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OPTIMALITY OF A GENERAL AD VALOREM TAX

by

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Abstract

Optimality of a general *ad valorem* tax is rigorously demonstrated under various assumptions which are in part familiar from the literature. Due to lack of convexity and compactness, calculus and first-order conditions are never used. The proofs rely on the concept of a revenue function as well as on techniques from majorization theory.

1. Introduction

The possible (second-best) optimality of a uniform commodity tax system in a one-consumer model has been investigated many times in the literature. Ramsey (1927) pointed out that uniform taxation generally will not be optimal. Atkinson and Stiglitz (1972), Sandmo (1974) and Sadka (1977) considered restrictions on preferences leading to a uniform tax being a solution of the first-order conditions. As emphasized by Sadka, these results do not allow the conclusion that uniform taxation be optimal in these cases. This is due to two basic difficulties in the literature on optimal taxation: (i) lack of convexity and (ii) lack of compactness. Whereas (i) has been frequently noticed, (ii) seems to have been overlooked.

Problem (i) is summarized by Deaton (1981, p. 1250) in the introduction to his analysis:

> "...we make no attempt to go beyond the standard first-order conditions which are familiar in the literature. As has been emphasized by Mirrlees ... these conditions are not only not sufficient for a tax optimum, they may not even be necessary. The implications deduced are thus properties of the conventional formulae and not necessarily of the tax optimum. Nevertheless, one might hope that some tax optima, at least, satisfy these conditions."

Problem (ii) arises because the system of first-order conditions may possess no solution at all. Usually, given the necessary differentiability properties, the very existence of an optimal tax system is derived, tacitly, from a compactness argument; from what other argument I cannot imagine. However, unlike ordinary consumer maximization the tax problem apparently cannot without any further ceremony be posed as one of maximization of a continuous function on a compact set. The difficiulty is that tax systems with arbitrarily large tax rates for some commodities may be arbitrarily close to optimality. In order to exclude this possibility one needs additional assumptions, for example that the indirect utility function of prices have bounded upper level sets. Combined with boundary conditions guaranteeing interior solutions of commodities such a requirement seems to be quite restrictive. Fortunately, as the following analysis does not involve first-order conditions and draw on compactness, there is no need to discuss further this rather intricate issue here.

In this paper I confine the analysis to the particular case where a uniform tax system will be optimal. Three conditions will be considered: (a) commodity preferences being homothetic and independent of labour, (b) commodities being symmetric, and (c) labour supply being fixed. Conditions (a) and (c) are familiar from the literature.

Section 2 contains preliminary concepts and definitions, in particular of the revenue function. The main results are proven in Sections 3,4,5. The proofs all draw on the concept of majorization. In Section 6 possible extensions to non-uniform taxation are briefly discussed.

2. Preliminaries

The model to be studied here is the usual one-consumer model of optimal taxation. For given $n \geq 1$ let $x = (x_1,\ldots,x_n)$ denote the commodity vector and let L denote labour supply. Throughout these quantities are restricted by $x_1,\ldots,x_n \geq 0$, $0 \leq L \leq L_o$. Pre-tax prices are determined by the production side of the economy and are given by the positive vector $p = (p_1,\ldots,p_n)$; the wage rate is $w > 0$. The consumer's preferences are given by a continuous strictly quasi-concave function U of (x,L). Furthermore U is strictly increasing in x and decreasing in L.

For a prescribed tax revenue $r > 0$ the <u>tax system</u> $t = (t_1,\ldots,t_n) \geq 0$ is <u>feasible</u> if the maximand (x',L') of $U(x,L)$ subject to $(p+t)x \leq wL$ satisfies $tx' \geq r$. The <u>value</u> V of the tax system t is $V(t) = U(x',L')$. Let T(r) denote the set of feasible tax systems. It is assumed that r is so small that $T(r)$ is non-empty. If $V(t)$ attains a maximal value over $t \in T(r)$ at some t^*, then t^* is an _optimal_ tax system. If $t = \theta p$, some $\theta > 0$, then θ corresponds to a general _ad valorem_ tax and t is called a _uniform_ tax system.

