Discussion Paper No. 726

DEMAND FOR DIFFERENTIATED PRODUCTS, DISCRETE CHOICE MODELS, AND THE ADDRESS APPROACH

by

Simon P. Anderson*
André de Palma**
Jacques-François Thisse***

April 1987

---

*CEME, Universite Libre de Bruxelles, Bruxelles, Bruxelles, BELGIUM.

**Department of Civil Engineering, Northwestern University, Evanston, Illinois 60201, USA.

***Core, Universite Catholique de Louvain, Louvain la Neuve, BELGIUM.

The authors thank J. Dreze and C. Michelot for stimulating discussions. The research of the first two authors was supported by the "Projet d'Action Concertee" of the Belgian government under contract 84/8965.

1. INTRODUCTION

Aggregate demand systems (a.d.s.) are widely used in both theoretical and empirical work on product differentiation. The a.d.s. can be taken as a primitive which obeys some a priori "reasonable" properties such as the gross substitutes property, some kind of Chamberlinian symmetry, etc. (see, e.g. Shubik (1959) and Friedman (1977)). However, for many purposes, it is useful to have a theory of the microeconomic underpinnings, from the viewpoint of individual consumers, of such an a.d.s. In particular, with a disaggregated model, further insight may be gleaned about the rôle of individual preferences and incomes, and moreover, exact welfare analysis becomes possible.

The two prevalent utility-theoretic approaches used in product differentiation are the representative consumer and address models. The representative consumer model (see, e.g. Spence (1976), and Dixit and Stiglitz (1977)) leaves open the question as to the individual preferences it represents since it is itself an a priori aggregate preference ordering. The utility function posited under this approach (most typically a CES-type function) embodies aggregate preference for diversity via a parameter in the utility function (ρ in the CES). The justification of such a model as a description of consumers with specific tastes has not been made explicit.

In the address model, products are described by the bundle of characteristics they embody (see e.g. Lancaster (1979) and Archibald et al. (1986)). Individual preferences are defined over characteristics directly (in the previous approach they are defined over goods) and aggregate preference for diversity is captured by endowing each consumer with a different most preferred product.

A third model was recently suggested by Perloff and Salop (1985) who claimed to synthesize the two previous ones via a probabilistic choice framework, although the precise nature of the synthesis was not made explicit. Their approach is reminiscent of discrete choice theory developed in econometrics (see e.g. Amemiya (1981) and McFadden (1984)). In this theory, individuals face mutually exclusive choice and are assumed to maximize a stochastic utility function. As a result, each individual has a positive probability defined on each choice.

In this paper, we want to explore the linkages between these different approaches. To this end, we choose a simple and specific address model and we identify a set of conditions to be imposed on a given a.d.s. satisfying the gross substitutes property to be generated from our address model. In particular, we show that the dimension of the characteristics space must be large enough compared to the number of variants of the differentiated product for the reconciliation to be possible. We then state some additional conditions to be satisfied by the a.d.s. in order to make it fully consistent with our address model. As an illustration, we consider the *logit, probit* and *linear probability* models and derive their explicit address (and deterministic) representations, thus confirming Manski (1977)'s hypothesis that randomness in observed consumer behavior lies (at least partially) in unobservable characteristics influencing consumer choices. This enables us to provide an alternative to the standard stochastic utility interpretation of these models. In addition, this highlights their implicit restrictions, especially in regard to the meaning of their main parameters.

Finally, we take the a.d.s. generated by the *CES representative consumer* and show explicitly that it can be derived from a specific address model similar to that considered previously. Whereas the CES a.d.s. was loosely based from a consumer-theoretic viewpoint, and criticized for this fact, we now have a concrete disaggregated preference structure which generates it. On top of this, we uncover a strong conceptual link between the CES and logit models.

2. THE MODELS

Model I. Let us consider a demand system defined over n variants of a differentiated product. The demand function of variant i is given by

$$D_i(p_1, \ldots, p_n), \quad i = 1, \ldots, n \tag{1}$$

where p_i is the price of variant i. The corresponding a.d.s. is assumed to obey the following properties :

(A1) D_i is strictly positive for all nonnegative prices, $n-1$ times continuously differentiable and satisfies the gross substitutes property (GSP)

$$\frac{\partial D_i}{\partial p_j} > 0 \quad \text{for all} \quad p_h \geq 0, \quad j \neq i, \quad h, i, j = 1, \ldots, n. \tag{2}$$

Further assumptions will be introduced in the course of the analysis in order to cast the above demand system into the address framework.

Model II. The address model we want to compare to Model I is defined as follows :

(B1) There are m characteristics and $\mathbb{R}^m$ is the characteristics space. The n variants are located at $\underline{z}_1 = (z_1^1, \ldots, z_1^m) \ldots \underline{z}_n = (z_n^1, \ldots, z_n^m)$, with $\underline{z}_i \neq \underline{z}_j$ for all $i, j = 1, \ldots, n$ and $i \neq j$. [1]

(B2) There is a continuum of consumers distributed in $\mathbb{R}^m$ according to a continuous and strictly positive density function $f(\underline{z})$, where $\underline{z} = (z^1, \ldots, z^m)$, with $\int_{\mathbb{R}^m} f(\underline{z}) d\underline{z} = N$, the total population.

