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A DIRECT MECHANISM CHARACTERIZATION OF SEQUENTIAL BARGAINING WITH ONE-SIDED INCOMPLETE INFORMATION

by

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B1-Ausubel, Lawrence M. and Deneckere, Raymond J.--

B2-A Direct Mechanism Characterization of Sequential Bargaining with One-Sided Incomplete Information

C2-We characterize infinite-horizon bargaining games with one-sided incomplete information, under three extensive forms. When the seller (the uninformed party) makes all the offers, we prove a folk theorem: *every* incentive compatible, individually rational, direct bargaining mechanism is implementable by sequential equilibria. When the seller and buyer alternate in making offers, direct mechanisms must satisfy an additional individual-rationality-like constraint to be implementable. When the buyer makes all the offers, there exists a unique sequential equilibrium. Thus, depending on the offer structure, sequentiality may (or may not) impose restrictions beyond static incentive compatibility.

B6-*J. Econ. Theory*.

B7-

B8-

C4-

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1. Introduction

Much recent work has sought to clarify our understanding of the bargaining process by analyzing noncooperative, sequential, explicitly game-theoretic models of bilateral trade. Rubinstein [20] found a unique subgame perfect equilibrium in the alternating-offer bargaining game with complete information; subsequent authors have searched for related equilibrium concepts in games with similar extensive forms but containing incomplete information. In particular, considerable effort has been devoted to the study of infinite-horizon, discrete-time bargaining models where the seller's valuation s is common knowledge, the buyer's valuation b is private information (but its distribution function over the interval $[\underline{b},\bar{b}]$ is common knowledge), and time enters agents' utility functions through discounting.

It will be the purpose of this paper to provide an essentially complete description of the sequential equilibria of this bargaining game with one-sided incomplete information, when $\underline{b} = s$ and when the time interval z between successive offers is allowed to tend to zero. We examine sequential bargaining under the three extensive forms which are commonly analyzed in the literature: (a) the seller makes all the offers; (b) the seller and buyer alternate in making offers; and (c) the buyer makes all the offers. In each case, we find the entire set of sequential equilibria, in terms of (static) direct mechanisms, as z approaches zero.

A substantial number of authors have analyzed the extensive form where the uninformed party makes all the offers, and the related problem of durable goods monopoly. A line of papers which includes Coase [9], Bulow [4], Stokey [22], Sobel-Takahashi [21], Fudenberg-Levine-Tirole [12], and Gul-Sonnenschein-Wilson [15] has developed three valuable insights. First, there exist sequential equilibria--some of which form limits of equilibria of finite-horizon games--which satisfy a weak-Markov restriction on strategies. Second, in the case that $\underline{b} > s$, this sequential equilibrium is generically unique. Third, whenever $\underline{b} \geq s$, and the time interval between successive offers is short, the weak-Markov equilibria satisfy the "Coase Conjecture" property that the initial offer is close to $\underline{b}$. Previous work by the present authors (Ausubel and Deneckere [2]) amended this understanding by considering non-Markovian equilibria in the case where $\underline{b} = s$. There we reversed the Coase Conjecture by demonstrating the existence of "reputational equilibria" which mimic static monopoly pricing but are nevertheless sequential.

Some authors have also analyzed sequential bargaining with one-sided incomplete information where the two parties alternate in making offers.[1] Grossman and Perry [13] showed that there exists at most one so-called "perfect sequential equilibrium," under the assumption $\underline{b} > s$. Gul and Sonnenschein [14] proved that with certain stationarity assumptions and $\underline{b} > s$, trade occurs arbitrarily quickly as the time interval between offers approaches zero. Admati and Perry [1] obtained delay by modifying the rules of the game. A thorough survey of this sequential bargaining literature, as well as a treatment of the third extensive form (where the informed party makes all the offers), is contained in Wilson [24].

In characterizing the set of sequential equilibria of these games, we make use of a second line of work which is often seen as at odds with the sequential bargaining approach. The mechanism design literature (see Myerson [16, 17, 18]), Chatterjee and Samuelson [8], Myerson and Satterthwaite [19], and Williams [23]) analyzes the bargaining problem by utilizing the revelation principle: <u>any</u> equilibrium of <u>any</u> static bargaining game may be rewritten as a direct mechanism for which truth telling must be a Nash equilibrium. The reason for the tension between the two lines of literature is that the mechanism approach is explicitly one-shot, while the sequential approach allows infinitely many opportunities for the parties to reach agreement. If a sequential equilibrium in these infinite-horizon games has trade occurring in finite time, the finiteness is endogenously generated; whereas the finite time in a direct mechanism is assumed.

Nevertheless, we find the two strands of literature complementary. First, static mechanisms give us a compact language for describing the set of sequential equilibria. Any conceivable equilibrium is payoff equivalent to an element of the set of all individually rational, incentive compatible bargaining mechanisms; our main theorems tell us which elements of this set are implementable by sequential equilibria of the three infinite-horizon bargaining games. Second, the fact that certain direct mechanisms are implementable by sequential equilibria provides a justification for studying direct mechanisms. The "Nash program," in general, asks if there is a noncooperative justification for various solution concepts in cooperative game theory. The most conspicuous success along these lines is the work associated with Binmore [3] and Rubinstein [20]: take the alternating-offer, complete information bargaining game and let the time interval z between offers approach zero. With equal discount factors, the unique sequential equilibrium converges (in payoff terms) to split-the-difference; with unequal discount factors, one obtains the asymmetric Nash bargaining solution. Similarly, here, mechanisms have a cooperative flavor, as players are assumed to be able to commit to them. Mechanisms also have an axiomatic development--see, for example, Myerson [17]. Our folk theorem for seller-offer, sequential bargaining games provides a noncooperative justification.

Throughout this paper, we will be assuming that $\underline{b} = s$, which many authors have considered to be the most "relevant" case (e.g., Gul, Sonnenschein and Wilson [15], p. 162). We do <u>not</u> impose any "refinements," but rather we seek to find the full set of sequential equilibria. As in Coase Conjecture studies and in Binmore [3], we perform the limiting exercise of letting the time interval between offers approach zero.

In Section 2, we discuss direct mechanisms. Section 3 formulates the sequential model and defines "implementable by sequential equilibria." Section 4 reviews some earlier results on sequential bargaining games. In Section 5, we prove the Folk Theorem for sequential bargaining where the uninformed seller makes all the offers: <u>every</u> individually rational, incentive compatible bargaining mechanism is implementable by sequential equilibria. In Section 6, we analyze alternating-offer bargaining with the same information structure and find that the equilibrium set is <u>distinctly smaller</u> than in the previous case: a bargaining mechanism must satisfy an additional individual-rationality-like constraint to be implementable by alternating-offer equilibria. In Section 7, we consider the remaining case of buyer offers and show that the equilibrium set shrinks dramatically--to a single point. We conclude in Section 8.

2. <u>Static Mechanisms</u>

Consider a trading situation in which one individual (henceforth referred to as the seller) owns an object he would like to sell to another individual (referred to as the buyer). Let the seller's valuation, s, for the object be common knowledge, and let the buyer's valuation, b, be distributed according to the (common knowledge) distribution function $F(\bullet)$. The support of $F(\bullet)$ is a subset of $[\underline{b},\bar{b}]$ containing $\underline{b}$ and $\bar{b}$. For most of what follows, we will consider the case $\underline{b} = s$ and, without loss of generality, we thus set $s = \underline{b} = 0$.

Define a <u>bargaining mechanism</u> to be a game in which the buyer reports his private information, b, to a mediator, who then determines whether the good is transferred from the seller to the buyer and how much the buyer must pay the seller. A bargaining mechanism is characterized by two outcome functions: $p(b)$, which denotes the probability of trade; and $x(b)$, which denotes the expected transfer payment. A bargaining mechanism is <u>incentive compatible</u> if honest reporting forms a Bayesian Nash equilibrium, i.e.,

$$(1) \qquad bp(b) - x(b) \geq bp(b') - x(b'), \ \forall\, b,b' \in \text{supp } F.$$

A bargaining mechanism is (ex post) <u>individually rational</u> iff:

$$(2) \qquad bp(b) - x(b) \geq 0, \text{ and } x(b) \geq 0, \ \forall\, b \in \text{supp } F.$$

Mechanisms satisfying both the individual rationality and the incentive constraints will be referred to as <u>incentive compatible bargaining mechanisms (ICBMs)</u>. Let $U(\bullet)$ and V defined by:

$$U(b) = bp(b) - x(b), \text{ and}$$

(3)

$$V = \int_0^{\bar{b}} x(b)dF(b),$$

be the buyer's and seller's utilities, respectively. Then we may define the set of <u>incentive feasible points</u>, M, as:

$$M = \{(U(\bullet),V)\text{: } U(\bullet) \text{ and } V \text{ satisfy (3) for some ICBM } (p,x)\}.$$

Assuming that the support of the distribution function $F(\bullet)$ is the entire interval $[0,\bar{b}]$, the following theorem provides a complete characterization of the set of ICBMs and incentive feasible points:

<u>Theorem 1</u>: Let $F(\bullet)$ be any strictly monotone distribution function. Then $\{p,x\}$ is an ICBM if and only if $p\colon [0,\bar{b}] \to [0,1]$ is (weakly) increasing and x is given by the Stieltjes integral:

$$x(b) = \int_0^b r\, dp(r).$$

Furthermore, $U(0) = 0$ and, for all $0 \leq b \leq \bar{b}$:

$$U(b) = \int_0^b p(r)dr.$$

<u>Proof</u>: An adaptation of Myerson-Satterthwaite [19]. The analogue to Myerson and Satterthwaite's inequality (2) holds for all (weakly) increasing functions p. Ex post individual rationality implies $U(0) = 0$. []

Consider a static bargaining game in which the seller makes a single take-it-or-leave-it offer. After the buyer's decision whether or not to accept the offer, the game concludes. All equilibria of this game have the seller offer a price q_*, where $q_* \in \arg\max_r \{r[1 - F(r)]\}$; the buyer accepts if and only if his valuation is greater than or equal to q_* (see Example 2 for the associated bargaining mechanism). While such an outcome seems perfectly reasonable in a static context, it may appear much less so in a dynamic world, because of the associated ex-post inefficiency: if $0 < b < q_*$, traders do not transact even though gains from trade exist.