Even though the set $T(r)$ will be closed, it is not in general true that $T(r)$ is bounded. Thus as noted in Section 1, existence of an optimal tax system cannot be established by a standard compactness argument. Here, existence will be demonstrated by a "constructive" proof; thus the possible lack of compactness of $T(r)$ does not affect the following analysis.

In order to characterize the set of optimal tax systems it is convenient to introduce the _revenue function_ R. For given t and real number α define

$$R(t,\alpha) = \min\{tx \,|\, U(x,L) \geqslant \alpha,\ (p+t)x \leqslant wL\}$$

with the convention that minimum over the empty set equals zero. This function may look similar to various functions known from "duality" theory but there is an important difference: since t occurs in the constraint, one cannot conclude that R is concave in t. This contrasts with the concavity of the expenditure function in consumer analysis.

Let $x(t)$ denote the commodity consumption given t. Since U is strictly quasiconcave, $x(t)$ is a continuous function of t. Furthermore, by strict quasi-concavity it follows that for any $t \geqslant 0$,

$$R\big(t,V(t)\big) = t\,x(t) \tag{1}$$

and $R\big(0,V(0)\big) = 0$.

Lemma 1. Consider two tax systems t, t' such that $R(t',V(t)) \geq R(t,V(t)) = r$. Then there exists $\delta \in [0,1]$ satisfying $R(\delta t', V(\delta t')) = r$ and $V(\delta t') \geq V(t') \geq V(t)$.

Proof: By the definition of V and R, $V(t') \geq V(t)$. Obviously, $V(\delta t') \geq V(t')$ for $\delta \in [0,1]$. Since $R(t',V(t')) \geq R(t',V(t)) \geq r$, then by continuity one can find δ having the required property. q.e.d.

Given the hypothesis of Lemma 1 it is natural to say that moving from t to t' represents an (weak) increase in welfare. In fact, the revenue function can in some situations be used as a device for computing an optimal tax system. This is the case here where certain conditions will be imposed on U implying that R is well-behaved. Since "well-behaved" cannot mean concave, some other properties must be used. In confining the analysis to cases where uniform taxation is optimal, one can restrict the attention to properties related to the concept of majorization (see Marshall and Olkin (1979), for a general discussion of majorization).

3. **Optimal Taxation**

For non-negative n-vectors q and z such that $qz > 0$ define the $n \times n$ matrix $C = (c_{ij})$ by $c_{ij} = z_i q_j / qz$. Thus for any $x \geq 0$, the vector Cx belongs to the ray generated by z and $qC = q$, $Cz = z$. Let I be the $n \times n$ identity matrix. The partial ordering $<^q_z$ is defined by $x <^q_z x'$ if $x' = Bx$ for some $\gamma \in [0,1]$, $B = \gamma C + (1-\gamma)I$. In this case x' directionally majorizes x (with respect to q, z). Clearly, if for some s, $s' \geq 0$ one has $s' = sB$, then $s <^z_q s'$. Note that for any non-negative x and t

$$x \prec_z^q (qx/qz)z; \; t \prec_q^z (tz/qz)q.$$

The motivation for introducing these concepts is that if $t \prec_q^z t'$, then a move from t to t' will increase welfare given some additional assumptions to be specified in the following.

Lemma 2. Suppose there exist α and a non-negative matrix A such that $pA = p$ and for all x, L: $U(Ax,L) \geqslant \alpha$ implies $U(x,L) \geqslant \alpha$. Then for all t, $R(tA,\alpha) \geqslant R(t,\alpha)$.

Proof: For given α and t one has

$$R(t,\alpha) = \min\{tx \mid U(x,L) \geqslant \alpha, \; (p+t)x \leqslant wL\} \tag{2}$$

$$\leqslant \min\{t(Ax) \mid U(Ax,L) \geqslant \alpha, \; (p+t)Ax \leqslant wL\}$$

$$\leqslant \min\{(tA)x \mid U(x,L) \geqslant \alpha, \; (p+tA)x \leqslant wL\} = R(tA,\alpha). \quad \text{q.e.d.}$$

This lemma is the key result leading to optimality of a uniform tax system. By quasi-concavity of U one readily verifies that (see Fig. 1):

Lemma 3. Suppose the non-zero vector $y \geqslant 0$ has the following property. For every $\delta > 0$ and L, δy maximizes $U(x,L)$ subject to $px = \delta py$. Then every $U(\cdot,L)$ preserves $\prec_y^p$: if $x \prec_y^p x'$, then $U(x,L) \leqslant U(x',L)$.