(B3) Each consumer purchases one unit of the variant which offers the greatest utility. (This assumption is relaxed in Section 5.) The utility of a consumer located at $\underline{z}$ (where $\underline{z}$ corresponds to its most preferred brand) and purchasing variant i is given by

$$U_i(\underline{z}) = V_i - c \sum_{k=1}^{m} (z^k - z_i^k)^2, \quad i = 1, \ldots, n \tag{3}$$

where c is a positive constant and

$$V_i = \alpha_i - p_i \tag{4}$$

with α_i representing a (one-dimensional) quality index of variant i. The second term on the RHS of (3) is the disutility from not buying the ideal brand $\underline{z}$, which is proportional to the square of the Euclidean distance between $\underline{z}$ and $\underline{z}_i$. (In the geographical context, this disutility corresponds to the transportation cost.) This particular specification of the utility function, which has been used recently in several models of product differentiation (see, e.g. Eaton and Wooders (1985)), has been chosen to allow us to determine in a simple way the basic components of the address framework consistent with Model I. [2]

The *market space* of the variant i is defined as

$$M_i = \{\underline{z} \in \mathbb{R}^m; \quad U_i(\underline{z}) \geq U_j(\underline{z}), \quad j = 1, \ldots, n\}$$

which can be rewritten as (using (4))

$$M_i = \{\underline{z} \in \mathbb{R}^m; V_j - V_i \leq c \sum_{k=1}^{m} (z_j^k - z_i^k)(z_j^k + z_i^k - 2z^k), j = 1, \ldots, n\}. \tag{5}$$

Hence, M_i is the intersection of $n-1$ closed half-spaces, the boundaries of which are hyperplanes orthogonal to the straight lines passing through $\underline{z}_i$ and $\underline{z}_j$, with $j = 1, \ldots, n, \quad j \neq i$.

The *demand for variant i* is therefore defined as

$$X_i = \int_{M_i} f(\underline{z}) d\underline{z}. \tag{6}$$

Our objective in the next section is to find a function $f(\underline{z})$ and a set of points $\{\underline{z}_1, \ldots, \underline{z}_n\}$ consistent with the demand system (A1). The address approach also implies further restrictions on the demand system satisfying (A1).

3. EQUIVALENCE OF THE MODELS

For the demands X_i to be identical to the demands D_i, we must impose some conditions both on the dimension m of the characteristics space and on the positions of the variants $\underline{z}_1, \ldots, \underline{z}_n$. For example, if $m = 1$ and $n = 3$ with $z_1 < z_2 < z_3$, then either $M_2 \neq \emptyset$ and hence $\partial X_1/\partial p_3 = \partial X_3/\partial p_1 = 0$, or $M_2 = \emptyset$ and hence $X_2 = 0$. In both cases, assumption (A1) is violated. Secondly, suppose that $m = 2, n = 4$ and $\underline{z}_1, \ldots, \underline{z}_4$ are as shown in Figure 1. Then, for $p_1 = p_3 < p_2 = p_4$, the market spaces are represented in the figure so that $\partial X_2/\partial p_4 = \partial X_4/\partial p_2 = 0$.

[Insert here Figure 1]

On the other hand, if $m = 2$ and $n = 3$ with $\underline{z}_1, \underline{z}_2$ and $\underline{z}_3$ noncollinear, then all demands are strictly positive and the cross derivatives are always positive. This is illustrated in Figure 2.

[Insert here Figure 2]

More generally, we have the following result :

PROPOSITION 1. For the demands X_i to satisfy the GSP, it must be that $\{\underline{z}_1, \ldots, \underline{z}_n\}$ contains $n-1$ linearly independent points.

The proof is relegated to Appendix I.

This proposition has an important implication : a necessary condition for the GSP to hold is that the number of variants does not exceed the number of characteristics by more than one (the characteristics space is rich enough). When the condition of the proposition is met, we have something akin to the Chamberlinian assumption that a change of the price of one variant is spread out over all others. Otherwise, the characteristics space is crowded by variants and it is not possible for all to be "neighbors" for all prices (the GSP is essentially a neighbor condition in that it requires $M_i \cap M_j \neq \emptyset$ for all prices and all i, j).

Accordingly, we can limit ourselves to $m \geq n-1$. Let us first consider the case $m = n-1$. Given Proposition 1, we may assume without loss of generality that $\underline{z}_1, \ldots, \underline{z}_{n-1}$ form a basis of $\mathbb{R}^m$; furthermore, $\underline{z}_n$ can be chosen arbitrarily. A natural candidate for the basis is the orthonormal one with $\underline{z}_n = \underline{0}$. However, to derive a simple expression for the consumer density function, we make a change of unit and shift the origin in order to have $\underline{z}_i$ defined as follows :

$$z_i^j = \begin{cases} a & \text{if} \quad i = j \\ & \\ -a & \text{otherwise} \end{cases} \quad , i, j = 1, \ldots, n-1 \tag{7}$$

whereas

$$\underline{z}_n = (-a, \ldots, -a) \tag{8}$$

with a being a positive constant reflecting the proximity of variants. For (7) and(8), the market space for variant n is therefore given by

$$M_n = \{\underline{z} \in \mathbb{R}^m; \quad z^j \leq \frac{V_n - V_j}{4ac}, \quad j = 1, \ldots, n-1\} \tag{9}$$

which shows that the set of consumers indifferent between variants n and j is a hyperplane orthogonal to the j-axis at

$$\hat{z}^j = \frac{V_n - V_j}{4ac}. \tag{10}$$

As $m = n$, $\hat{z}$ is univocally determined by $n-1$ linear independent equations. In other words, given (4), there exists a one-to-one correspondence between the vectors $(p_1 - p_n, \ldots, p_{n-1} - p_n)$ and $\hat{\underline{z}}$ in $\mathbb{R}^m$. Observe that consumers located at $\hat{\underline{z}}$ are indifferent among the n variants and that $M_1, \ldots, M_n$ have the form of n polyhedral cones with a common vertex at $\hat{z}$.