Indeed, Peter Cramton [11] and others (e.g., Chatterjee and Samuelson [6, 7], Chatterjee [5], and Fudenberg, Levine and Tirole [12]) have criticized the static ICBM approach on account that it "implicitly assumes that the players are able to commit to walking away [from the bargaining table] without trading, after it has been revealed that substantial gains from trade exist" (Cramton [11], p. 150). In fact, these authors take the position that any bargaining game that is modeled as having an exogenously given, finite number of stages violates a broad interpretation of sequential rationality: commitments not to reopen or continue the bargaining process after the prescribed number of stages has passed are not credible. Thus, "a useful bargaining game must allow the number of potential stages to be infinite, with the actual length of bargaining determined endogenously by the bargaining process" (Chatterjee and Samuelson [6], p. 5).

In this paper, we will follow these authors' implicit suggestions, and study the set of sequential equilibria of infinite-horizon bargaining games.

3. Sequential Games and Sequential Mechanisms

In our study of infinite-horizon bargaining games, three types of trading rules will be of interest:[2]

Seller offers:	the seller makes all the offers, and the buyer waits to accept an appropriate price.
Alternating offers:	the seller and buyer alternate in making offers until one of the parties accepts the other's offer.
Buyer offers:	the buyer makes all the offers, and the seller waits to accept an appropriate price.

With each of these three trading rules, price offers will be assumed to occur at discrete moments in time, spaced equally apart. Let z $(z > 0)$ denote the time interval between successive offers. Then bids are made at times $t = 0, z, 2z, 3z, \ldots$. Both players exhibit impatience,[3] which is specified by a common discount rate r $(r > 0)$. Hence, if the good is traded at time t for the price p, the seller obtains surplus of $e^{-rt}p$ and the buyer derives surplus of $e^{-rt}(b - p)$.

Corresponding to any equilibrium of any of these infinite-horizon bargaining games, there exists a direct revelation sequential bargaining mechanism.[4] Such a mechanism specifies a pair of outcome functions $t(\bullet)$ and $x(\bullet)$, where $t(b)$ is the time that the good will be transferred to the buyer, and $x(b)$ is the discounted expected transfer to the seller, given that the buyer reports b. The incentive compatibility constraints now become:

$$U(b) = be^{-rt(b)} - x(b) \geq be^{-rt(b')} - x(b'),\ \forall\ b, b' \in \text{supp } F, \tag{4}$$

and the (ex post) individual rationality constraints are:

$$(5) \qquad be^{-rt(b)} - x(b) \geq 0 \text{ and } x(b) \geq 0, \ \forall \ b \in \text{supp } F.$$

Observe the close resemblance between (4)-(5) and (1)-(2). In fact, if we make the transformation $p(b) = e^{-rt(b)}$, the conditions for the sequential mechanism match up exactly with those for the static mechanism. Thus, every sequential equilibrium of the infinite-horizon bargaining games induces, through the above transformation, a static bargaining mechanism which is incentive compatible.

The next lemma shows that with one-sided incomplete information every sequential equilibrium must also be ex post rational. This explains why we chose to impose ex post individual rationality in our definition of an ICBM, rather than the more common (see, for example, Myerson and Satterthwaite [19]) interim rationality.

Lemma 1: Every sequential equilibrium of the infinite-horizon bargaining game with one-sided incomplete information is ex post individually rational, under any of the three offer rules.[5]

Proof: Let q denote the infimum of all prices offered with positive probability by the seller in any sequential equilibrium and after any history. We claim that $q \geq 0$. For suppose not. Define $\delta = e^{-rz}$. Then there exists a sequential equilibrium σ and a history h such that the seller offers a price q_h with positive probability where $q_h < \delta q < 0$. Observe that all buyers must accept this offer, since even the zero type prefers trade at q_h in the current period to trade at q in the next period. But, then, the seller benefits by deviating from the equilibrium and offering a price of zero forever. We conclude $q \geq 0$.

We have just shown that the seller never offers the good to the buyer at a negative price. Clearly, the seller also never accepts a buyer's counteroffer to purchase at a negative price, since rejection dominates. Thus, the seller's utility is nonnegative in all states of the world.

Finally, we will show that $U(0) = 0$. (This immediately implies that $U(b) \geq 0$ for all b.) We argued above that the seller never offers or accepts a negative price. Consequently, the buyer of type zero can have no reason to counteroffer a positive price which has positive probability of acceptance; nor to accept a positive price. We conclude $U(0) = 0$. [].

We have just observed that every sequential equilibrium of an infinite-horizon bargaining game may be represented as a static ICBM. The interesting question which remains to be asked is: When can we go the other direction? When can a static ICBM be implemented as a sequential equilibrium of an infinite-horizon bargaining game?

Before answering this question we need to make precise what we mean by "implementation as a sequential equilibrium." Suppose, given a mechanism $\{p,x\}$ and a positive number z, we can find an equilibrium σ such that: (a) σ is a sequential equilibrium of the infinite-horizon bargaining game; and (b) σ, written in mechanism terms, yields $\{p,x\}$. Then we could certainly say that "$\{p,x\}$ is implemented by σ in the game with time interval z.". There are two related reasons why we will adopt a somewhat different definition of implementation. First, we are working in a discrete-time model and, under the proposed definition, changing the time interval z between periods is likely to change which mechanisms are implementable and which are not. We would prefer a criterion for implementability which is essentially independent of z. Second, our discrete-time game is meant to model a situation where there is basically no constraint on the rate at which parties can make successive offers. The meaningful question to ask is then: What can we say about equilibrium behavior as the time interval between successive offers approaches zero?

These considerations motivate the following definitions:

Definition 1: Let σ be a sequential equilibrium in pure strategies of an infinite-horizon bargaining game. Suppose that, in equilibrium, the buyer of type b trades at time t(b) and at price q(b). Then we define the ICBM corresponding to σ by:

$$p_\sigma(b) = e^{-rt(b)}, \text{ and}$$

(6)

$$x_\sigma(b) = q(b)e^{-rt(b)}.$$

Definition 2: Let {p,x} be any (static) direct mechanism, not necessarily IR or IC. For any offer rule (e.g., seller-offer, buyer-offer, or alternating-offer), we will say that the bargaining mechanism {p,x} is implementable by sequential equilibria of the infinite-horizon bargaining game if $p(0) \le \lim_{b\downarrow 0} p(b)$,[6] and if there exists a sequence $\{\sigma_n, z_n\}_{n=1}^{\infty}$ such that:

(i) $z_n \downarrow 0$ and, for every $n \geq 1$, σ_n is a sequential equilibrium of the infinite-horizon bargaining game where z_n is the time interval between successive offers; and

(ii) The ICBMs $\{p_n, x_n\}$ corresponding to σ_n converge to $\{p, x\}$, in the following sense: for every $\varepsilon > 0$, there exists $\bar{n}(\varepsilon)$ such that

(7) $$|p_n(b) - p(b)| < \varepsilon, \ \forall \ n \geq \bar{n}(\varepsilon) \text{ and } \forall \ b \in \text{supp } F \text{ s.t. } b > \varepsilon, \text{ and}$$

(8) $$|x_n(b) - x(b)| < \varepsilon, \ \forall \ n \geq \bar{n}(\varepsilon) \text{ and } \forall \ b \in \text{supp } F.$$

The asymmetric treatment of $b = 0$ in the above definition deserves some comment. A symmetric treatment would have required that $\{p_n, x_n\}$ converge uniformly to $\{p, x\}$.[7] While such a definition would be satisfactory in the sense that it also implies uniform convergence of buyer and seller utilities, it is too harsh in that it would preclude implementation of mechanisms in which all trade occurs in finite "time," i.e., those for which $\lim_{b \downarrow 0} p(b) > 0$ (see Example 3, below). This is because any sequential equilibrium in the seller-offer or alternating-offer game has the property that sales occur over infinite time.[8] Hence, if p_n is derived from such a sequential equilibrium, we must have $\lim_{b \downarrow 0} p_n(b) = 0$.

Our definition of "implementation" is a very strong one: (7) and (8) together immediately imply <u>uniform convergence of buyer and seller utilities</u>, i.e., $\forall \ n \geq \bar{n}(\varepsilon)$ and $\forall \ b \in$ supp F:

$$|[bp_n(b) - x_n(b)] - [bp(b) - x(b)]| < (\bar{b} + 1)\varepsilon, \text{ and}$$

$$\left|\int_0^{\bar{b}} x_n(b)dF(b) - \int_0^{\bar{b}} x(b)dF(b)\right| < \varepsilon.$$

In fact, if every ICBM is implementable, every incentive feasible point can be arbitrarily closely approximated, and we will say that the Folk Theorem holds. More precisely:

Folk Theorem: For every $m = (U(\cdot),V) \in M$, there exist sequences $\{\varepsilon_n,\sigma_n,z_n\}_{n=1}^{\infty}$ with $\varepsilon_n,z_n \downarrow 0$, such that σ_n is a sequential equilibrium for the infinite-horizon bargaining game when z_n is the time interval between successive offers,

$$|[bp_n(b) - x_n(b)] - U(b)| < \varepsilon_n, \quad \forall\ b \in \text{supp } F, \text{ and}$$

$$\left|\int_0^{\bar{b}} x_n(b)dF(b) - V\right| < \varepsilon_n,$$

where each $\{p_n,x_n\}$ is the ICBM corresponding to σ_n.