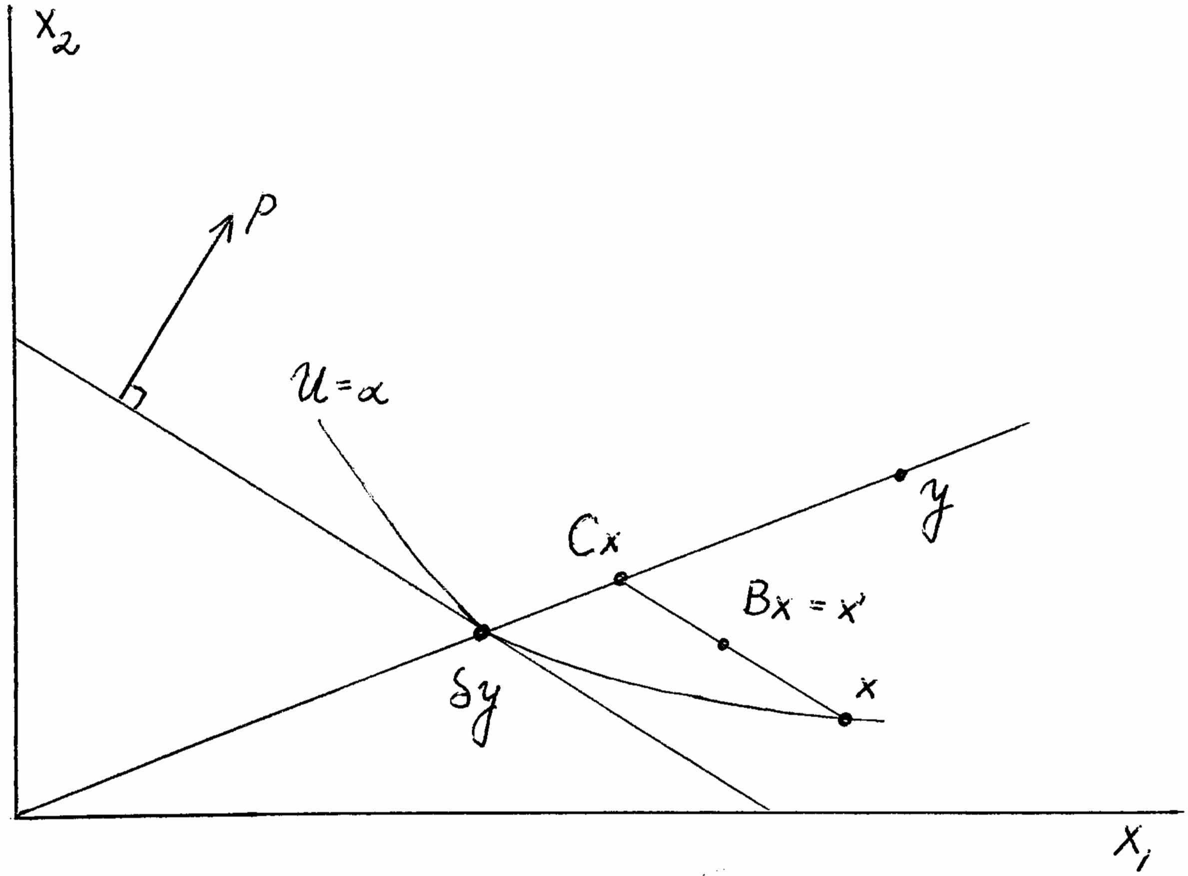

Figure 1

By Lemma 1, 2 and 3 one can now conclude that uniform taxation is optimal for U being homothetic in x and weakly separable in the sense that commodity preferences do not depend on L.

Theorem 1. Suppose U is homothetic in x and weakly separable in x and L. Let y denote a maximand of U(x,L) subject to px = py. Then for any α, $R(t,\alpha) \leq R(t',\alpha)$ for $t \overset{y}{\underset{p}{<}} t'$. In particular, uniform taxation is optimal.