We can now construct the demand X_n for variant n as :

$$X_n = \int_{-\infty}^{\hat{z}^1} \cdots \int_{-\infty}^{\hat{z}^{n-1}} f(\underline{z}) dz^1 \ldots dz^{n-1}. \tag{11}$$

In view of (10) and (11), it should be apparent that X_n is only a function of $p_j - p_n, \quad j = 1, \ldots, n-1$. Hence, for $X_n = D_n$, it must be that D_n can be written as $D_n(p_1 - p_n, \ldots, p_{n-1} - p_n)$. In fact, we shown in Appendix II that a similar characterzation holds for $X_i, \quad i = 1, \ldots, n-1$ (see (39)). Consequently, the address framework imposes the following :

(A2) The demand function of variant i has the form

$$D_i(p_1 - p_i, \ldots, p_n - p_i), \quad i = 1, \ldots, n. \tag{12}$$

Furthermore, from (11), we have

$$\frac{\partial^{n-1} X_n}{\partial p_1 \ldots \partial p_{n-1}} = (4ac)^{1-n} f(\hat{z}). \tag{13}$$

Given (13), for $X_n = D_n$, it must be that $\partial^{n-1} D_n / \partial p_1 \ldots \partial p_{n-1}$ exists and, as $f(\hat{z}) > 0$, this expression must be strictly positive. It is shown in Appendix II that the $(n-1)$-th derivative of variant i's demand w.r.t. all other prices is the same for all variants $i = 1, \ldots, n$ (see (40)). Hence we have :

$$\text{(A3)}\ \varphi(p_1 - p_n, \ldots, p_{n-1} - p_n) \stackrel{\text{def}}{=} \frac{\partial^{n-1} D_i}{\partial p_1 \ldots [\partial p_i] \ldots \partial p_n} > 0, \quad i = 1, \ldots, n. \tag{14}$$

Finally, as each consumer buys one unit of a single variant, the total demand for the variants must equal N, or

(A4) $\quad \sum_{i=1}^{n} D_i(p_1 - p_i, \ldots, p_n - p_i) = N.$

Replacing $p_j - p_n$ by $4ac\hat{z}^j + \alpha_j - \alpha_n$ in (14) and equating to (13) then yields the following result (where we henceforth replace $\hat{z}$ by z since any point in $\mathbb{R}^m$ can be reached by appropriate choices of prices) :

PROPOSITION 2. Assume that the variant locations $z_1, \ldots, z_n$ are given by (7) and (8). The consumer density function $f(z)$ consistent with (A1) - (A4) *is unique and given by*

$$f(z) = (4ac)^{n-1} \varphi(4acz^1 + \alpha_1 - \alpha_n, \ldots, 4acz^{n-1} + \alpha_{n-1} - \alpha_n). \tag{15}$$

Intuitively, the construction of $f(z)$ can be understood as follows. For any $z \in \mathbb{R}^m$, a vector $(p_1 - p_n, \ldots, p_{n-1} - p_n)$ can be found such that consumers at z are indifferent between all variants. A marginal increase in p_1 induces consumers on the boundary of M_1 (including therefore those at $\underline{z}$) not to buy variant 1. A similar increase in p_2 leads some of these consumers (including again those at $\underline{z}$) not to purchase variant 2 either. Repeating the argument for variants $3 \ldots n-1$, we are left with the consumers at $\underline{z}$ who now want to purchase only variant n.

Note that locations (7) and (8) have been chosen for simplicity. Other locations (subject to the restriction imposed by Proposition 1) would again lead to a unique density function (but, of course, different from (15)). Furthermore, when $n = 2$ and $\alpha_1 = \alpha_2$, the density function takes the simple form

$$f(z) = 4ac \, \frac{\partial D_2}{\partial p_1}(4acz) = 4ac \, \frac{\partial D_1}{\partial p_2}(-4acz) \tag{16}$$

with $z \in \mathbb{R}$.

It remains to discuss the case where $m > n-1$, that is, the space of characteristics is rich with respect to the number of variants. Following the same procedure as in the previous case, we must specify the variant locations subject to the restriction imposed by Proposition 1. This leads us to define $\underline{z}_n$ as in (8), whereas $\underline{z}_1, \ldots, \underline{z}_{n-1}$ are now defined as follows :

$$z_i^j = \begin{cases} a & \text{if} \quad i = j, \\ -a & \text{otherwise} \end{cases} \qquad i = 1, \ldots, n-1; \; j = 1, \ldots, m. \tag{17}$$

The difference between (7) and (17) is that j can take values larger than $n-1$: for the same characteristics space, we have less variants than before.

In view of (17), all the variants offer an identical amount of each of the characteristics $n, n+1, \ldots, m$. As a result, what matters for consumer choice is the amount of each of the characteristics $1, 2, \ldots, n-1$. Hence, consumers with identical preferences on characteristics $1, \ldots, n-1$ can be bunched together. Formally, this means that we can reduce the dimension of the characteristics space from m to $n-1$ and that we can replace the density function $f(\underline{z})$ defined on $\mathbb{R}^m$ by the marginal density function

$$g(z^1, \ldots, z^{n-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\underline{z}) dz^n, \ldots, dz^m \tag{18}$$

defined on $\mathbb{R}^{n-1}$. Consequently, we can state :

PROPOSITION 3. Assume that the variant locations $\underline{z}_1, \ldots, \underline{z}_n$ are given by (8) and (17). The consumer density function $f(\underline{z})$ is consistent with the demand system **(A1) - (A4)** *if and only if its marginal density function (18) is given by (14).*

Because $m > n-1$, we get only $n-1$ equations similar to (11) which precludes us from determining a unique point $\hat{\underline{z}}$ in $\mathbb{R}^m$ where consumers are indifferent among variants. Instead, we have a linear variety of dimension $m-(n-1)$ of points in $\mathbb{R}^m$ satisfying this condition. Therein lies the reason for the nonuniqueness of the density function $f(\underline{z})$ which is able to generate the demand system satisfying **(A1) - (A4)**.

4. EXAMPLES

a. **The multinomial logit model.** The demand function of variant i associated with the logit is given by (with $\alpha_i = \alpha$)

$$D_i(p_1 - p_i, \ldots, p_n - p_i) = \frac{N}{1 + \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} \exp\left(-\frac{(p_j - p_i)}{\mu}\right)} \tag{19}$$

where μ is a positive constant which is interpreted in discrete choice theory as the standard-deviation of consumer tastes. It is readily verified that (19) satisfied conditions **(A1) - (A4)**.