4. Preliminaries

This section collects some existing results on the seller-offer game, which will be useful in our subsequent analysis. For more formal statements of these definitions and theorems, and for complete proofs, we urge the reader to consult the references mentioned. Two assumptions will henceforth be made on the distribution function $F(\cdot)$ of buyer valuations:

(9) There exist $L \geq M > 0$ such that $Mb \leq F(b) \leq Lb$ for all $b \in [0,\bar{b}]$.

(10) There exists $\varepsilon > 0$ such that F is strictly increasing on the interval $[0,\varepsilon]$.

For example, if F has density in a neighborhood of 0, and this density is bounded above and below by positive numbers, then it satisfies the above conditions. Observe, however, that (9) and (10) also allow mass points. It is helpful to define,[9] for fixed L and M ($L \geq M > 0$), the family $\mathcal{F}_{L,M} = \{F\text{: F is a distribution function and } Mb \leq F(b) \leq Lb \text{ for all } b \in [0,\bar{b}]\}$.

Many of the results in the literature concern sequential equilibria in which the buyer's strategy is history-independent. To be more precise:

Weak-Markov Equilibrium: Let σ_s be a strategy for the seller and σ_b be a strategy for the buyer. (σ_s,σ_b) is a weak-Markov equilibrium if (σ_s,σ_b) is a sequential equilibrium and σ_b depends only on the seller's most recent offer.

One of the principal reasons to be interested in these equilibria is that the celebrated Coase Conjecture [9] holds: the seller is forced to offer the good at an introductory price barely above his valuation. This was first proven for a linear demand example by Stokey [22] and was subsequently generalized to the whole class of weak-Markov equilibria by Gul-Sonnenschein-Wilson [15, Theorem 3]. Here, we will need a somewhat stronger version of this theorem.

Uniform Coase Conjecture [2, Theorem 5.4]: For every L ($L \geq 1/\bar{b}$), every M ($0 < M \leq 1/\bar{b}$), and every ε ($\varepsilon > 0$), there exists $\bar{z}(L,M,\varepsilon)$ such that for every $F \in \mathcal{F}_{L,M}$, for every z satisfying $0 < z < \bar{z}(L,M,\varepsilon)$, and for every weak-Markov equilibrium in the seller-offer game where the distribution function of buyers is F and the time interval between periods is z, the seller's initial offer is less than ε.

In fact, the above theorem is not vacuous, as under assumption (10), one may also prove that the set of weak-Markov equilibria is nonempty. The first such existence theorem is due to Fudenberg, Levine and Tirole [12, Proposition 2]; we use a somewhat stronger version here:

<u>Existence of Weak-Markov Equilibria</u> [2, Theorem 4.2]: Let F be any distribution function satisfying (10). Then for any time interval z between periods, there exists a weak-Markov equilibrium of the seller-offer game.

The weak-Markov equilibria are decidedly unfavorable to the seller. However, relaxing the weak-Markov restriction drastically enlarges the equilibrium set:

<u>The Folk Theorem for Seller Payoffs</u> [2, Theorem 6.4]: Assume F satisfies (9) and (10). Define $\pi^* = \sup_b \{b[1 - F(b)]\}$. Then for every $\varepsilon > 0$ and every $\pi \in [\varepsilon, \pi^* - \varepsilon]$, there exists $\bar{z}(\varepsilon) > 0$ such that whenever the time interval z satisfies $0 < z < \bar{z}(\varepsilon)$, there exists a sequential equilibrium of the seller-offer game which yields the seller an expected surplus of π.

Let us explore the intuition for this result. A constant price path over time, such as charging the static monopoly price q_* forever, is not time-consistent by Coase's original critique. Instead, we utilize the exponentially-descending price path[10] $q(t) = q_* e^{-\eta t}$ (confined to the grid of times $\{0,z,2z,\ldots\}$). Observe first that if η and z are small positive numbers, then this price path becomes an excellent approximation to charging the monopoly price forever, in the sense of yielding almost the same expected seller surplus. Second, this price path generates sales over infinite time, since (10) implies that the expected net present value of seller surplus, evaluated at every time, remains positive. Third, using the Uniform Coase Conjecture, it can be shown that for sufficiently small z, the net present value of expected seller surplus along any weak-Markov equilibrium price path is uniformly (at all times) lower than along the price path $q_* e^{-\eta t}$.

Consider now a strategy in which the seller follows the proposed exponential price path unless he has deviated from it in the past, in which case a reversion to a weak-Markov equilibrium is triggered. Our reasoning above guarantees that this strategy, coupled with corresponding buyer acceptance behavior, forms a sequential equilibrium for sufficiently small z. Thus, $\pi^* - \varepsilon$ is attainable by such reputational equilibria. Continuously varying the initial price makes possible all other seller surpluses between ε and $\pi^* - \varepsilon$.

Observe that the above theorem says nothing about buyer payoffs, and deals only with the seller-offer game. In the next section, we will prove the Folk Theorem (as defined in Section 3) for the seller-offer game; in Sections 6 and 7 we analyze the alternating-offer and buyer-offer games.

5. The Seller-Offer Game

Consider any incentive compatible bargaining mechanism $\{p(\bullet),x(\bullet)\}$. In Definition 1, we wrote p and x as functions of t (the time of trade) and q (the price of trade). We may invert the transformation (6) to obtain:

$$(11) \quad \begin{aligned} t(b) &= -(1/r) \log p(b), \\ q(b) &= x(b)/p(b). \end{aligned}$$

Define $\mathcal{A} = \{t(b)\colon 0 \le b \le \bar{b}\}$, i.e., $\mathcal{A}$ is the range of $t(\bullet)$ in (11). Observe that (11) parametrically defines q in term of T for all $T \in \mathcal{A}$; call this function $Q(T)$. This may be extended to a function of all $t \ge 0$ by letting $T(t) = \sup\{t'\colon t' \in \mathcal{A} \text{ and } t' \le t\}$ and defining:

$$(12) \qquad q(t) = \begin{cases} Q(T(t)), & \text{if } t \ge \inf \mathcal{A} \\ \bar{b} & \text{, otherwise.} \end{cases}$$

The function $q(t)$ in equation (12) will be called the <u>price path associated with $\{p,x\}$</u>. It is consistent with the parametric representation (11) at all times belonging to $\mathcal{A}$, and it defines a "flat" price path at all other times. Roughly speaking, we will implement the ICBM $\{p,x\}$ in the seller-offer game by constructing sequential equilibria whose main price paths approximate $q(t)$. The following examples will both illustrate the basic procedure and demonstrate some of the difficulties that must be overcome.

<u>Example 1</u>: Exponentially-descending price path; uniform distribution.

Consider the direct mechanism:

$$p^1(b) = \begin{cases} 1 \\ \left[\frac{rb}{(r+\eta)q_0}\right]^{r/\eta} \end{cases} \quad x^1(b) = \begin{cases} q_0 & \text{if } b \geq \frac{r+\eta}{r} q_0, \\ \left[\frac{rb}{(r+\eta)}\right]^{1+(r/\eta)} q_0^{-(r/\eta)} & \text{if } b < \frac{r+\eta}{r} q_0, \end{cases}$$

where $0 < q_0 < 1$, $\eta > 0$, and buyer valuations are distributed according to the uniform distribution $F(b) = b$ $(0 \leq b \leq 1)$. Observe, using Theorem 1, that $\{p^1, x^1\}$ is an ICBM. By (11) and (12), the price path associated with $\{p^1, x^1\}$ is $q^1(t) = q_0 e^{-\eta t}$, for all $t \geq 0$.

Using our previous paper (Ausubel and Deneckere [2]), Example 1 is straightforward to handle. As observed in the next-to-last paragraph of Section 4, grid approximations to the price path $q^1(t)$ can be supported as the main equilibrium paths of sequential equilibria, for sufficiently short time intervals between offers. Furthermore, since $q^1(t)$ is continuous and because every buyer type has a single most-preferred purchase time, it is relatively easy to show that the purchase probabilities and expected transfers (associated with these sequential equilibria) converge uniformly to $p^1(\cdot)$ and $x^1(\cdot)$, respectively.

Example 2: Take-it-or-leave-it offer; uniform distribution.

Consider the direct mechanism:

$$p^2(b) = \begin{cases} 1 \\ 0 \end{cases} \quad x^2(b) = \begin{cases} q_0 & \text{if } b \geq q_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < q_0 < 1$. Observe that $\{p^2, x^2\}$ is an ICBM and that the associated price path is $q^2(t) = q_0$, for all $t \geq 0$.

It is worth noting, given a sequence $\eta_m \downarrow 0$, that the sequence of mechanisms $\{p_m^1, x_m^1\}$ (defined by using $\eta = \eta_m$ in Example 1) does not converge uniformly to $\{p^2, x^2\}$ in neighborhoods of q_0. Thus, a diagonal argument is not sufficient to establish that $\{p^2, x^2\}$ is implementable.

