the transformation Eq. 3.8 was determined, use the y_K term that has a_1 top ranked, a_2 bottom ranked, and the remaining alternatives ranked in the same way as given by the type for p_J. Solve for y_K; for this change of variables, the integrating factor is a scalar multiple of

3.18 $(2(N-2)(4(N-3)!+N+2)+2\sum_j u_j^2+\sum_{j<k}(u_j-u_K)^2)^{1/2}$

The constraint the $y_K \geq 0$ provides one of the extensions of Eq. 3.10, or

3.19 $\langle \underline{N}_2, \underline{x} \rangle \leq 2$.

Because this equation results from Eq. 3.17, it is immediate that $\underline{N}_2$ is obtained from $\underline{N}'$ by dropping the component corresponding to y_K. Now, either by direct algebraic substitution, or by symmetry arguments based on interchanging the order in which these two transformations are obtained, it follows that the extension of Eq. 3.16 is

3.20 $\langle \underline{N}_1, \underline{x} \rangle \leq 2$

where $\underline{N}_1$ is obtained from $\underline{N}_2$ by directly interchanging the first and the last $(N!-2)/2$ components.

It remains to compute the region defined by the volume of the region defined by $x_K \geq 0$ and Eqs. 3.19, 3.20. The regions defined by Eq. 3.19 and Eq. 3.20 are each defined by $\underline{0}$ and the $N!-2$ vertices resulting from $\langle \underline{N}_J, \underline{x} \rangle = 2$. The intersection gives two geometrically congruent regions where the appropriate vertex on a coordinate axis is the smaller of the two values computed above. This means that each of these regions has $\underline{0}$ and $(N!-2)/2$ defining points on certain coordinate axes. The remaining $(N!-2)/2$ defining points are given by the intersection of the two surfaces, and, by symmetry, they are points where two symmetric coordinates are 1/2 and all others are zero. (The symmetric coordinates are the ones transferred into each other in the construction of $\underline{N}_1$ and $\underline{N}_2$.) The important fact is that these intersection points are independent of the choice of $\underline{W}_N$. Therefore, the volume of each region is a common scalar multiple of the volume defined by the points on the coordinate axes. This is a scalar multiple of

3.21 $$1/\{\prod(9-u_j^2)\prod_{j<k}(u_j-u_k+3)\}^{(N-2)!}$$

Therefore, the measure of binary susceptibility is given by

3.22 $$u(\underline{W}_N) = \frac{SF\{(N-2)(N-1)-\sum u_j(N-2j-1)\}\{2(N-2)(4(N-3)!+N+2)+2\sum u_j^2+\sum_{j<k}(u_j-u_k)^2\}^{1/2}}{\{\prod(9-u_j^2)\prod_{j<k}(u_j-u_k+3)\}^{(N-2)!}\{2[N+\sum u_j^2]+(\sum(u_j-u_k)^2\}^{1/2}}$$

or

3.23 $$DSF\{(N-2)(N-1)-\sum u_j(N-2j-1)\}/\{\prod(9-u_j^2)\prod_{j<k}(u_j-u_k+3)\}^{(N-2)!}$$

where $D = \{-1+[[2(N-2)(4(N-3)!+N+2)-2N]/[2N+2\sum u_j^2+\sum_{j<k}(u_j-u_k)^2]]\}^{1/2}$. From this, the rest of Theorem 1 follows by use of elementary (but messy) calculus techniques. In determining the minimum values by gradient techniques, it follows from the proposition that we can assume $u_i \geq 0$. Some of the minimum points occur on the boundary $|u_j|=1$. This happens for N=4 where $u_1=-u_2=1$, or $\underline{W}_4=(1,1,0,0)$. By use of the symmetry, it follows for odd values of N that at a critical point, the middle term u_j, $j=(N-1)/2$, always equals zero. This is a consequence of the more general statement $u_j=-u_{N-j-1}$. For instance, N=5 involves both facts; $u_1=-u_3=1$, $u_2=0$, so $\underline{W}_5=(2,2,1,0,0)$. For $N \geq 6$, the critical point begins to involve interior values of u_j. In particular, when N=6, u_2 is a positive irrational number. Therefore we get $(1,1,u_2,-u_2,-1,-1)$ or $\underline{W}_6=(2,2,1+u_2,1-u_2,0,0)$. The value specified in the introductory section is a rational approximation of this vector.

For larger values of N, the term D in Eq. 3.23 plays only a minor role in determining the optimal point. It follows from the symmetry of the remaining terms that for all j, u_j-u_{j+1} tends to a fixed value as the value of N becomes larger. (The boundary conditions complicates the issue somewhat.) This is, of course, the Borda Count. This completes the proof of Theorem 1.

To complete the proof of Theorem 2, note that the integration of the generalized form of Eq. 3.4a is over the region $a_1=a_2$ where all other alternatives are ranked below these two alternatives. On the hypersurface $a_1=a_2$, there are $(N-1)!$ different rankings (without admitting ties between any other alternatives) and, by symmetry, each defines the same surface volume. Of these $(N-1)!$ regions, $(N-2)!$ of them have $a_1=a_2$ top ranked. Therefore the problem concerning the manipulation of the top two ranked alternatives gives a surface volume (and hence a susceptibility measure) $1/(N-1)$ the size of the original one. Thus, all relative results remain the same. (By ignoring regions where there are still other indifferences, I'm ignoring a finite number of regions of lower dimension, and hence a finite number of regions of zero $(N!-2)$ dimensional volume. Thus, this doesn't effect the answer.)

To prove Theorem 4, we need a different space. Assume a multiple system is given by the voting vectors $(\underline{W}_N^1,..,\underline{W}_N^K)$. That is, each voter has K ways to have the ballot tallied. Let q_j^s, s=1,..,k, be the fraction of voters of the j^{TH} type that choose to have the ballot tallied with $\underline{W}_N^s$. This defines a point $\underline{q}_j$ in Si(k). Therefore, the point $\underline{p}=(p_1,..,p_N)$ is replaced by $\underline{q}=(p_1\underline{q}_1,...,p_N\underline{q}_N)$. That is, $p_jq_j^s$ describes the fraction of all voters that are of the J^{TH} type and that choose $\underline{W}_N^s$. This defines a vector of dimension k(N!). So, this can be viewed as being a fiber space $Si(N!)x(Si(k)^N$ where the fiber describes how each type of voters split in their choice of tallying methods.

The indifference surface between a_1 and a_2 is a hyperplane in this fiber space. In describing this equation, the coefficient for the $p_jq_j^s$ variable is the difference between the weights assigned to a_1 and a_2 (as determined by the j^{TH} type of voter) by $\underline{W}_N^s$. The measure of susceptibility is given by the number of profiles that are close enough to this indifference surface to effect the outcome with a scalar multiple of $\underline{v}$. Thus, this is given by the projection of this hyperplane in the fiber space into the base space Si(N!). Trivially, this projection contains as <u>a proper subset</u> the indifference surface in Si(N!) defined by the simple system $\underline{W}_N^s$. As a result, it follows that Theorem 4 is true.

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