Computing $\frac{\partial^{n-1} D_n}{\partial p_1, \ldots, \partial p_{n-1}}$ for (19) yields

$$\varphi(p_1 - p_n, \ldots, p_{n-1} - p_n) = -$$

$$N\mu^{1-n}(n-1)! \prod_{i=1}^{n-1} \exp\left(-\frac{p_i - p_n}{\mu}\right) \left[1 + \sum_{j=1}^{n-1} \exp\left(-\frac{p_j - p_n}{\mu}\right)\right]^{-n}.$$

Hence, using Proposition 2 and (10), we obtain :

PROPOSITION 4. Let $m = n-1$. The consumer density function consistent with the logit demand (19) is given by

$$f(\underline{z}) = N\left(\frac{4ac}{\mu}\right)^{n-1} (n-1)\,! \; \frac{\prod_{i=1}^{n-1} \exp\left(-\frac{4ac}{\mu} z^i\right)}{\left[1 + \sum_{j=1}^{n-1} \exp\left(-\frac{4ac}{\mu} z^j\right)\right]^n} \tag{20}$$

This function has the following properties :

(i) $f(\underline{z})$ has a unique maximum at $\underline{z} = \underline{0}$ which is equal to $N\left(\frac{4ac}{n\mu}\right)^{n-1}\frac{(n-1)!}{n}$. To show this, it is sufficient to check that $z^i = 0,\ i = 1,\ldots,n$, is the only solution of the FOC. As $f(\underline{z}) > 0$ and $\lim_{z^i \to \infty} f(\underline{z}) = 0$, it must be that this solution is the only global maximizer;

(ii) the spatial distribution of consumers is more spread out as μ rises and as a and c fall;

(iii) for $\mu \to 0$, the density function reduces to an atomic distribution of value N at the origin; in this case, all consumers buy the variant with the lowest price. For $\mu \to \infty$, the density approaches zero everywhere (with the total mass remaining equal to N); in this case, consumers tend to buy the closest variant, regardless of its price so that $D_i = \frac{N}{n}$;

(iv) when $m > n-1$, the expression (20) in which $n-1$ is replaced by m is a possible density function in $I\!R^m$ since its marginal density function is precisely (20). In other words, the structure of (20) is invariant with respect to the number of variants;

(v) when $m = 1$ and $n = 2$, (20) reduces to

$$f(z) = N\frac{4ac}{\mu}\quad\frac{\exp\left(-\frac{4ac}{\mu}z\right)}{\left[1+\exp\left(-\frac{4ac}{\mu}z\right)\right]^2} \tag{21}$$

for $z \in I\!R$. This is the logistic formula with zero mean and standard-deviation equal to $\frac{\mu}{4ac}\ \frac{\pi}{\sqrt{3}}$, which is symmetric around the origin.

A final comment is in order. For a fixed consumer density function (20), any variation in μ is equivalent to a proportional variation in a or c. Intuitively, this means that a change in the standard-deviation of consumer tastes can be represented in the address model by a similar change either in the distance between variants or in the marginal transport rate. As a consequence, the terms *consumer heterogeneity* or *product heterogeneity* can be used interchangeably at the aggregate level. This is also true for the subsequent examples.

b. The multinomial probit model. The demand function of variant i for the probit can be written as follows (with $\alpha_i = \alpha$) :

$$D_i(p_1 - p_i, \ldots, p_n - p_i) =$$

$$N \int_{-\infty}^{p_1 - p_i} \cdots \left[\int_{-\infty}^{p_i - p_i} \right] \cdots \int_{-\infty}^{p_n - p_i} \mathcal{N}(\underline{x}; \underline{0}, \Omega) dx^1 \ldots [dx^i] \ldots dx^n \tag{22}$$

where $\mathcal{N}(\underline{x}; \underline{0}, \Omega)$ is the multivariate normal density with mean $\underline{0}$ and covariance matrix Ω (see Domenchich and McFadden (1975)). Using (10), D_n becomes

$$N(4ac)^{n-1} \int_{-\infty}^{\hat{z}^1} \cdots \int_{-\infty}^{\hat{z}^{n-1}} \mathcal{N}(4ac\underline{z}; \underline{0}, \Omega) dz' \ldots dz^{n-1}. \tag{23}$$

This form is directly comparable with (11) so that we can state

PROPOSITION 5. Let $m = n - 1$. The consumer density function consistent with the probit demand (22) is given by

$$f(\underline{z}) = N(4ac)^{n-1} \mathcal{N}(4ac\underline{z}; \underline{0}, \Omega). \tag{24}$$

Hence consumers are distributed according to a normal distribution with zero mean and covariance matrix $\Omega/(4ac)^2$. In particular, for $n = 2$, we have

$$f(z) = N \frac{4ac}{\sigma} \psi \left(\frac{4ac}{\sigma} \right) \tag{25}$$

where ψ is the density function of the standard normal distribution. In other words, consumers are distributed over $\mathbb{R}$ according to a normal distribution of mean zero and standard-deviation $\frac{\sigma}{4ac}$. From (21), it is clear that μ and σ play the same role.

c. **The linear probability model.** This model is only defined for the binary case ($n = 2$). The demands are given by :

$$D_i(p_j - p_i) = N \begin{cases} 0 & \text{if} \quad p_i - p_j > L \\ \frac{p_j - p_i + L}{2L} & \text{if} \quad -L \le p_i - p_j \le L; \\ 1 & \text{if} \quad p_i - p_j < -L \end{cases} \qquad i, j = 1, 2 \tag{26}$$

where L is a positive constant.