Similarly, a "naive approach" to constructing sequential equilibrium price paths will fail to establish implementability. Let T_n denote the time which players discount to $1/n$, i.e., $e^{-rT_n} = 1/n$. We might propose a sequence of equilibria σ_n in which, along the equilibrium path, the seller follows a grid approximation to the price path $q^2(t)$ until time T_n and then follows an exponentially-descending price path $q_0 e^{-\eta(t-T_n)}$. For sufficiently small z_n, we can show that σ_n are sequential equilibria.[11] Let b_n denote the lowest buyer type who purchases at time zero in σ_n. Then σ_n, expressed as an ICBM $\{\pi_n^2, \psi_n^2\}$, satisfies: $\pi_n^2(b) = 1$, if $b \geq b_n$; and $\pi_n^2(b) < 1/n$, if $b < b_n$. However, observe that $b_n > q_0$ for all n, and so $\pi_n^2(q_0) < 1/n$ for all n, implying that $\{\pi_n^2\}$ does <u>not</u> converge to p^2 at q_0.

We prove implementability by a more sophisticated argument. Let $\mu(n)$ denote the time of buyer q_0's most-preferred offer after time T_n. That is, let $\mu(n) \equiv \arg\max_{kz}\{e^{-rkz}[q_0 - q_0 e^{-\eta(kz-T_n)}]$: $kz \geq T_n$ and k is an integer$\}$ and let $\Phi(n)$ denote the price offered at $\mu(n)$. Also define $s_{n,z}$ by:

$$(13) \qquad q_0 - s_{n,z} = e^{-r\mu(n)}[q_0 - \Phi(n)].$$

Consider now a price path for σ_n given by: $q_n^2(t) = s_{n,z}$, if $0 \leq t < T_n$; and $q_n^2(t) = e^{-\eta(t-T_n)}$, if $t \geq T_n$. As before, this price path yields a sequential equilibrium when z is sufficiently small. Observe, however, that type q_0 is now exactly indifferent between purchasing at times 0 and $\mu(n)$, by (13), so our equilibrium can specify that type q_0 purchase at either time (or randomize between the two). In particular, we may impose that he purchase at time 0. Letting $\{p_n^2, x_n^2\}$ be the associated ICBM: $p_n^2(b) = 1$, if $b \geq q_0$; and $p_n^2(b) < 1/n$, if $b < q_0$. This converges uniformly to p^2.

The failure to converge at a single point, q_0, may have seemed fairly innocuous in Example 2. But modify that example by assuming a mass point in the distribution at q_0. Then the seller's expected surplus, V_n^2, computed using ICBM's $\{\pi_n^2, \psi_n^2\}$, fails to converge to V^2, computed from $\{p^2, x^2\}$. Thus, the "naive approach" fails to establish a folk theorem.

Example 2 has also illustrated how our general proof handles discontinuities in the function $p(\bullet)$. If p is discontinuous at $\hat{b}$, then for sufficiently large n, σ_n makes $\hat{b}$ exactly indifferent between purchasing at times approximately corresponding to $\lim_{b \uparrow \hat{b}} p(b)$ and $\lim_{b \downarrow \hat{b}} p(b)$.

Example 3: Two-step price path; uniform distribution.

Consider the direct mechanism:

$$p^3(b) = \begin{cases} 1 \\ 1/2 \end{cases} \qquad x^3(b) = \begin{cases} 1/4, & \text{if } b \geq 1/2, \\ 0, & \text{if } b < 1/2. \end{cases}$$

Observe that $\{p^3, x^3\}$ is an ICBM and that the associated price path is: $q^3(t) = 1/4$, if $1/2 < e^{-rt} \leq 1$; and $q^3(t) = 0$, if $e^{-rt} \leq 1/2$. This price path is more difficult to implement because it drops to zero in finite time, whereas in any sequential equilibrium, price stays forever positive.

Define $T^z \equiv \inf\{kz: e^{-rkz} \le 1/2$, for some integer $k\}$. Since we cannot set a zero price at time T^z, we instead set a price of $1/n$ at time T^z in the n^{th} equilibrium price path. Analogous to (13), define $s_{n,z}$ such that: $1/2 - s_{n,z} = e^{-rT^z}[1/2 - 1/n]$. We then specify a price path: $q_n^3(t) = s_{n,z}$, if $0 \le t < T^z$; and $q_n^3(t) = (1/n)e^{-\eta(t-T^z)}$, if $t \ge T^z$. As usual, $q_n^3(t)$ yields a sequential equilibrium when z is sufficiently small. Now, however, type 1/2 is exactly indifferent between purchasing at times 0 and T^z. Moreover, $p_n^3(b) = 1$ for $b \ge 1/2$, and $p_n^3(b) \approx 1/2$ for $b < 1/2$, except for b close to zero. We thus obtain uniform convergence to $p^3(\cdot)$ except in neighborhoods of zero, proving implementability.

Analogous arguments allow us to construct equilibrium price paths which mimic arbitrarily complicated ICBM's, so we have:

Theorem 2: A direct bargaining mechanism $\{p,x\}$ is implementable by sequential equilibria of the seller-offer game if and only if it is an ICBM.

Proof: See the Appendix.

6. The Alternating-Offer Game

In the previous section, we studied a very restricted infinite-horizon bargaining game: only the uninformed party was permitted to make offers. We will now consider a richer game form: the seller and buyer alternate in making offer and counteroffer. This would seem to open up a wealth of new possibilities for equilibrium behavior,[12] as the informed party is now allowed a much larger vocabulary (the interval $[0,\bar{b}]$, instead of just

{Y,N}).[13] Indeed, appearances have been that an assumption of one-sided offers artificially restricts the equilibrium set.

Of course, one should immediately observe that the addition of buyer counteroffers to the infinite-horizon bargaining game cannot possibly expand our set of sequential equilibrium outcomes: we already have a folk theorem for the seller-offer game.

More surprisingly, the introduction of buyer counteroffers actually reduces the equilibrium set of payoffs. The richer extensive form enables us to exclude some outcomes, because when the buyer has the opportunity to make counteroffers, he can observably deviate by making an unexpected demand. The most adverse inference which the seller can form is the ("optimistic") conjecture that the buyer's type is $\bar{b}$. But, in that event, the high-valuation buyer must earn at least what he obtains in the complete information, alternating-offer game (see Grossman and Perry's [13] Lemma 3.1). When the time interval between offers shrinks to zero, this amount converges to one-half of total surplus. We have:

Theorem 3: A direct bargaining mechanism {p,x} is implementable by sequential equilibria of the alternating-offer game if and only if {p,x} is an ICBM and:

$$(14) \qquad \bar{b}p(\bar{b}) - x(\bar{b}) \geq \bar{b}/2.$$

Proof: We first prove the "only if" part. Suppose {p,x} is not IC or IR. Then we obtain the same contradiction as in the proof of Theorem 2. Now, suppose $\bar{b}p(\bar{b}) - x(\bar{b}) < \bar{b}/2$. We will show that there exists a positive measure of buyer types who can break the alleged equilibria σ_n, for sufficiently large n. Observe that there exists $\varepsilon > 0$ and $N \geq 1$ such that:

$$\text{(15)} \quad bp(\sigma_n;b) - x(\sigma_n;b) < b/2, \text{ for all } n \geq N \text{ and } b \in [\bar{b} - \varepsilon, \bar{b}] \cap \text{supp } F.$$

Note that, by definition, $F(\bar{b}) - F(\bar{b} - \varepsilon) > 0$ for all $\varepsilon > 0$.

Let $\delta_n = e^{-rz_n}$, the discount factor between successive periods in the equilibrium σ_n. By Lemma 3.1(ii) of Grossman and Perry [13],[14] after any history in any sequential equilibrium, the seller will accept any buyer counteroffer greater than or equal to $\delta_n\bar{b}/(1 + \delta_n)$ (the price offered by buyer $\bar{b}$ in the Rubinstein [20] game). Hence, the buyer with valuation b has the option of rejecting the seller's initial offer in equilibrium σ_n, instead counteroffering $\delta_n\bar{b}/(1 + \delta_n)$. Discounting to the beginning of the game, this assures the buyer with valuation b a payoff of $\delta_n\{b - \delta_n\bar{b}/(1 + \delta_n)\}$. Hence, buyer b can break an alleged equilibrium σ_n unless:

$$\text{(16)} \qquad bp(\sigma_n;b) - x(\sigma_n;b) \geq \delta_n\{b - \delta_n\bar{b}/(1 + \delta_n)\}.$$

It is easy to see that (15) leads to a contradiction of (16) for appropriate choice of ε and N, for all $b \in [\bar{b} - \varepsilon, \bar{b}] \cap \text{supp } F$ and all $n \geq N$, since $\lim_{n\to\infty} \delta_n = 1$. This establishes the "only if" part of the theorem.

We now prove the "if" part of the theorem, by "embedding" the seller-offer equilibria (with time interval 2z) of Theorem 2 into the alternating-offer game (with time interval z).[15] If $\delta = e^{-rz}$, define the following candidate equilibrium:

<u>If there have been no prior buyer deviations</u>:

In even periods ($t = 0, 2z, 4z, \ldots$):

Seller offers $q_{n,2z}(t)$, as defined in the proof of Theorem 2, if there have been no prior seller deviations.

Seller offers the price from a weak-Markov equilibrium (with time interval 2z), if there have been prior seller deviations.

Buyer responses and seller beliefs also correspond exactly to the seller-offer equilibrium (with time interval 2z).

In odd periods ($t = z, 3z, 5z, \ldots$):

Buyer counteroffers zero.

Seller rejects.