A second set of assumptions on U and p yielding uniform taxation can be obtained by considering a particular form of U. Let q and z be given positive vectors. Then U is defined by

$$U(x,L) = \sum_{i=1}^{n} q_i z_i \ v(x_i/z_i, L) \tag{3}$$

where every $v(\cdot,L)$ is increasing and strictly concave. Let the set of $n\times n$ matrices M be defined by $M(q,z) = \{A = (a_{ij}) | a_{ij} \geq 0,\ qA = q,\ Az = z\}$. Then one can prove that U in (3) has the property that $U(Ax,L) \geq U(x,L)$ for $A \in M(q,z)$. Consequently, the hypothesis of Lemma 3 is satisfied for $y = z$. It is not essential that (3) is additive. For example if

$$U(x,L) = F\left(q_1 v(x_1,L),\ldots,q_n v(x_n,L);L\right) \tag{4}$$

where $F(\xi_1,\ldots,\xi_n;L)$ is increasing, quasi-concave and symmetric in $\xi_1,\ldots,\xi_n$, then $U(Ax,L) \geq U(x,L)$ for $A \in M(q,e)$, $e = (1,\ldots,1)$. The functions (3) and (4) are discussed in a companion paper, Thorlund-Petersen (1987). Again, by Lemma 1, 2 and 3 one can conclude:

Theorem 2. Given the utility function (3) with $p = q$ and a tax system t. Then for any α, $R(tA,\alpha) \geq R(t,\alpha)$ for $A \in M(p,z)$; in particular, uniform taxation is optimal.

Under the conditions of Theorem 2, replacing t by tA will increase welfare; in particular, the set of directions of improvements of welfare will be larger than the one described in Theorem 1. This aspect will be further discussed in the next section.

One can easily formulate a variant of Theorem 2 corresponding to the utility function (4). Although the functional forms (3) and (4) do not imply homotheticity and separability as in Theorem 1, elements of these properties are still present. For $q = p$ the consumer will choose some x collinear to z and the parameters of U are independent of L. Furthermore, the assumption q =

p is unusual as it imposes a joint assumption about consumption and production. Consequently, it is not claimed that (3) or (4) are particularly realistic specifications. However, it is of interest to be able to identify functional forms which by assumption lead to uniform taxation; see the discussion in Deaton (1981). A further specialization of (4) is to the case where commodities are symmetric; not surprisingly, uniform taxation will then be optimal but a number of other interesting properties can be obtained.

4. Symmetric Commodities

Suppose commodities are symmetric in both production and consumption. This means that $p_1 = \dots = p_n$, and $U(Px,L) = (x,L)$ for any n×n permutation matrix P. Thus two vectors having identical distributions are equivalent. Note that this does not necessitate that all commodities be perfect substitutes, in which case the utility of x depends only on the sum $x_1 + \dots + x_n$.

It is well-known that the set of all doubly stochastic matrices M(e,e) equals the convex hull of the set of n×n permutation matrices. Consequently, as U is quasi-concave, one has $U(Ax,L) \geq U(x,L)$ for any $A \in M(e,e)$. In other words, $U(\cdot,L)$ is <u>Schur-concave</u> for any L, see Marshall and Olkin (1979). The revenue function R does not in general inherit quasi-concavity from U but in this case $R(\cdot,\alpha)$ inherits Schur-concavity for every α. This remarkable fact follows from the inequalities (2) above.

Concerning the directions of improvement of welfare one gets particularly neat results in the symmetric case. The function $R(\cdot,\alpha)$ does not only preserve directional majorization $<_e^e$ but the much stronger ordering of majorization. For $a \in \mathbb{R}^n$ let $a_{(1)} \leq \dots \leq a_{(n)}$ denote the increasing rearrangement of a. Then for $a,b \in \mathbb{R}^n$ one say that b <u>majorizes</u> a, $a \prec b$, if for $k = 1,\ldots,n$

$$\sum_{i=1}^{k} a_{(i)} \leqslant \sum_{i=1}^{k} b_{(i)} \tag{5}$$

with equality for $k = n$, see Marshall and Olkin (1979). It is well-known that (5) is equivalent to the existence of some $A \in M(e,e)$ such that $b = Aa$. This leads to the following conclusion.