Note that **(A1)** is not satisfied since $\frac{\partial D_i}{\partial p_j} = 0$ for some prices. However, our analysis remains applicable. Given (7) and (8), the two firms are located at $z_1 = a$ and $z_2 = -a$. Taking the derivative of (26) and using (10) yields

$$\frac{\partial D_2}{\partial p_1} = N \begin{cases} 0 & \text{if} \quad z < -L/4ac \\ 1/2L & \text{if} \quad -L/4ac < z < L/4ac \\ 0 & \text{if} \quad z > L/4ac \end{cases} \tag{27}$$

Applying (16) then enables us to write

PROPOSITION 6. Let $m = n - 1 = 1$. The consumer density function consistent with the linear demand (26) is given by

$$f(z) = \begin{cases} 0 & \text{for} \quad |z| > L/4ac \\ \frac{2acN}{L} & \text{for} \quad |z| \leq L/4ac \end{cases} \tag{28}$$

This is the standard rectangular density function most commonly used in spatial competition since Hotelling (1929). Here the role of μ in the logit is played by L.

5. THE CES UTILITY MODEL

In the foregoing analysis, it has been assumed that consumers make indivisible purchases (see assumption (**B3**)). This fits some commodities well (like durables) but not commodities that are (perfectly) divisible. In the latter context, (**B3**) should be replaced in order to allow each consumer to buy his optimal *quantity* of a certain variant.

Model III. We consider a representative consumer with income Y and a CES-type utility function

$$U\left\{\left[\sum_{i=1}^{n} Q_i^{\rho}\right]^{1/\rho}, Q_0\right\} = \left[\sum_{i=1}^{n} Q_i^{\rho}\right] Q_0^{\alpha} \tag{29}$$

where Q_i is the quantity consumed of variant i, Q_0 the numéraire, ρ a constant such that $0 \leq \rho \leq 1$ and α a positive constant. The demand function of variant i is then given by

$$D_i(p_1, \ldots, p_n) = \frac{Y}{1+\alpha} \cdot \frac{p_i^{1/(\rho-1)}}{\sum_{j=1}^{n} p_j^{(\rho/\rho-1)}}, \quad i = 1, \ldots, n. \tag{30}$$

Clearly, D_i satisfies assumption (**A1**). Furthermore, the derivative of D_i w.r.t. $p_1 \ldots [p_i] \ldots p_n$ is positive and the same for all i which implies that the analogous condition to (**A3**) holds.

Model IV. The address model we wish to compare to Model III is defined by (**B1**), (**B2**) and

(**B3 bis**). The utility of a consumer located at $\underline{z}$ and purchasing quantity q_i of variant i is given by

$$U_i(\underline{z}; q_i) = ln\ q_i + y - p_i q_i - c \sum_{k=1}^{m} (z^k - z_i^k)^2, \quad i = 1, \ldots, n \tag{31}$$

where $y > 1$ is the income of the consumer.

In this expression, the first three terms define the gross utility while the last term corresponds to a lump-sum loss to be interpreted as in Model II. [3] Consumer choice can now be viewed as a two-stage process : in the first stage he chooses the variant i to buy and, in the second, the amount purchased, i.e. $q_i^* = 1/p_i$.

Clearly, Proposition 1 remains valid in the context of Model IV so that we must have $m \geq n - 1$. For simplicity, we limit ourselves to $m = n - 1$. The case $m > n - 1$ can be treated as in Section 3. A straightforward calculation using the indirect utilities shows that, for (7) and (8) unchanged, the market space of variant n is given by (cf. (9))

$$M_n = \{\underline{z} \in I\!R^n; z^k \leq \frac{-ln p_n + ln p_j}{4ac}, \quad j = 1, \ldots, n-1\}. \tag{32}$$

Thus, the set of consumers indifferent between purchasing variants n and j is a hyperplane orthogonal to the j-axis at

$$\hat{z}^j = \frac{ln p_j / p_n}{4ac}, \quad j = 1, \ldots, n-1 \tag{33}$$

As every consumer in M_n buys $1/p_n$ units of variant n, the demand X_n is now equal to

$$X_n = \frac{1}{p_n} \int_{-\infty}^{\hat{z}_1} \cdots \int_{-\infty}^{\hat{z}_{n-1}} f(\underline{z}) d\underline{z}. \tag{34}$$

Unlike (11), X_n is no longer a function of $p_i - p_n$ only, so that assumption (**A2**) becomes irrelevant.

Following the approach of Section 3, we take the derivative of X_n w.r.t. $p_1, \ldots, p_{n-1}$ and obtain

$$\frac{\partial^{n-1} X_n}{\partial p_1 \ldots \partial p_{n-1}} = \frac{1}{p_1 \ldots p_n} (4ac)^{1-n}\ f(\hat{\underline{z}}). \tag{35}$$

We now claim that the consumer density function (20)derived for the multinomial logit can be made consistent with the a.d.s. (30). Indeed, substituting (33) into (20), we get

$$\frac{\partial^{n-1} X_n}{\partial p_1 \ldots \partial p_{n-1}} = N\mu^{1-n}(n-1)! \frac{\prod_{j=1}^{n} p_j^{-1/\mu-1}}{\left[\sum_{j=1}^{n} p_j^{1/\mu}\right]^{n}}. \tag{36}$$

Integrating (36) and neglecting the constant of integration at each stage yield

$$X_n = N\ p_n^{-1/\mu-1}\left[\sum_{j=1}^{n} p_j^{-1/\mu}\right]. \tag{37}$$

Using again the argument of Appendix III, the comparison of (30) and (37) shows the following :

PROPOSITION 7. Let $m = n-1$. *If* $\mu = \frac{1-\rho}{\rho}$ *and* $N = \frac{Y}{1+\alpha}$, *then the a.d.s. generated by the CES-type utility function (29) is consistent with Model IV in which the consumer density function is given by (20).*