<u>If the buyer has ever counteroffered a positive price</u>:

The seller updates his beliefs to:

$$\hat{F}(b) = \begin{cases} 1, & \text{if } b = \bar{b} \\ 0, & \text{if } b < \bar{b}, \end{cases}$$

and never changes his beliefs again.

In even periods:

Seller offers $\bar{b}/(1 + \delta)$.

Buyer of type b accepts any offer $\leq \min\{\bar{b}/(1 + \delta), b\}$.

In odd periods:

Buyer of type b offers $\min\{\delta\bar{b}/(1 + \delta), b\}$.

Seller accepts any price $\geq \delta\bar{b}/(1 + \delta)$.

Observe that, in the case of no prior buyer deviations, the seller follows the equilibrium by the same argument as in the proof of Theorem 2. The buyer of type b follows the equilibrium in period zero if and only if (16) is satisfied, since the left side gives the buyer's utility from following the equilibrium path and the right side gives profits from optimally deviating. Suppose that (14) is satisfied with strict inequality. Then there exists $\alpha > 0$ such that:

(17) $\bar{b}p(\bar{b}) - x(\bar{b}) > \bar{b}/2 + \alpha.$

The incentive compatibility of $\{p,x\}$ implies, for all $b < \bar{b}$,[16] that: $bp(b) - x(b) \geq bp(\bar{b}) - x(\bar{b})$. Subtracting $\bar{b}p(\bar{b}) - x(\bar{b})$ from each side yields:

(18) $[bp(b) - x(b)] - [\bar{b}p(\bar{b}) - x(\bar{b})] \geq [b - \bar{b}]p(\bar{b}) \geq b - \bar{b},$

since $p(\bar{b}) \leq 1$ and $[b - \bar{b}]$ is negative. But adding (17) and (18) yields:

(19) $bp(b) - x(b) \geq b - \bar{b}/2 + \alpha$, for all $b \leq \bar{b}$.

As in Theorem 2, $[bp(\sigma_n;b) - x(\sigma_n;b)]$ converges uniformly to $[bp(b) - x(b)]$, implying (16) for all $b \leq \bar{b}$ and for sufficiently large N. Thus, no buyer type deviates from the equilibrium σ_n, for $n \geq N$.

Now suppose that (14) is only satisfied with equality. Observe that there exists a sequence of ICBMs $\{p^m,x^m\}_{m=1}^{\infty}$, uniformly converging to $\{p,x\}$, such that $\bar{b}p^m(\bar{b}) - x^m(\bar{b}) > \bar{b}/2$ for every m. For every m there exists $\{\sigma_n^m\}_{n=1}^{\infty}$ and N(m) such that no buyer type deviates from σ_n^m whenever $n \geq N(m)$.

We conclude that the sequence $\{\sigma^m_{N(m)}\}^{\infty}_{m=1}$ has the desired properties.

Finally, suppose that the buyer has ever counteroffered a positive price. The prescribed strategies are a minor adaptation of the Rubinstein [20] (complete information) strategies, and can be shown to form a sequential equilibrium by a similar argument. []

7. The Buyer-Offer Game

We have just seen that in the alternating-offer and seller-offer games, sequential equilibria implement a rather large set of ICBMs, including both relatively efficient and inefficient mechanisms. It may thus come as a surprise that in the buyer-offer game, the inefficiency as well as the multiplicity necessarily disappear. In Theorem 4, we prove that the unique sequential equilibrium has the buyer offer zero, and the seller accept:

Theorem 4: A direct bargaining mechanism $\{p,x\}$ is implementable by sequential equilibria of the buyer-offer game if and only if:

$$p(b) = 1 \text{ and } x(b) = 0, \text{ for all } b \in \text{supp } F.$$

Proof: Let $\bar{Q}$ denote the supremum of all prices (offered by buyers) which are ever rejected with positive probability (by the seller), in any equilibrium after any history. We will first prove the "only if" part of the theorem by showing that $\bar{Q} = 0$; recall, by Lemma 1, that the buyer never offers $q < 0$.

Suppose, to the contrary, that $\bar{Q} > 0$. Observe that for every $\alpha > 0$ there exists a price q, greater than $\bar{Q} - \alpha$, and a history such that if the buyer offers q after this history, the seller will reject with positive probability. In particular, we may pick $\alpha = (1 - \delta)\bar{Q}$, where $\delta = e^{-rz}$. For the seller to reject the price $q > \delta\bar{Q}$ in equilibrium, it must be that he expects a (higher) price q' with positive probability in the future. Since q' is offered at least one period later, it must be that $\delta q' > q$. But then $\delta q' > \delta\bar{Q}$, and $q' > \bar{Q}$. However, no rational buyer will ever offer such a price, since the offer $(q' + \bar{Q})/2$ would be accepted with probability one. We conclude that $\bar{Q} = 0$.

To prove the "if" part of the theorem, observe that for <u>any</u> $z > 0$, it is an equilibrium for the buyer always to offer zero, and the seller to accept any price that is at least zero.[17] Thus $p(b) \equiv 1$ and $x(b) \equiv 0$ on supp F is implementable. [ ]

Observe that the proof of Theorem 4 depends in no way on the assumption that $\underline{b} = s$.

8. Conclusion

In this paper, we examined three incomplete information bargaining games which, while sequential and infinite-horizon, were not literally "repeated" games. We have proven, in the seller-offer game, that a folk theorem <u>literally</u> holds, as the time interval between successive offers approaches zero. In the alternating-offer game, there is also a continuum of sequential equilibria, but <u>not</u> a folk theorem: only a strict subset of all ICBMs are implementable. Finally, in the buyer-offer game, we are much further distanced from the realm of folk theorems: there exists one unique sequential equilibrium.

Observe that our results coincide nicely with the notion that the exclusive right to make offers confers bargaining power. The set of sequential equilibria of the seller-offer game is quite large, but includes the ICBM which maximizes the seller's ex ante expected surplus. Addition of buyer counteroffers to the seller-offer game removes from the equilibrium set those elements which are most one-sidedly unfavorable to the buyer--including, under certain distributions (but not the uniform distribution), the ICBM which maximizes ex ante seller surplus. Meanwhile, the unique equilibrium of the buyer-offer game is the unique ICBM which is most favorable to all buyer types, and coincides with the unique subgame perfect equilibrium of the buyer-offer game under complete information.

An interesting question that remains for future research is: What characterization theorems (expressed in direct mechanism terms) can be established for other sequential games of incomplete information, and when do folk theorems hold?

The immediate place to ask such a question is in other sequential bargaining games. First, this paper explicitly leaves open the characterization of the set of sequential equilibria of the alternating-offer game when $\underline{b} > s$.[18] Second, one would like to know whether the Folk Theorem holds in sequential bargaining games with two-sided incomplete information. We will specifically address the latter question in a future paper.

Appendix

Proof of Theorem 2

If {p,x} is implementable, then there exists a sequence $z_n \downarrow 0$ and a sequence of sequential equilibria inducing, for each n, an ICBM $\{p_n, x_n\}$ such that $\lim_{n\to\infty} (p_n(b), x_n(b)) = (p(b), x(b))$ for all $b > 0$. Observe that ICBM's satisfy (1) and (2). Since these inequalities are preserved by the limiting operation, $U(b|b') \le U(b)$ for all b, $b' \neq 0$, where $U(b|b')$ is the utility to b posing as type b'. Obviously, $U(b|b') \le U(b)$ at $b = 0$. Finally, observe that $U(b|0) \le \lim_{b'\downarrow 0} U(b|b') \le U(b)$, since $p(0) \le \lim_{b'\downarrow 0} p(b')$. Thus {p,x} is incentive compatible. Lemma 1 implies that it is also (ex post) individually rational.

Conversely, consider any ICBM {p,x}. We will show the existence of a sequence $z_n \downarrow 0$, and a sequence of sequential equilibria σ_n (corresponding to a time interval between offers of z_n), with the property that for all $\varepsilon > 0$, there exists $\bar{n}(\varepsilon)$ such that for all $n \ge \bar{n}(\varepsilon)$: $|p_n(b) - p(b)| < \varepsilon$ for all $b \in \text{supp } F$ such that $b > \varepsilon$; and $|x_n(b) - x(b)| < \varepsilon$ for all $b \in \text{supp } F$.

We first need the following definitions:

$$\mathcal{P} = \text{closure } \{p(b)\text{: } b \in \text{supp } F \text{ and } b > 0\}, \text{ and}$$

$$R_{k,n} = \min\{\pi \in \mathcal{P}\text{: } \pi \ge k/n\} \cup \max\{\pi \in \mathcal{P} \cap [1/n, 1]\text{: } \pi \le k/n\},$$

where n is a positive integer, and min(•) and max(•) are taken to be the empty set if their arguments are empty. Also let $\hat{\mathcal{P}}_n = \cup_{k=1}^{n} R_{k,n}$. Observe that $\hat{\mathcal{P}}_n$ has at most 2n elements. Furthermore, if there is a "gap" in $\mathcal{P}$ with length at least $1/n$, then both endpoints of that gap are contained in $\hat{\mathcal{P}}_n$. Also define $b(\pi) = \{b \in [0,\bar{b}]: p(b) = \pi\}$, for all π in the range of $p(\bullet)$, and $b(\pi) = \lim_{n\to\infty} b(\pi_n)$, for those π that are limits of $\pi_n \in \mathcal{P}$, but are not contained in the range of $p(\bullet)$. Observe that $b(\pi)$ is set-valued.