Theorem 3. If commodities are symmetric, then the revenue function $R(\cdot,\alpha)$ is Schur-concave for any α. Whenever $t \prec t'$, then replacing t by t' increases welfare. In particular, uniform taxation is optimal.

By way of illustration, consider $t = (t_i)$ with $t_1 \leqslant \ldots \leqslant t_n$ such that $t_2 - t_1 \geqslant 2\varepsilon > 0$. Then if

$$t' = (t_1+\varepsilon,\ t_2-\varepsilon,\ t_3,\ldots,t_n)$$

one has $t \prec t'$, see (5). Thus making the tax system "more uniform" in commodity 1 and 2 leads to a welfare improvement. Generally it is quite natural to say that t' is more uniform than t when $t \prec t'$. A very particular case is directional majorization: $t \overset{e}{\underset{e}{<}} t'$ entails that $t' = \gamma t + (1-\gamma)e$ for some $\gamma \in [0,1]$ if $\sum_{i=1}^{n} t_i = n$. Thus $t \overset{e}{\underset{e}{<}} t'$ implies $t \prec t$ whereas the opposite implication obviously is false.

5. **Fixed Labour Supply**

One of the most well-established assertions in optimal tax theory is that uniform taxation is optimal for labour supply being fixed. This statement has been explained and interpreted many times. In order to prove it, suppose that $U(x,L) = U(x)$ does not depend on L.

For a given L_o denote by z a solution to the problem of maximizing $U(x)$ subject to $px = pz = wL_o$ where L_o is the upper bound on L stated in Section 2. Let $\alpha = U(z)$. Of course, z will generally depend on L_o. By quasi-concavity of U, it follows that $U(x) \geqslant \alpha$ implies $U(x') \geqslant \alpha$ whenever $x <^p_z x'$, see Lemma 3 and Fig. 1. Thus reasoning as in Lemma 2 we see that

$$t <^z_p t' \text{ implies } R(t,\alpha) \leqslant R(t',\alpha). \tag{6}$$

Thus, again, we conclude that $t = \theta p$, some $\theta > 0$, is optimal. This case differs from the previous ones in one respect: $R(\bullet,\alpha)$ preserves directional majorization $<^z_p$ as before but now $<^z_p$ may depend on α via z.

6. **Conclusion**

The results of the previous sections show how one in some cases can compute an optimal tax system. More specifically, one need only determine $\theta > 0$ such that $t = \theta p$ yields the desired revenue; then t is optimal. Without assumptions leading to uniformity, the issue of computation becomes far more complex.

Generally, one cannot reasonably expect to be able to express an optimal tax system explicitly as a function of p, r and other parameters. Thus if the ultimate purpose of studying the tax model is to calculate tax rates in practice (as emphasized by Deaton (1981)), then one is challenged to find computational procedures which converge to an optimal tax system.

One particular such procedure is an iterative scheme of the following kind. For given initial t^1 with $R(t^1, V(t^1)) = r$, a sequence $(t^\nu)_{\nu=1}^\infty$ is constructed satisfying $R(t^{\nu+1}, V(t^{\nu+1})) = r$ and $V(t^{\nu+1}) \geqslant V(t^{\nu})$, all ν. This sequence can be constructed in many ways; in particular the step-length must be suitably chosen. Technicalities aside, the crucial problem will be whether R has the property:

$$\text{If for some } t^*,\ R(t,(V(t^*)) \leqslant R(t^*, V(t^*)) \text{ for all } t \text{ in a neighbourhood of } t^*, \text{ then } t^* \text{ is optimal.} \tag{7}$$

If (7) holds, then one probably can choose $(t^{\nu})_1^{\infty}$ such that any limit point of this sequence is optimal. On the other hand, if (7) is violated, then an iterative procedure as sketched above generally will not converge to an optimal tax system.

It is therefore natural to look for conditions on U and p which imply that R satisfies (7). This is no doubt a difficult task but there are many guidelines in the literature. For example, certain remarks about "moving towards the optimum" in the classic paper by Corlett and Hague (1953) seem to be related to this problem although their results need be translated into terms more suitable for practical computations. Thus concerning the possibility of such computations, even in the simple one-consumer tax model, the situation still is in an uncertain state.

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