In other words, the CES-representative consumer does indeed represent the aggregate preferences of consumers distributed in a certain manner over a characteristics space, provided that the number of characteristics is large enough compared to the number of variants. Furthermore, $N = \frac{Y}{1+\alpha}$ is just an accounting condition which says that the sum of the individual expenses on the variants ($Nq_i^* p_i$) is equal to the income $\frac{Y}{1+\alpha}$ of the representative consumer available for buying the differentiated product. Also, $\mu = \frac{1-\rho}{\rho}$ shows that the parameters μ and ρ are inversely related. In particular, when $\mu = 0$ and $\rho = 1$, the n variants are perfect substitutes. On the other hand, when $\mu \rightarrow \infty$, the density $f(\tilde{z})$ approaches zero with a mass equal to N and $X_i = \frac{N}{np_i}$; equivalently, for $\rho \rightarrow 0$, the utility function (29) reduces to a Cobb-Douglas function $\prod_{i=1}^{n} x_i^{1/n} Q_0^{\alpha}$ with $D_i = \frac{Y}{n(1+\alpha)p_i}$. Finally, the individual demand for the numéraire is $y-1$ so that the aggregate demand is $(y-1)N$. The representative consumer's demand is $\frac{\alpha Y}{1+\alpha}$. Hence, consistency imposes the following additional condition :

$$(y-1)N = \frac{\alpha}{1+\alpha} Y. \tag{38}$$

The above proposition reveals some particular and interesting connections between the CES and logit demands. More specifically, the CES demand (30) can be viewed as a "fractional" version of the logit demand (19) in which the deterministic utility term $-p_i$ is replaced by $-lnp_i$. Indeed, using this replacement in (19), we obtain from the logit formula $p_i^{-1/\mu} / \sum_{j=1}^{n} p_j^{-1/\mu}$ which gives the CES demand function for variant i when multiplied by the fraction $1/p_i$. Within the context of the address framework, the consumer density functions is the same under the logit and CES models, but the market spaces, for a given price vector, are not the same (compare (9) and (32)). In addition, consumers choosing variant i buy a single unit under the logit while they buy $1/p_i$ under the CES.

6. CONCLUSIONS

In this paper, we have shown that several a.d.s. commonly used in theoretical and empirical work on product differentiation do represent the consumer preferences over a certain characteristics space. To this effect, we have chosen a simple and specific address model and identified a set of conditions that guarantee consistency between our address model and the a.d.s. under study.

This is true, in particular, for the main discrete choice theory models. In a criticism of Perloff and Salop (1985)'s paper, it was claimed by Archibald et al.(1986, pp.15-16) that the "cost" of interpreting the probabilistic choice model as an address model was tantamount to assuming characteristics spaces to be individual-specific, thus making the primitives of the model unclear. We have shown here that this latter position is not correct. From Proposition 1, we require that the characteristics space have the dimension at least $n-1$, where n, is the number of variants, for the reconciliation to be possible. Furthermore, for the probabilistic choice model to twin with our address model, we need specific restrictions on variant locations. For an m-dimensional characteristics space, with $m \geq n-1$, variant locations must form an $(n-1)$-dimensional basis for a consumer density function to exist which is consistent with the GSP.

The logit model generates an a.d.s. which can be shown to be consistent with a representative consumer model (see, e.g. Anderson et al. (1986)). Hence, *the conditions stated in this paper are also sufficient to reconcile, at least for a class of models including the logit and the CES, the representative consumer and address approaches to product differentiation.*

For a given a.d.s., if an additional variant is added to the market, it must locate at a specific point of an "unused" dimension of the characteristics space in order to preserve the structure of the a.d.s. (see, e.g. the logit or the probit). If not, we may end up with a completely different a.d.s. While any given form of an a.d.s. imposes a restrictive assumption on the horizontal location decisions, it should be noted that the model allows generality in the decision of vertical differentiation through the choice of qualities α_i.

REFERENCES

Amemiya, T. (1981), "Qualitative Response Models : A Survey," *Journal of Economic Literature* 19, 1483-1536.

Anderson, S.P., A. de Palma and J.-F. Thisse (1986), "A Representative Consumer Theory of the Logit Model," CORE DP 8643, Université Catholique de Louvain, Louvain-la-Neuve, Belgium.

Archibald, G.C., B.C. Eaton and R.G. Lipsey (1986), "Address Models of Value Theory," in *New Developments in the Analysis of Market Structure*, J.E.Stiglitz and G.F. Mathewson (eds.), The MIT Press, Cambridge, Mass.

Dixit, A.K. and J.E. Stiglitz (1977), "Monopolistic Competition and Optimum Product Diversity," *American Economic Review* 67, 297-308.

Domenchich, T. and D. McFadden (1975), *Urban Travel Demand : A Behavioral Analysis*, North-Holland, Amsterdam.

Eaton, B.C. and M.H. Wooders (1985), "Sophisticated Entry in a Model of Spatial Competition," *Rand Journal of Economics*, 16, 282-297.

Friedman, J.W. (1977), *Oligopoly and the Theory of Games*, North-Holland, Amsterdam.

Hanjoul, P. and J.-C. Thill (1987), "Elements of Planar Analysis in Spatial Competition," *Regional Science and Urban Economics*, forthcoming.

Hotelling, H. (1929), "Stability in Competition," *Economic Journal*, 39, 41-57.

Lancaster, K. (1979), *Variety, Equity and Efficiency*, Columbia University Press, New York.

Manski, C.F. (1977), "The Structure of Random Utility Models," *Theory and Decision*, 8, 229-254.

McFadden, D. (1984), "Econometric Analysis of Qualitative Response Models," in *Handbook of Econometrics*, North-Holland, Amsterdam.

Perloff, J.M. and S.C. Salop (1985), "Equilibrium with Product Differentiation," *Review of Economic Studies* 52, 107-120.

Shubik, M. (1959), *Strategy and Market Structure*, J. Wiley, New York.

Spence, A.M. (1976), "Product Selection, Fixed Costs, and Monopolistic Competition," *Review of Economic Studies* 43, 217-235.

Stahl, K. (1987), "Theories of Urban Business Location," in *Handbook of Regional and Urban Economics, Volume II : Urban Economics*, North-Holland, Amsterdam.

Footnotes

[1] If $\underline{z}_i = \underline{z}_j$ the GSP cannot hold.

[2] Even in the two-dimensional case, Hanjoul and Thill (1987) have shown that alternative transport cost specifications lead to extremely complex expressions for market areas.