Let $b^1(\pi) = \sup b(\pi)$ and $b^2(\pi) = \inf b(\pi)$. Define: $\mathcal{P}_n = \{\pi \in \hat{\mathcal{P}}_n: U(b^2(\pi)) > b^2(\pi)/n$ and $x(b^2(\pi))/p(b^2(\pi)) > 1/n\}$; and $\mathcal{B}_n = \{\beta: \beta = b^1(\pi)$ or $\beta = b^2(\pi)$, for some $\pi \in \mathcal{P}_n\}$. For any $z > 0$ and $\pi \in \mathcal{P}_n$, let $t_z^1(\pi) = \sup\{kz: e^{-rkz} \geq \pi$, for integer $k\}$ and $t_z^2(\pi) = t_z^1(\pi) + z$. Also, for all $\pi \in \mathcal{P}_n$ and $i = 1,2$, define $s_z^i(\pi)$ by:

$$(20) \qquad U(b^i(\pi)) = e^{-rt_z^i(\pi)}[b^i(\pi) - s_z^i(\pi)].$$

Select $\bar{z}_n^1 > 0$ such that for all z $(0 < z < \bar{z}_n^1)$: $|t_z^1(\pi) - t_z^1(\pi')| > 2z$ for all $\pi \neq \pi'$ $(\pi,\pi' \in \mathcal{P}_n)$; $s_z^i(\pi) > 1/n$ for all $\pi \in \mathcal{P}_n$ and $i = 1,2$; and $1 - e^{-rz} < 1/n$. Define $\tilde{\pi}_n = \min \mathcal{P}_n$, if $\mathcal{P}_n \neq \emptyset$, and $\tilde{\pi}_n = 1/n$ otherwise. Also let $\tilde{b}_n = \min \mathcal{B}_n$, if $\mathcal{B}_n \neq \emptyset$, and $\tilde{b}_n = 0$ otherwise. Furthermore, let $T_{n,z} = \inf\{kz: e^{-rkz} \leq 1/n$, for some integer $k\} + t_2^z(\tilde{\pi}_n)$.

We now construct a sequence of offers which the seller is supposed to make at times t, where $t \in \{0,z,2z,3z,\ldots\}$:

$$q_{n,z}(t) = \begin{cases} s_z^i(\pi) & \text{, if } t = t_z^i(\pi) \text{ and } \pi \in \mathcal{P}_n\text{, for } i = 1,2 \\ (1 - e^{-rz}) \sum_{k=0}^{\infty} e^{-rkz} F^{-1}[e^{-(m+k)z}F(1/n)] & \text{, if } t = T_{n,z} + mz\text{, for some nonnegative integer } m \\ \bar{b} & \text{, otherwise,} \end{cases}$$

where $F^{-1}(y) \equiv \inf\{b: F(b) \geq y\}$.

Consider a strategy for the seller in which he charges $q_{n,z}(t)$ if he has not deviated from that path in any prior period, and in which he reverts to a weak-Markov equilibrium otherwise. Buyers optimize given their predictions derived from the seller's strategy and, in case of indifference, choose the purchase time closest to what the mechanism $\{p,x\}$ prescribes. To be precise, let $\Upsilon(n,z;\beta) = \text{argmax}_t\{e^{-rt}(\beta - q_{n,z}(t))\}$, for $\beta \in (0,\bar{b}]$ $\cap$ supp F, and $\Upsilon(n,z;\beta) = +\infty$ for $\beta = 0$. For β such that $\Upsilon(n,z;\beta)$ is single-valued define the purchase time $\tau(n,z;\beta) = \Upsilon(n,z;\beta)$, and for β such that $\Upsilon(n,z;\beta)$ is multiple-valued mix between the lowest and highest members of Υ, to yield a purchase probability as close to $p(\beta)$ as possible. Concretely, choose $\theta \in [0,1]$ such that $|\theta e^{-rc} + (1 - \theta)e^{-rd} - p(\beta)|$ is minimal, where c and d are the lowest and highest element in $\Upsilon(n,z;\beta)$, respectively.[19] By Ausubel and Deneckere [2], there exists $\bar{z}_n^2 > 0$ such that for all $0 < z \leq \bar{z}_n^2$, $q_{n,z}(t)$ is a sequential equilibrium price path for all periods starting at time $T_{n,z}$. Let Π be the seller's expected profits along the sequential equilibrium starting at $T_{n,z}$, discounted back to period zero. Observe that $\Pi > 0$. By the Coase Conjecture there exists $\bar{z}_n^3 > 0$ such that for all $0 < z \leq \bar{z}_n^3$, and for every weak-Markov equilibrium of the seller-offer game (in which all potential buyers are still present), the seller's expected profits in the initial period are less than Π. Hence the seller has no incentive to deviate from $q_{n,z}(t)$ in any period t provided z is no larger than $\min(\bar{z}_n^1,\bar{z}_n^2,\bar{z}_n^3,\bar{z}_n^4)$, where $\bar{z}_n^4 > 0$ will be defined below.

Next, we show that for every $\varepsilon > 0$ there exists $\bar{n}(\varepsilon)$ such that for every $n \geq \bar{n}(\varepsilon)$, $|p_n(b) - p(b)| < \varepsilon$ if $b \geq \varepsilon$. In other words, we show that $p_n(\bullet)$ converges uniformly to $p(\bullet)$ on $[\varepsilon,\bar{b}]$, and this for every $\varepsilon > 0$. Recall, however, that any sequential equilibrium has the property that sales occur over infinite time, and thus that the graph of $p_n(\bullet)$ is anchored at the origin. Define $\tilde{b} = \inf\{b \in \text{supp } F: p(b) > 0\}$, and observe that $\tilde{b}_n \rightarrow \tilde{b}$. If $\lim_{b \downarrow \tilde{b}} p(b) = 0$, no special problems are encountered in obtaining $p_n(\bullet)$'s which have the right convergence properties, since the graph of the limiting $p(\bullet)$ is also anchored at the origin (this is Case 1). In Case 2, however, there is a discontinuity in $p(b)$ at the point $\tilde{b} > 0$. In order to obtain uniform convergence, then, we must construct equilibria which ensure that all $b < \tilde{b}$ have a small probability of trade, whereas all $b > \tilde{b}$ must have a large probability of trade. We accomplish this by making $\tilde{b}$ indifferent between an offer at a time approximately corresponding to $\tilde{p}$, and the next price accepted. In Case 3, a somewhat similar problem arises since there is (essentially) a discontinuity in $p(b)$ at the origin. The previous approach no longer works, however, since $\tilde{b}$ cannot be made to accept any positive offers. We circumvent this difficulty by letting a type $\hat{b}_n$ be the indifferent type, where $\hat{b}_n \rightarrow 0$ as $n \rightarrow \infty$. We now consider the three cases in sequence.

<u>Case 1</u>: Suppose $\lim_{b \downarrow \tilde{b}} p(b) = 0$.

Observe that if $b \in \mathcal{B}_n$, then b can attain $U(b)$ by purchasing at one of the prices offered before $T_{n,z}$. Indeed, <u>every</u> price offered before $T_{n,z}$ gives some type $b \in \mathcal{B}_n$ the utility he achieves under the mechanism $\{p,x\}$. Furthermore, since $U(b) > b/n$, type b does better purchasing before $T_{n,z}$ than after, for all z, i.e., $\tau(n,z;b) < T_{n,z}$.

Suppressing z, define $p_n(\beta) = e^{-r\tau(n,z;\beta)}$. We will now prove the following claim: there exists $\bar{z}_n^4 > 0$ s.t. for all z ($0 < z < \bar{z}_n^4$) and for all $\beta \in \text{supp } F \cap [\tilde{b}_n, \bar{b}]$, we have $|p_n(\beta) - p(\beta)| < 3/n$. First, we will demonstrate the claim for all $\beta \in \mathcal{B}_n$.

Let $U(\beta|t)$ denote the payoff to β of purchasing at time t. We need to show that if $\pi \in \mathcal{P}_n$ and $|\pi - p(\beta)| > 1/n$, then:

$$(21) \qquad U(\beta|t_z^i(\pi)) \le \max_{j=1,2} \{U[\beta|t_z^j(p(\beta))]\}.$$

Observe that the right side equals $U(\beta)$, since $\beta \in \mathcal{B}_n$, and that the left side converges to $\pi\beta - x(b^i(\pi))$, as $z \to 0$.

(i) Suppose $\pi\beta - x(b^i(\pi)) < U(\beta)$. Then there exists $\bar{z}(\beta,\pi)$ such that for all z $(0 < z < \bar{z}(\beta,\pi))$, (21) is satisfied with strict inequality.

(ii) Suppose $\pi\beta - x(b^i(\pi)) = U(\beta)$, and $|\pi - p(\beta)| > 1/n$. Without loss of generality, let $\pi > p(\beta)$. Observe that $b^2(\pi) = \beta = b^1(p(\beta))$, since any $b > \beta$ prefers π over $p(\beta)$ and any $b < \beta$ prefers $p(\beta)$ over π. By construction, β is indifferent between purchasing at $t_z^2(\pi)$ and $t_z^1(p(\beta))$, since both give $U(\beta)$. It is also fairly straightforward to show that β does strictly worse by purchasing at $t_z^1(\pi)$. We conclude that $U(\beta|t_z^i(\pi)) \le U(\beta|t_z^1(p(\beta)))$, for i = 1,2, and (21) follows.

(iii) The case $\pi\beta - x(b^i(\pi)) > U(\beta)$ is vacuous.

Finally, observe that $\Lambda \equiv \{(\beta,\pi)\colon \beta \in \mathcal{B}_n,\ \pi \in \mathcal{P}_n \text{ and } \pi\beta - x < U(\beta)\}$ is finite. Define $\bar{z}_n^4 = \min\{\bar{z}(\beta,\pi)\colon (\beta,\pi) \in \Lambda\}$. We have just shown that β purchases at a time which corresponds to a probability in the mechanism within 1/n of $p(\beta)$. We previously assumed that z is such that $(1 - e^{-rz}) \le 1/n$. We may conclude that $|p_n(\beta) - p(\beta)| \le 2/n$, for $\beta \in \mathcal{B}_n$, and $0 < z \le \bar{z}_n^4$.

Second, we will demonstrate the claim for all $\beta \in \text{supp } F \cap [\tilde{b}_n, \bar{b}]$. Let $\beta' = \max\{b \in \mathcal{B}_n \text{ s.t. } b \le \beta\}$ and $\beta'' = \min\{b \in \mathcal{B}_n \text{ s.t. } b \ge \beta\}$. By Fudenberg, Levine and Tirole's [12] Lemma 1, β purchases no later than β' and no earlier than β'' in <u>any</u> sequential equilibrium. We have already shown that β' purchases no later than a time yielding a purchase probability of $p(\beta') - 2/n$, and β'' purchases no earlier than a time associated with a probability of $p(\beta'') + 2/n$. Observe also that since $p(\beta) \in \mathcal{P}$, and by the definition of $\mathcal{P}_n$, we must have $p(\beta') > p(\beta) - 1/n$ and $p(\beta'') < p(\beta) + 1/n$. We conclude that the time at which β purchases in the sequential equilibrium translates to a probability between $p(\beta) - 3/n$ and $p(\beta) + 3/n$.

Finally, choose arbitrary $\varepsilon > 0$. We may pick $\bar{n}(\varepsilon)$ such that $3/n < \varepsilon$ and $p(\tilde{b}_n) < \varepsilon$ for all $n \geq \bar{n}(\varepsilon)$. Every equilibrium constructed above satisfies $|p_n(b) - p(b)| \leq \varepsilon$, for all $b \in \text{supp } F$ and $n \geq \bar{n}(\varepsilon)$. This concludes Case 1.

<u>Case 2</u>: Suppose $\tilde{b} > 0$ and $\tilde{p} \equiv \lim_{b \downarrow \tilde{b}} p(b) > 0$.

Restrict attention to n such that $1/n < \min(\tilde{p},\tilde{b})$. Repeat the construction of the previous price path, <u>except</u> define $s_z^2(\tilde{\pi}_n)$ by:

$$e^{-rt_z^2(\tilde{\pi}_n)}[\tilde{b} - s_z^2(\tilde{\pi}_n)] = e^{-rT_{n,z}}[\tilde{b} - q_{n,z}(T_{n,z})].$$

First, we claim that the buyer of type $\tilde{b}$ is indifferent between purchasing at $t_z^2(\tilde{\pi}_n)$ and $T_{n,z}$, and for sufficiently small z, strictly prefers these two times to all others. Observe:

(i) $\tilde{b}$ gets positive surplus buying either at $T_{n,z}$ or $t_z^2(\tilde{\pi}_n)$, and is indifferent between them, by construction.

(ii) $\tilde{b}$ strictly prefers $T_{n,z}$ to all later times. This follows from our construction of the endgame which makes the buyer of type 1/n purchase at either $T_{n,z}$ or $T_{n,z} + z$ (he is indifferent between them). By hypothesis,

$\tilde{b} > 1/n$, so type $\tilde{b}$ strictly prefers $T_{n,z}$ to $T_{n,z} + z$ and later times.

(iii) $\tilde{b}$ strictly prefers $t_z^2(\tilde{\pi}_n)$ to all earlier times, for sufficiently small z. Observe that prices charged at all times before $t_z^2(\tilde{\pi}_n)$ are defined to give some type b ($b \geq \tilde{b}$) the payoff he receives in the mechanism $\{p,x\}$. Also note that all prices in the mechanism associated with positive purchase probabilities are greater than or equal to $\tilde{b}$. Hence, for any $\varepsilon > 0$, there exists $\bar{z}_\varepsilon > 0$ such that for all z ($0 < z < \bar{z}_\varepsilon$), the prices charged in the price path are greater than $\tilde{b} - \varepsilon$, yielding $\tilde{b}$ surplus less than ε. Meanwhile, the price charged at $t_z^2(\tilde{p})$ has been chosen to give $\tilde{b}$ the same surplus he would receive as by purchasing at $T_{n,z}$ (at a price less than $1/n$, since type $1/n$ purchases at $T_{n,z}$). Note that $T_{n,z}$ is uniformly bounded in z, so purchasing at $t_z^2(\tilde{p})$ gives $\tilde{b}$ surplus which is bounded away from zero. We conclude that for sufficiently small ε (and $\bar{z}_\varepsilon$), $\tilde{b}$ strictly prefers to purchase at $t_z^2(\tilde{p})$ to earlier. This concludes the proof of the claim.

The claim establishes that for all n, all types $\beta < \tilde{b}$ prefer to purchase at time $T_{n,z}$ or later. Note that agents discount time $T_{n,z}$ by a factor of $1/n$, so: $|p_n(\beta)| < 1/n$, for all $\beta < \tilde{b}$, i.e., $|p_n(\beta) - p(\beta)| < 1/n$, for all $\beta < \tilde{b}$. The claim also establishes that all types $\beta > \tilde{b}$ prefer to purchase at time $t_z^2(\tilde{\pi})$ or earlier.

Define $\hat{b}_n = \sup\{b\colon t_z^2(\tilde{\pi}_n) = \tau(n,z;b)\}$. Observe that $\hat{b}_n$ converges to $b^1(\tilde{p})$, as $n \to \infty$. Choose arbitrary $\varepsilon > 0$. There exists $\bar{n}_1(\varepsilon)$ such that $|\tilde{\pi}_n - \tilde{p}| < \varepsilon/2$, and $|p(\hat{b}_n) - \tilde{p}| < \varepsilon/2$, for all $n \geq \bar{n}_1(\varepsilon)$. Hence $|p_n(b) - p(b)| = |\tilde{\pi}_n - p(b)| \leq |\tilde{\pi}_n - \tilde{p}| + |\tilde{p} - p(b)| < \varepsilon$ for all $b \in (\tilde{b}, \hat{b}_n]$.

By reasoning identical to Case 1, there exists $\bar{z}$ sufficiently small such that $|p_n(b) - p(b)| < 3/n$, for $b \in (\hat{b}_n, \bar{b}]$, and $0 < z < \bar{z}$. Finally, note that $\tilde{b}$ can be made to purchase with probability near $p(\tilde{b})$ by mixing between purchasing at $t_z^2(\tilde{\pi}_n)$ and $T_{n,z}$, since $0 \le p(\tilde{b}) \le \tilde{p}$. We may thus conclude that there exists $\bar{z}_n^4 > 0$ such that for all z ($0 < z < \bar{z}_n^4$) and for all $\beta \in \text{supp } F \cap [[0,\tilde{b}] \cup (\hat{b}_n,\bar{b}]]$, $|p_n(\beta) - p(\beta)| < 3/n$. Finally, let $\varepsilon > 0$ be arbitrary. Pick $n \ge \bar{n}(\varepsilon)$ such that $3/n \le \varepsilon$. We then have constructed equilibria that satisfy, for $n \ge \bar{n}(\varepsilon)$ and $0 < z \le \bar{z}^4_{n(\varepsilon)}$, $|p_n(\beta) - p(\beta)| < \varepsilon$, for all $\beta \in \text{supp } F$. This concludes Case 2.

<u>Case 3</u>: Suppose $\tilde{b} = 0$ and $\tilde{p} \equiv \lim_{b \downarrow \tilde{b}} p(b) > 0$.

We restrict attention to the case where $b^1(\tilde{p}) > 0$, i.e., there is a mass point of people buying at a zero price (if $b^1(\tilde{p}) = 0$, the proof proceeds just as in Case 1). Let n be sufficiently large that $b^1(\tilde{p}) > 1/n$, and $U(b^1(\tilde{p})) > b^1(\tilde{p})/n$. In what follows, assume that $\mathcal{P}_n \ne \emptyset$ for some $n < \infty$.[20]

Let $t^* = \min\{kz\colon e^{-rkz}[b^1(\tilde{p}) - 1/n] < U(b^1(\tilde{p}))$, for some nonnegative integer $k\}$, and define $q(t^*)$ such that $U(b^1(\tilde{p})) = e^{-rt^*}[b^1(\tilde{p}) - q(t^*)]$. Note that $q(t^*) < 1/n$, but very close to $1/n$ when z is small. Also, let $T_{n,z} = \inf\{kz\colon e^{-rkz} \le 1/n$, for some integer $k\} + t^*$. Then, for sufficiently large n, $\mathcal{B}_n \ne \emptyset$, and $\tilde{b}_n > b^1(\tilde{p})$. Observe now that:

(i) If $\beta < b^1(\tilde{p})$, β purchases at time t^* or later, provided z is sufficiently small. Indeed, for such z, t^* is an optimal purchase time for $b^1(\tilde{p})$. Purchasing at t^* yields $U(b^1(\tilde{p}))$. Purchasing after t^* yields a surplus of at most $b^1(\tilde{p})/n$, which is less than $U(b^1(\tilde{p}))$. Finally, all earlier prices either yield exactly $U(b^1(\tilde{p}))$, or approximately $U(b^1(\tilde{p})|b)$ for some $b \in \mathcal{B}_n$. By incentive compatibility of the mechanism, the latter options yield $b^1(\tilde{p})$ strictly lower utility, for sufficiently small z. Hence, if $\beta < b^1(\tilde{p})$, β strictly prefers to buy at t^* or later.