[3] This specification is not the standard one used in spatial models, where it is frequently assumed that transportation cost varies with quantity. Stahl (1987) forceably argues that the lump-sum transportation cost assumption makes more sense in several spatial contexts. In the characteristics interpretation, both assumptions seem a priori conceivable.

APPENDIX

I. Proof of Proposition 1. (An illustration of the proof is contained in Figure 3.)

For (2) to hold, it must be that $M_i \cap M_j \neq \emptyset$ for all $i, j = 1, \ldots, n$ whatever $V_1, \ldots, V_n$. Suppose, then, that $\{\underline{z}_1, \ldots, \underline{z}_n\}$ contains $\ell < n-1$ linearly independent points.

Let $\underline{z}_1$ be an extreme point of the convex hull P of $\{\underline{z}_1, \ldots, \underline{z}_n\}$. W.l.o.g. set $\underline{z}_1 = \underline{0}$. Let also C be the cone $\{\underline{z} \in \mathbb{R}^m; \underline{z} = \sum_{i=2}^n \lambda_i \underline{z}_i, \lambda_i \geq 0\}$. This cone is generated by ℓ independent points of $\{\underline{z}_1, \ldots, \underline{z}_n\}$, say $\{\underline{z}_2, \ldots, \underline{z}_{\ell+1}\}$.

Denote by $\underline{z}_1\underline{z}_i$ (with $\underline{z}_1 = \underline{0}$) the straight line passing through $\underline{z}_1$ and $\underline{z}_i$ for $i = 2, \ldots, \ell+1$. We can always choose $V_1, \ldots, V_{\ell+1}$ such that (i) the hyperplane $H_{1i}, i = 2, \ldots, \ell+1$, orthogonal to $\underline{z}_1\underline{z}_i$ passes through $\underline{z}_1$, and (ii) the consumers located in the closed half-space containing $\underline{z}_i$ prefer $\underline{z}_i$ to $\underline{z}_1$. Consequently, the intersection of the half-spaces that do not contain $\{\underline{z}_2, \ldots, \underline{z}_{\ell+1}\}$ defines a convex polyhedron that must include M_1. By construction of C, $\underline{z}_j \in C$ for $j = \ell+2, \ldots, n$. Then, for any such $\underline{z}_j$ we can always choose V_j for the hyperplane H_{1j} orthogonal to the line $\underline{z}_1\underline{z}_j$ to be such that P is strictly on one side of H_{1j} and the consumers who prefer $\underline{z}_j$ to $\underline{z}_1$ are on the other side. Hence, it must be that M_j is included in the latter closed half-space. Consequently, we have $M_1 \cap M_j = \emptyset$, a contradiction.

∎

[Insert here Figure 3]

II. Using (7) and (8), we may rewrite $M_i, i \neq n$, as follows :

$$M_i = \{\underline{z} \in \mathbb{R}^m; V_j - V_i \leq -4ac(z^j - z^i), j = 1, \ldots, n-1, \text{and } V_n - V_i \leq 4acz^i\}$$
$$= \{\underline{z} \in \mathbb{R}^m; z^j \leq \hat{z}^j + z^i - \hat{z}^i, j = 1, \ldots, n-1, \text{and} z^i \geq \hat{z}^i\}.$$

Accordingly, we have :

$$X_i = \int_{\hat{z}^i}^{\infty} \int_{-\infty}^{\hat{z}^1 + z^i - \hat{z}^i} \cdots \int_{-\infty}^{\hat{z}^{n-1} + z^i - \hat{z}^i} f(\underline{z}) d\underline{z}. \tag{39}$$

Taking the derivative of X_i w.r.t. $p_1 \ldots [p_i] \ldots p_{n-1}$ then yields

$$\frac{\partial^{n-2} X_i}{\partial p_1 \ldots [\partial p_i] \ldots \partial p_{n-1}} = (4ac)^{2-n} \int_{\hat{z}^i}^{\infty} f(\hat{z}^1 + z^i - \hat{z}^i, \ldots, z^i, \ldots, \hat{z}^{n-1} + z^i - \hat{z}^i) dz^i$$

from which it follows that

$$\frac{\partial^{n-1} X_i}{\partial p_1 \ldots [\partial p_i] \ldots \partial p_n} = (4ac)^{1-n} f(\hat{z}).$$

Thus, by (13), we get

$$\frac{\partial^{n-1} X_n}{\partial p_1 \ldots \partial p_{n-1}} = \frac{\partial^{n-1} X_i}{\partial p_1 \ldots [\partial p_i] \ldots \partial p_n}. \tag{40}$$

FIGURE 1.