(ii) If $\beta > \tilde{b}_n$, β purchases at a time corresponding to $\tilde{\pi}_n$ or earlier, by similar reasoning as in (i).

Consequently, if $(\bar{b} + 1)/n < \beta < b^1(\tilde{p})$, β purchases at t*. Indeed, since the price at t* is less than 1/n, buying at t* gives a surplus of at least: $e^{-rt*}[\beta - 1/n] > e^{-rt*}(\bar{b}/n)$. The next price after t* is not offered until $T_{n,z}$, and thus, buying after t* yields surplus of at most $e^{-rt*}(\beta/n) < e^{-rt*}(\bar{b}/n)$. Also, if $\beta > \tilde{b}_n$, we obtain $|p_n(\beta) - p(\beta)| < 3/n$, by reasoning similar to that of Case 1.

Finally, consider $\beta \in [b^1(\tilde{p}),\tilde{b}_n]$. First, we treat the case where $\lim_{b \downarrow b^1(\tilde{p})} p(b) = \tilde{p}$. Then, for any $\varepsilon > 0$, there exists $\bar{n}(\varepsilon)$ such that for all $n \geq \bar{n}(\varepsilon)$, $|p(\tilde{b}_n) - \tilde{p}| < \varepsilon/2$ and $2/n < \varepsilon/2$. Since β purchases at either t* or a time corresponding to $p(\tilde{b}_n)$, we obtain $|p_n(\beta) - p(\beta)| < \varepsilon$.

Second, consider $\beta \in [b^1(\tilde{p}),\tilde{b}_n]$, and assume $\lim_{b \downarrow b^1(\tilde{p})} p(b) \equiv \hat{p} > \tilde{p}$. By incentive compatibility of the mechanism, $b^1(\tilde{p})\tilde{p} \geq b^1(\tilde{p})\hat{p} - \hat{x}$, where $\hat{x}$ is the discounted price associated with $\hat{p}$, and 0 is associated with $\tilde{p}$. Thus, $\hat{x} \geq b^1(\tilde{p})[\hat{p} - \tilde{p}] > 0$, and the price $\hat{x}/\hat{p}$ is bounded away from zero. Hence, for sufficiently large n, $\hat{p} \in \mathcal{P}_n$ and so $\tilde{b}_n = b^2(\hat{p})$. Our argument above has treated all $\beta \in [(\bar{b} + 1)/n,\bar{b}] \cap \text{supp } F$, except possibly $b^1(\tilde{p})$ and $b^2(\hat{p})$. If $b^1(\tilde{p}) = b^2(\hat{p})$, observe that for sufficiently large n, $b^1(\tilde{p})$ attains the same payoff at t* as at $t_z^2(\hat{p})$, and via mixed strategies, can be made to buy with any probability p satisfying $\tilde{p} \leq p \leq \hat{p}$. If $b^1(\tilde{p}) < b^2(\hat{p})$, then by definition, $b^1(\tilde{p})$ strictly prefers $\tilde{p}$ to $\hat{p}$ in the mechanism, and the reverse for $b^2(\hat{p})$. Since the constructed sequential equilibrium approximates the mechanism, the same strict preference relationship will hold here, for sufficiently large n. This concludes the proof of Case 3, as we have shown that for arbitrary $\varepsilon > 0$ there exists sufficiently large $\bar{n}(\varepsilon)$ such that

$|p_n(b) - p(b)| < \varepsilon$, for all $n \geq \bar{n}(\varepsilon)$ and all $b \in [(\bar{b} + 1)/n, \bar{b}] \cap \text{supp } F$.

Note that n can be chosen sufficiently large that $(\bar{b} + 1)/n < \varepsilon$. Hence, we conclude that for every $\varepsilon > 0$, there exists $\bar{n}_1(\varepsilon)$ such that for all $n \geq \bar{n}_1(\varepsilon)$ and for all $b \in \text{supp } F$ satisfying $b > \varepsilon$, $|p_n(b) - p(b)| < \varepsilon$. Lemma 2 below then implies there exists $\bar{n}_2(\varepsilon)$ such that for all $n \geq \bar{n}_2(\varepsilon)$ and all $b \in \text{supp } F$, $|x_n(b) - x(b)| < \varepsilon$. This ends the proof of Theorem 2. []

<u>Lemma 2</u>. Suppose $|p_n(b) - p(b)| < \varepsilon$, for all $b \in \text{supp } F$ such that $b > \varepsilon$, and $bp_n(b) - x_n(b) = U(b)$ for all $b \in \mathcal{B}_n$. Then $|x_n(b) - x(b)| < \bar{b}(3\varepsilon + 2/n)$, for all $b \in \text{supp } F$.

<u>Proof</u>: First observe that for all $b \leq \varepsilon$, $|x_n(b) - x(b)| \leq \varepsilon$, since both $x_n(b)$ and $x(b)$ are in the interval $[0,b]$. Second, for $b \in \mathcal{B}_n$, $U(b) - bp_n(b) + x_n(b) = b(p(b) - p_n(b)) - (x(b) - x_n(b)) = 0$, implying $|x_n(b) - x(b)| = b|p_n(b) - p(b)| \leq \bar{b}\varepsilon$. Third, by incentive compatibility of the sequential equilibrium, for any b and $b' \in \text{supp } F$: $bp_n(b) - x_n(b) \geq bp_n(b') - x_n(b')$; and $b'p_n(b') - x_n(b') \geq b'p_n(b) - x_n(b)$. These two inequalities imply that $|x_n(b) - x_n(b')| \leq \bar{b}|p_n(b) - p_n(b')|$.

Finally, consider any $b \in \text{supp } F$ such that $b > \varepsilon$. Observe that there exists $\beta \in \mathcal{B}_n$ such that $|p(b) - p(\beta)| < 1/n$. Note that: $|x_n(b) - x(b)| \leq |x_n(b) - x_n(\beta)| + |x_n(\beta) - x(\beta)| + |x(\beta) - x(b)|$. The first term on the right side is bounded by $\bar{b}|p_n(b) - p_n(\beta)|$, which in turn is bounded above by $\bar{b}(2\varepsilon + 1/n)$. The second term is bounded above by $\bar{b}\varepsilon$, and the last by $\bar{b}/n$. Thus, $|x_n(b) - x(b)| \leq \bar{b}(3\varepsilon + 2/n)$. []

Notes

1. Two-sided incomplete information is not considered here. For some recent treatments, see Cramton [10] and Chatterjee-Samuelson [6, 7].
2. We do not study simultaneous-move trading rules, since even in the one-shot complete information case these rules permit every division of the pie as an equilibrium.
3. This assumption is what makes the game form dynamic: not only do the probabilities of trade and the expected payment matter, but also the time at which the trade is concluded!
4. The terminology here is from Cramton [11].
5. Fudenberg, Levine and Tirole's [12] Lemma 2 implies this result, for the seller-offer game.
6. Without the condition $p(0) \leq \lim_{b\downarrow 0} p(b)$, our definition would impose no restrictions on the behavior of $p(\bullet)$ at 0, due to the asymmetric treatment of $b = 0$ in (7) and (8) below. Observe that any incentive compatible bargaining mechanism necessarily satisfies the condition.
7. Our definition of implementation is equivalent to uniform convergence if and only if $\lim_{b\downarrow 0} p(b) = 0$.
8. In fact, the same is true with any offer structure which allows the seller to make offers infinitely often, whenever $s = \underline{b}$.
9. We depart here slightly from the notation in [2]. The distribution function $F(b)$ here is related to the indirect demand function $f(q)$ in [2] through the transformation $F(b) = 1 - f^{-1}(b)$. Thus, $\mathcal{F}_{L,M}$ in the present paper corresponds to $\mathcal{F}_{1/M,1/L}$ in [2].
10. The proof in [2] literally uses an exponential quantity path in order

to establish the result in full generality.

11. For sufficiently small $z_n^{'}$, the "endgame" is supported as a sequential equilibrium. For sufficiently small $z_n^{''}$, expected surplus in the endgame (discounted to the beginning of the game) exceeds expected surplus along the punishment path. Let $z_n = \min(z_n^{'}, z_n^{''})$.

12. If the seller makes all the offers and if trade occurs over infinite time, there is <u>no</u> (observably) out-of-equilibrium behavior for the informed player.

13. For example, Grossman and Perry [13] state in footnote 2, "We do not analyze a concession game, i.e., a game where only one party can make offers and the other only accepts or rejects, because the party that is unable to make counteroffers is artificially restricted."

14. We specifically make use of the assumption in the definition of sequential equilibrium that the support of the seller's (updated) beliefs about the buyer's valuation never moves outside supp F.

15. Fudenberg, Levine and Tirole ([12], Section 5.4) used a similar technique. See also Section 4 of Gul and Sonnenschein [14].

16. This technique owes to Myerson and Satterthwaite ([19], Section 2).

17. See Wilson [24], pp. 43-44 and 66-67.

18. We have explicitly treated the case $\underline{b} = s$. Section 7 also solves the buyer-offer game when $\underline{b} > s$; Fudenberg-Levine-Tirole [12] and Gul-Sonnenschein-Wilson [15] solve the seller-offer game when $\underline{b} > s$.

19. Mixed strategies along the equilibrium path could be avoided by introducing extra prices offered between c and d, but only at considerable notational expense.

20. If $\mathcal{P}_n = \emptyset$ for all n, use the construction below, with $q(t^*) = 1/n$.

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