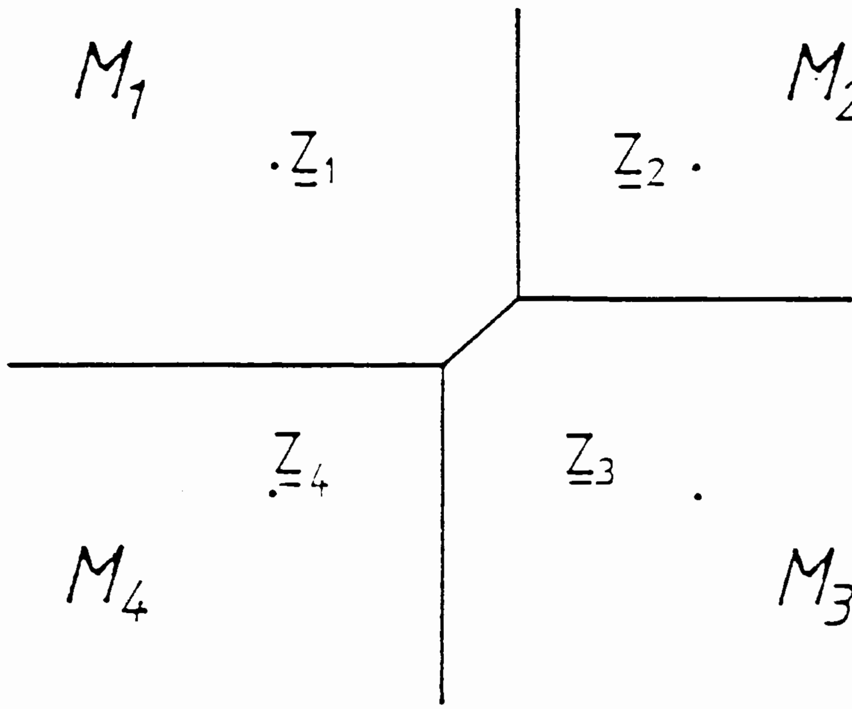
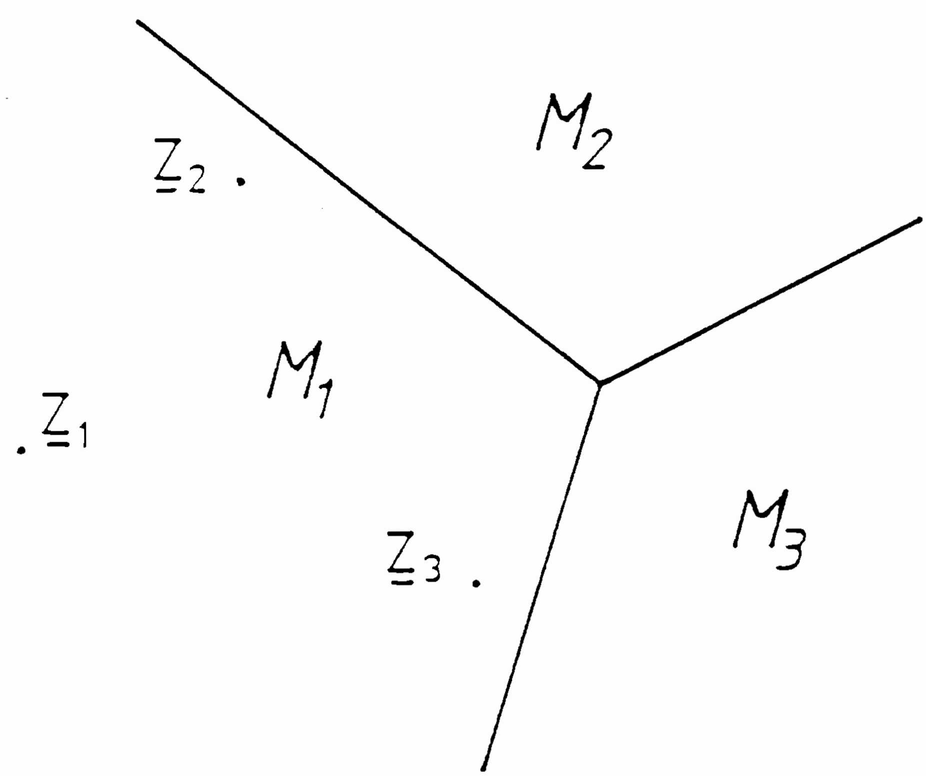


FIGURE 2.



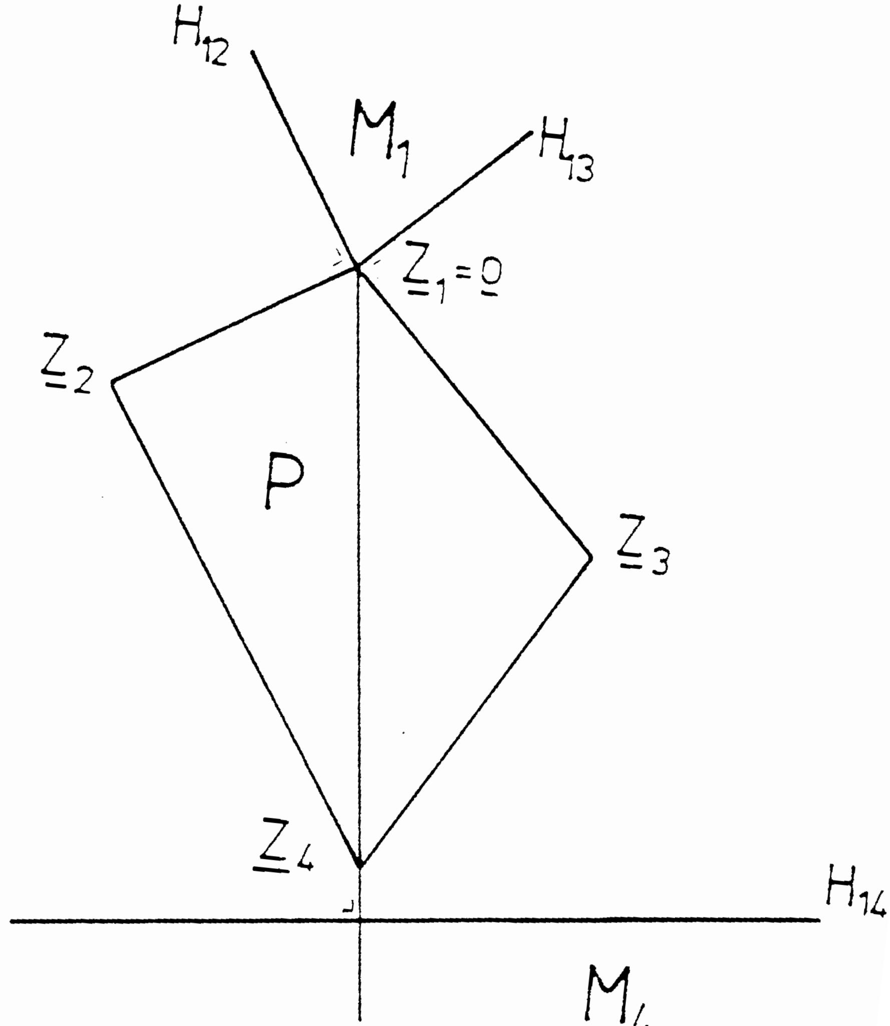


FIGURE 3.